GLOBAL-IN-TIME SMOOTH SOLUTIONS TO A THREE-DIMENSIONAL CAHN-HILLIARD-STOKES MODEL

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Abstract. The analysis for a Cahn-Hilliard-Stokes model, a modified Cahn-Hilliard equation coupled with a Stokes flow, is provided in this article. Such an equation can be viewed as a specialized conserved gradient flow with respect to the standard Cahn-Hilliard energy. The velocity vector is determined by the phase variable by either the Darcy law or a static Stokes equation. If the Darcy law is imposed, a global-in-time weak solution and a local-in-time strong solution have been established for the three-dimensional model in the existing literature. Instead, if a static Stokes equation is formulated to update the velocity, we present a global-in-time strong solution and smooth solution for the 3D Cahn-Hilliard-Stokes model in this paper, and the uniqueness of the strong solution is also proven. Such an analysis is accomplished with the help of an $L^\infty$ a-priori estimate of the velocity vector, in terms of the phase variable.

Key words. Cahn-Hilliard-Stokes, Stokes equation, Sobolev embedding, Galerkin procedure, $L^\infty$ estimate

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1. Introduction. The Cahn-Hilliard-Flow diffuse interface family of models – for example, the Cahn-Hilliard-Navier-Stokes, Cahn-Hilliard-Brinkman, Cahn-Hilliard-Stokes, and Cahn-Hilliard-Hele-Shaw models – describe the process of phase separation of a viscous, binary fluid into domains in which the fluid is very nearly pure in the respective components. We refer the readers to the related references [2, 20, 24, 30, 32, 33, 34, 35, 36, 41, 47, 48, 49] for physical background, numerical approximations, and some PDE analyses, such as existence, uniqueness and regularity.

The Cahn-Hilliard energy functional is given by [8]

$$E(\phi) := \int_{\Omega} \left[ \frac{1}{4\varepsilon} \phi^4 - \frac{1}{2\varepsilon} \phi^2 + \frac{\varepsilon}{2} |\nabla \phi|^2 \right] dx,$$

and the Cahn-Hilliard equation takes the form

$$\partial_t \phi = \varepsilon \Delta \mu, \quad \mu = \delta_\phi E = \varepsilon^{-1} \left( \phi^3 - \phi \right) - \varepsilon \Delta \phi, \quad \text{in } \Omega_T,$$
with appropriate boundary conditions to close the system. Here \( \phi \) denotes the concentration of a binary fluid, \( \mu \) is the chemical potential, and \( \Omega_T := \Omega \times (0, T] \), where \( \Omega \subset \mathbb{R}^3 \) is a bounded domain. Formally, the parameter \( \varepsilon > 0 \) gives the thickness of the transition region, i.e., the diffuse interface thickness, between the two stable phases of the fluid. In certain asymptotic limits, as \( \varepsilon \) tends to zero, solutions to the Cahn-Hilliard equation tend to solutions of an appropriate sharp interface problem. It is well-known that with periodic boundary conditions, for example, the Cahn-Hilliard system is energy dissipative: \( \frac{d}{dt} E = -\varepsilon \| \nabla \mu \|_{L^2}^2 \leq 0 \).

Herein we shall consider the following Cahn-Hilliard-Flow model:

\[
\begin{align*}
\partial_t \phi &= \varepsilon \Delta \mu - \mathbf{u} \cdot \nabla \phi \quad \text{in } \Omega_T, \\
\mu &= \varepsilon^{-1} (\phi^3 - \phi) - \varepsilon \Delta \phi \quad \text{in } \Omega_T, \\
-\sigma \Delta \mathbf{u} + \kappa \mathbf{u} &= -\nabla p - \gamma \phi \nabla \mu \quad \text{in } \Omega_T, \\
\nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega_T,
\end{align*}
\]

where \( \mathbf{u} \) is the advective velocity, \( p \) is the pressure. The flow parameters are as follows: \( \gamma \geq 0 \) is related to surface tension, \( \sigma \geq 0 \) is a kinematic viscosity, and \( \kappa \geq 0 \) is the fluid permeability. The term \( -\gamma \phi \nabla \mu \) is a diffuse interface approximation of the singular surface force describing surface tension. For \( \sigma > 0 \), the model has been utilized for studying various viscous two-phase diffuse interface flows [11, 12, 20, 30, 32, 35, 36, 41]. The system can have several names, for instance, Cahn-Hilliard-Darcy-Stokes, Cahn-Hilliard-Brinkman, or Cahn-Hilliard-Stokes. Herein we use the latter, the Cahn-Hilliard-Stokes (CHS) system to refer to (1.3) – (1.6). In the singular case \( \sigma = 0 \), the Brinkman (or Darcy-Stokes) equation (1.5) becomes Darcy’s law, and the corresponding system (1.3) – (1.6) is referred to as the Cahn-Hilliard-Hele-Shaw (CHHS) system of equations, following [33, 34, 49].

The initial and boundary conditions are assumed to be

\[
\begin{align*}
\phi(\cdot, 0) &= \phi_0 \quad \text{in } \Omega, \\
\partial_n \phi &= \partial_n \mu = 0 \quad \text{on } \partial \Omega_T := \partial \Omega \times (0, T], \\
\mathbf{u} \cdot \mathbf{n} &= 0, \quad \partial_n (\mathbf{u} \cdot \mathbf{\tau}) = 0 \quad \text{on } \partial \Omega_T := \partial \Omega \times (0, T],
\end{align*}
\]

where \( \mathbf{n} \) is the unit outward normal vector, \( \partial_n := \mathbf{n} \cdot \nabla \), and \( \mathbf{\tau} \) is any unit tangential vector on the boundary, \( \partial \Omega \), which we assume is smooth. The boundary condition \( \mathbf{u} \cdot \mathbf{n} = 0 \) is called no-penetration, and \( \partial_n (\mathbf{u} \cdot \mathbf{\tau}) = 0 \) is called free-slip. Note that, for the CHHS system (the singularly perturbed case \( \sigma = 0 \)), only the no-penetration boundary condition is needed in (1.9). One can show that, for these particular flow
boundary conditions, if the fields are sufficiently regular, it follows that

\[-\Delta p = \gamma \nabla \cdot (\phi \nabla \mu) \quad \text{in } \Omega_T, \quad (1.10)\]

\[-\partial_n p = \gamma \phi \partial_n \mu = 0 \quad \text{on } \partial \Omega_T. \quad (1.11)\]

In short, one can separate the pressure and velocity calculations. The system (1.3) – (1.6) is mass conservative, i.e., \(\int_\Omega \phi(x,t) \, dx = \int_\Omega \phi(x,0) \, dx\), and energy dissipative [24, 49], i.e.,

\[d_t E(\phi) + \varepsilon \int_\Omega \nabla \mu \cdot \nabla \mu \, dx + \frac{1}{\gamma} \int_\Omega (\sigma \nabla u : \nabla u + \kappa u \cdot u) \, dx = 0. \quad (1.12)\]

The PDE analysis for the CHS system (1.3) – (1.9) has attracted some attention in recent years. In particular, the well-posedness of the CHHS system (\(\sigma = 0\)) can be found in [24]. In more detail, a convex splitting numerical scheme was formulated, with a mixed finite element approximation in space. Such an approximate construction assures an unconditional energy stability. As a result, using certain energy/compactness arguments, the authors obtained a weak convergence of the finite element numerical approximation to a global-in-time weak solution, with an \(L^\infty(0,T;H^1) \cap L^2(0,T;H^3)\) regularity for the phase variable and an \(L^2(0,T;L^2)\) regularity for the velocity. They proved uniqueness for sufficiently regular solutions, but did not establish the existence of such regular solutions.

The CHHS solution with higher order regularities was discussed in [48], using more advanced Littlewood-Paley theory. In more details, the regularity of \(L^\infty(0,T;H^s) \cap L^2(0,T;H^{s+2})\) for the phase variable, assuming initial data in \(H^s (s > \frac{d}{2} + 1)\), was established. The estimates are global-in-time for the 2-D CHHS system and local-in-time for the 3D model. In fact, several blow-up criteria in the 3D case were also stated. Meanwhile, in another recent work [47], global-in-time classical solutions were proven for the 3D CHHS system, if the initial data is close to an energy minimizer or the Péclet number is sufficiently small.

A careful calculation reveals that, the primary difficulty to establish a global-in-time strong solution for the 3D CHHS equation (with \(\sigma = 0\)) is due to the lack of any high-order norm estimates of the velocity variable when Darcy’s law is imposed. This difficulty makes an estimate for the nonlinear convection term very challenging. In this paper, we provide an analysis to establish a global-in-time strong solution for the CHS equation, with \(\sigma > 0\) in the fluid equation (1.5), with the regularity of \(L^\infty(0,T;H^s) \cap L^2(0,T;H^{s+2})\) for the phase variable, for initial data in \(H^s (s \geq 1)\).

This analysis is accomplished by the help of an \(L^\infty\) estimate of the velocity variable in terms of an \(L^2\) norm of \(-\gamma \phi \nabla \mu\), which comes from an application of elliptic
regularity and using the Helmholtz projection in the Darcy-Stokes (Brinkmann) equation (1.5). Such an $L^\infty$ estimate is also crucial to the uniqueness analysis of the strong solution. In addition, the Sobolev estimates for the nonlinear terms, both the one in the chemical potential and the one in the nonlinear convection, lead to a uniform-in-time $H^s$ estimate for the phase variable, for any $s \geq 1$. As a result, using the standard Galerkin procedure, we can construct approximate solutions, with uniform estimates in certain $H^s$ space, and the limit function turns out to be the unique strong solution of the CHS system.

The rest of the paper is organized as follows. Some basic tools and notation are introduced in Section 2. The weak solution is defined and analyzed in Section 3. In Section 4, a global-in-time strong solution, with regularity of $L^\infty(0,T;H^2) \cap L^2(0,T;H^4)$, is established for the CHS system (1.3) – (1.6), with any $H^2$ initial data for $\phi$. The uniqueness of the strong solution is also proved in that section. The existence of a solution with the arbitrary regularity of $L^\infty(0,T;H^s) \cap L^2(0,T;H^{s+2})$ (for $\phi$), is proven in Section 5, if the initial data for $\phi$ are in $H^s$. These estimates indicate a global-in-time (spatially) smooth solution for the CHS system (1.3) – (1.6), with sufficiently smooth initial data.

2. Notation and Basic Tools. We use the standard symbols for the basic Sobolev spaces and their norms. Let us also define the following spaces:

- $\tilde{L}^2(\Omega) := \{ u \in L^2(\Omega) \mid (u, 1) = 0 \}$, $\hat{H}^1(\Omega) := \tilde{L}^2(\Omega) \cap H^1(\Omega)$,
- $H_0^2(\Omega) := \{ u \in H^2(\Omega) \mid \partial_n u|_{\partial \Omega} = 0 \}$, $\hat{H}_N^2(\Omega) := \tilde{L}^2(\Omega) \cap H_0^2(\Omega)$,
- $H^{-1}(\Omega) := (H^1(\Omega))^*$, $\hat{H}^{-1}(\Omega) := \{ v \in H^{-1}(\Omega) \mid \langle v, 1 \rangle = 0 \}$,
- $V := \{ v \in [H^1(\Omega)]^3 \mid \nabla \cdot v = 0 \text{ in } \Omega, \ v \cdot n = 0 \text{ on } \partial \Omega \}$,

where $\langle \cdot, \cdot \rangle$ is the duality paring between $H^{-1}$ and $H^1$.

We prove several of our results using the Galerkin procedure. To this end, define the operator $A$ to be $-\Delta$ paired with homogeneous Neumann boundary conditions on $\partial \Omega$. We define the range of $A$ as $R(A) := \tilde{L}^2(\Omega)$. Then, since $\partial \Omega$ is smooth, the domain of $A$ is naturally $D(A) = \hat{H}_N^2(\Omega)$, and $A : D(A) \to R(A)$ is a positive, self-adjoint linear operator that admits a compact inverse.

Let $\mathcal{B} := \{ \Phi_j \}_{j=1}^\infty \subset \tilde{L}^2(\Omega)$ be the set of eigenfunctions of $A$, i.e., $A \Phi_j = \lambda_j \Phi_j$, for $j = 1, 2, \cdots$. Recall, $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, and $\lim_{n \to \infty} \lambda_n = \infty$. Since $\partial \Omega$ is smooth, the eigenfunctions satisfy $\Phi_j \in \hat{H}_N^2(\Omega) \cap C^\infty(\overline{\Omega})$, and they may be chosen so that $\mathcal{B}$ is an orthonormal basis for $\tilde{L}^2(\Omega)$. With this choice, $\mathcal{B}$ is an orthogonal (not orthonormal) basis for both of the spaces $\hat{H}^1(\Omega)$ and $\hat{H}_N^2(\Omega)$. We can increase $\mathcal{B}$ so the resulting set is an orthonormal basis for all of $L^2(\Omega)$. This is accomplished by
setting \( \Phi_0 \equiv |\Omega|^{-1/2} \in H_2^N(\Omega) \) and \( \lambda_0 := 0 \). Observe that, after a trivial modification of the domain of \( A \), we have \( A\Phi_0 = \lambda_0 \Phi_0 \). The set \( \mathcal{B} := \{ \Phi_j \}_{j=0}^{\infty} = \tilde{\mathcal{B}} \cup \{ \Phi_0 \} \) is an orthonormal basis for \( L^2(\Omega) \).

Set \( \mathcal{G}_M := \text{span} \left( \{ \Phi_j \}_{j=0}^{M} \right) \). The operator \( \mathcal{P}_M : L^2(\Omega) \to \mathcal{G}_M \) is the canonical orthogonal projection:

\[
\mathcal{P}_M f := \sum_{j=0}^{M} \hat{f}_j \Phi_j, \tag{2.1}
\]

where \( \hat{f}_j = (f, \Phi_j) \). Of course, if \( f \in \hat{L}^2(\Omega) \), then \( \hat{f}_0 = 0 \). We can extend the domain of definition \( \mathcal{P}_M \) to \( \hat{H}^{-1}(\Omega) \) as follows: if \( f \in \hat{H}^{-1}(\Omega) \) then

\[
\langle \mathcal{P}_M f, g \rangle := \langle f, \mathcal{P}_M g \rangle, \quad \forall g \in \hat{H}^1.
\]

Recall that \( \left( \hat{H}^1(\Omega), \|u\|_{\hat{H}^1} := \|\nabla u\|_{L^2} \right) \) and \( \left( \hat{H}_N^2(\Omega), \|u\|_{\hat{H}_N^2} := \|\Delta u\|_{L^2} \right) \) are Hilbert spaces. Likewise, \( \left( \hat{H}^{-1}(\Omega), \|\cdot\|_{\hat{H}^{-1}} \right) \) is a Hilbert space using the standard operator norm [12]. We have the following basic properties of the orthogonal projection that we state without proof [38]:

**Lemma 2.1.** Let \( X = \hat{H}^{-1}(\Omega), \hat{L}^2(\Omega), \hat{H}^1(\Omega), \) or \( \hat{H}_N^2(\Omega) \). Then, for any \( f \in X \),

\[
\|\mathcal{P}_M f\|_X \leq \|f\|_X, \quad \text{and} \quad \|f - \mathcal{P}_M f\|_X \xrightarrow{M \to \infty} 0. \tag{2.2}
\]

The results can be modified in a trivial way to accommodate functions that are not of mean zero.

**Lemma 2.2.** Suppose that \( k \in \{0,1,2,\cdots\} \), \( f \in \mathcal{G}_M \), and \( g \in H^1(\Omega) \). Then,

\[
(\nabla \Delta^{k} f, \nabla g) = - (\Delta^{k+1} f, g). \tag{2.3}
\]

If \( k \in \{0,1,2,\cdots\} \), \( f \in \mathcal{G}_M \), and \( g \in H_N^2(\Omega) \). Then,

\[
(\Delta^{k} f, \Delta g) = (\Delta^{k+1} f, g). \tag{2.4}
\]

**Proof.** Since \( \Phi_j \in C^\infty(\overline{\Omega}) \), it follows that, for any \( 0 \leq j, k < \infty \), \( \partial_n(\Delta^{k} \Phi_j) = 0 \) on \( \partial \Omega \). Consequently, \( \partial_n(\Delta^{k} f) = 0 \), on \( \partial \Omega \), for any non-negative integer \( k \). Utilizing the first and second Green identities, the result follows. \( \square \)

**Lemma 2.3.** Let \( m \) be a positive integer, and suppose that \( \psi \in H^{2m}(\Omega) \) satisfies the following boundary compatibility conditions:

\[
\partial_n(\Delta^k \psi)|_{\partial \Omega} = 0, \quad \forall \ k \in \{0,1,\cdots, m-1\}. \tag{2.5}
\]

Then the following stability holds:

\[
\|\Delta^m(\mathcal{P}_M \psi)\|_{L^2} \leq \|\Delta^m \psi\|_{L^2}. \tag{2.6}
\]
If $\psi \in H^{2m+1}(\Omega)$ satisfies the boundary compatibility conditions (2.5), then the following stability holds:

$$\|\nabla\Delta^m(P_M\psi)\|_{L^2} \leq \|\nabla\Delta^m\psi\|_{L^2}. \quad (2.7)$$

**Proof.** Since $\Delta^j(P_M\psi) \in G_M \subset H^2_N(\Omega)$, for each $j \in \{0, 1, 2, \ldots, m-1\}$, using the identity (2.4) we have

$$\|\Delta^m(P_M\psi)\|_{L^2}^2 = (\Delta^m(P_M\psi), \Delta^m(P_M\psi)) = (\Delta^{m+1}(P_M\psi), \Delta^{m-1}(P_M\psi))$$

$$= \cdots = (\Delta^{2m}(P_M\psi), \psi). \quad (2.8)$$

The last equality follows since $\Delta^{2m}(P_M\psi) \in G_M$, using the basic properties of the $P_M$ projection. Now, we need to work backwards, and for this direction we need to explicitly use the compatibility conditions (2.5). Since $\Delta^j\psi \in H^2_N(\Omega)$, for each $j \in \{0, 1, 2, \ldots, m-1\}$, and again using the identity (2.4), we have

$$(\Delta^{2m}(P_M\psi), \psi) = (\Delta^{2m-1}(P_M\psi), \Delta\psi) = (\Delta^{2m-2}(P_M\psi), \Delta^2\psi)$$

$$= \cdots = (\Delta^m(P_M\psi), \Delta^m\psi). \quad (2.9)$$

Using the Cauchy-Schwartz and Young inequalities, we have

$$\|\Delta^mP_M\psi\|_{L^2}^2 = (\Delta^mP_M\psi, \Delta^m\psi) \leq \frac{1}{2} \|\Delta^mP_M\psi\|_{L^2}^2 + \frac{1}{2} \|\Delta^m\psi\|_{L^2}^2, \quad (2.10)$$

and (2.6) follows.

The proof of (2.7) is quite similar. To start with, using (2.3), we observe that

$$\|\nabla\Delta^m(P_M\psi)\|_{L^2}^2 = - (\Delta^{m+1}(P_M\psi), \Delta^m(P_M\psi)). \quad (2.11)$$

From here, we can proceed as before; we skip the details for the sake of brevity. \box{

**Lemma 2.4.** Let $M \geq 1$ be fixed. For any $\psi \in G_M$ we have

$$\|\nabla\psi\|_{L^2}^2 \leq \|\psi\|_{L^2} \|\Delta\psi\|_{L^2}, \quad (2.12)$$

$$\|\Delta\psi\|_{L^2} \leq \|\nabla\psi\|_{L^2}^{\frac{1}{2}} \|\nabla\Delta\psi\|_{L^2}^{\frac{1}{2}}, \quad \|\nabla\Delta\psi\|_{L^2} \leq \|\Delta\psi\|_{L^2} \|\Delta^2\psi\|_{L^2}^{\frac{1}{2}}, \quad (2.13)$$

$$\|\Delta^2\psi\|_{L^2} \leq \|\nabla\psi\|_{L^2}^{\frac{1}{2}} \|\Delta^2\psi\|_{L^2}^{\frac{1}{2}}, \quad \|\nabla\Delta^2\psi\|_{L^2} \leq \|\nabla\psi\|_{L^2}^{\frac{1}{2}} \|\Delta^2\psi\|_{L^2}^{\frac{1}{2}}. \quad (2.14)$$

$$\|\Delta^3\psi\|_{L^2} \leq \|\psi\|_{L^2} \|\nabla\Delta\psi\|_{L^2}^{\frac{3}{2}}, \quad \|\nabla\Delta\psi\|_{L^2} \leq \|\nabla\psi\|_{L^2}^{\frac{3}{2}} \|\Delta\psi\|_{L^2}^{\frac{1}{2}}. \quad (2.15)$$

Frequent use will be made of following Gagliardo-Nirenberg interpolation inequality:

**Theorem 2.5 (Nirenberg [37]).** Let $\Omega \subset \mathbb{R}^d$ be a bounded, connected, open set with Lipschitz boundary, $1 \leq q, r \leq \infty$, $\frac{d}{m} \leq \theta \leq 1$ and

$$\frac{1}{p} - \frac{j}{d} = \left(\frac{1}{r} - \frac{m}{d}\right) \theta + \frac{1 - \theta}{q}. \quad (2.16)$$
Suppose that $\psi \in L^q(\Omega)$ with $\partial^\alpha \psi \in L^r(\Omega)$ for all $|\alpha| = m$. Then $\partial^\beta \psi \in L^p(\Omega)$ for all $|\beta| = j$, and there exists a constant $C = C(d, j, m, p, q, r, \Omega) > 0$ such that

$$|\psi|_{W^{j,p}} \leq C \left( |\psi|_{W^{m,r}} \|\psi\|_{L^q}^{1 - \theta} + \|\psi\|_{L^q} \right).$$

(2.17)

Since the boundary is smooth, the following elliptic regularity estimates are available [26]:

**Lemma 2.6.** There is a constant $C > 0$, such that, for any $\psi \in H^{m+2}(\Omega)$,

$$\|\psi\|_{H^{m+2}} \leq C (\|\Delta \psi\|_{H^m} + \|\psi\|_{L^2}), \quad \forall \psi \in H^{m+2}(\Omega),$$

(2.18)

$$\|\psi\|_{H^{2m}} \leq C (\|\Delta^m \psi\|_{L^2} + \|\psi\|_{L^2}), \quad \forall \psi \in H^{2m}(\Omega),$$

(2.19)

$$\|\psi\|_{H^{2m+1}} \leq C (\|\nabla \Delta^m \psi\|_{L^2} + \|\psi\|_{L^2}), \quad \forall \psi \in H^{2m+1}(\Omega).$$

(2.20)

As a consequence of the last lemma, the Gagliardo-Nirenberg inequality (2.17), and the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, the $L^\infty$ norm of $\psi \in H^3(\Omega)$ can be bounded by

$$\|\psi\|_{L^\infty} \leq C \left( \|\psi\|_{L^2}^{\frac{3}{2}} \|\nabla \psi\|_{H^1}^{\frac{1}{2}} + \|\psi\|_{L^6} \right) \leq C \left( \|\psi\|_{H^1}^{\frac{3}{2}} \|\nabla \Delta \psi\|_{L^2}^{\frac{1}{2}} + \|\psi\|_{H^1} \right).$$

(2.21)

Finally, we define a Helmholtz-type projection,

$$\mathcal{P}_H : \left\{ \mathbf{f} \in [H^1(\Omega)]^3 \mid \mathbf{f} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega \right\} \rightarrow \mathbb{V}.$$

(2.22)

Precisely, $\mathcal{P}_H(\mathbf{f}) := \mathbf{f} + \mathbf{\nabla} p$, where $p \in \dot{H}^2_N(\Omega) \cap H^1(\Omega)$ is the unique solution to $-\Delta p = \nabla \cdot \mathbf{f}$ in $\Omega$, as in (1.10) – (1.11). Clearly, $(\mathcal{P}_H(\mathbf{f}), \mathbf{f} - \mathcal{P}_H(\mathbf{f}))_{L^2} = 0$. From this we can prove the $L^2$ stability of the projection, which we use in the proof of Theorem 4.3 and elsewhere. Of course, sufficiently regular solutions to the CHS system (1.3) – (1.6) satisfy $-\sigma \Delta \mathbf{u} + \kappa \mathbf{u} = -\gamma \mathcal{P}_H (\phi \mathbf{\nabla} \mu)$, assuming the no flux, no-penetration, and free-slip boundary conditions.


We prove the existence of global-in-time weak solutions using the Galerkin method. From this point on we will assume that the viscosity is scaled to one, i.e., $\sigma = 1$. We define weak solutions as follows.

**Definition 3.1.** Let $T > 0$. The triple $(\phi, \mu, \mathbf{u})$, with

$$\phi \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^3(\Omega)) \cap C^0([0, T], L^2(\Omega)),$$

$$\partial_t \phi \in L^2(0, T; H^{-1}(\Omega)), \quad \mu \in L^2(0, T; H^1(\Omega)), \quad \mathbf{u} \in L^2(0, T; \mathbb{V}),$$

$$\partial_t \mu \in L^2(0, T; \mathbb{V}), \quad \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega.$$
is called a weak solution to the CHS system (1.3) – (1.6) on the interval [0, T] if and only if, for almost every \( t \in [0, T] \),

\[
\begin{align*}
& (\partial_t \phi, \psi) - (u \phi, \nabla \psi) + \varepsilon (\nabla \mu, \nabla \nu) = 0, \quad \forall \nu \in H^1(\Omega), \\
& (\mu, \psi) = \varepsilon^{-1} \left( \phi^3 - \phi \right) + \varepsilon (\nabla \phi, \nabla \psi), \quad \forall \psi \in H^1(\Omega), \\
& (\nabla u, \nabla v) + \kappa (u, v) = -\gamma (\phi \nabla \mu, v), \quad \forall \psi \in \mathcal{V},
\end{align*}
\]

and \( \phi(0) = \phi_0 \in H^1(\Omega) \). We have the following:

**Theorem 3.2.** Let \( \phi_0 \in H^1(\Omega) \). Then there exists at least one global-in-time weak solution for the CHS equation (1.3) – (1.6), such that for any \( T > 0 \)

\[
\begin{align*}
& \| \phi \|_{L^\infty(0,T;H^1)}^2 + \| \nabla \mu \|_{L^2(0,T;H^1)}^2 + \| u \|_{L^2(0,T;H^1)}^2 \leq \widetilde{C}_1, \\
& \| \phi \|_{L^2(0,T;H^2)}^2 + \| \partial_t \phi \|_{L^2(0,T;H^{-1})}^2 + \| \mu \|_{L^2(0,T;H^1)}^2 \leq \widetilde{C}_2(T + 1),
\end{align*}
\]

where \( \widetilde{C}_1, \widetilde{C}_2 > 0 \) are independent of \( T \).

**Proof.** We construct an approximate solution of the CHS system (1.3) – (1.6) as follows: find

\[
\phi_M(x, t) := \sum_{j=0}^M \alpha_{M,j}(t) \Phi_j(x), \quad \mu_M(x, t) := \sum_{j=0}^M \beta_{M,j}(t) \Phi_j(x),
\]

and \( \tilde{u}_M \in \mathcal{V} \), such that, for all \( t \in [0, T] \),

\[
\begin{align*}
& (\partial_t \phi_M, \nu) + \varepsilon (\nabla \mu_M, \nabla \nu) + (\tilde{u}_M \cdot \nabla \phi_M, \nu) = 0, \quad \forall \nu \in \mathcal{G}_M, \\
& \varepsilon^{-1} \left( \phi^3_M - \phi_M \right) + \varepsilon (\nabla \phi_M, \nabla \psi) + (\mu_M, \psi) = 0, \quad \forall \psi \in \mathcal{G}_M, \\
& (\nabla \tilde{u}_M, \nabla v) + \kappa (\tilde{u}_M, v) + \gamma (\phi_M \nabla \mu_M, v) = 0, \quad \forall v \in \mathcal{V},
\end{align*}
\]

and \( \phi_M(0) = P_M(\phi_0) \). The approximation scheme (3.7) – (3.9) represents a system of ordinary differential equations. It is straightforward to conclude at least the local in-time existence of the approximate solution up to some finite time \( T > 0 \), with

\[
\alpha_M(\cdot) := \left( \alpha_{M,1}(\cdot), \cdots, \alpha_{M,M}(\cdot) \right) \in [C^1([0,T])]^M.
\]

We will show that, in fact, \( T \) can be made arbitrarily large. For all \( t \in [0, T] \) and any \( M \geq 1 \), it follows that \( \tilde{u}_M(t) \in \left[ C^\infty(\Omega) \right]^3 \cap \mathcal{V} \).

By setting \( \nu = 1 \) in (3.7) we next observe that, for any \( t \geq 0 \), and any \( M \geq 1 \),

\[
(\phi_M(t) - \bar{\phi}_0, 1) = (\phi_M(0) - \bar{\phi}_0, 1) = (\phi_0 - \bar{\phi}_0, 1) = 0,
\]

where \( \bar{\phi}_0 := \left| \Omega \right|^{-1} \int_\Omega \phi_0(x) \, dx \). Furthermore, setting \( \nu = \mu_M \) in (3.7), \( \psi = \partial_t \phi_M \) in
(3.8), and \( \mathbf{v} = \mathbf{u}_M \) in (3.9), we have

\[
(\partial_t \phi_M, \mu_M) + (\mathbf{u}_M \cdot \nabla \phi_M, \mu_M) + \varepsilon \| \nabla \mu_M \|_{L^2}^2 = 0, \tag{3.12}
\]

\[
(\partial_t \phi_M, \mu_M) = (\partial_t \phi_M, \varepsilon^{-1} (\phi_M^3 - \phi_M) - \varepsilon \Delta \phi_M) = \frac{d}{dt} E(\phi_M). \tag{3.13}
\]

\[
(\mathbf{u}_M \cdot \nabla \phi_M, \mu_M) = \frac{1}{\gamma} \left( \| \nabla \mathbf{u}_M \|_{L^2}^2 + \kappa \| \mathbf{u}_M \|_{L^2}^2 \right). \tag{3.14}
\]

Putting these together, it follows that

\[
\frac{d}{dt} E(\phi_M) + \varepsilon \| \nabla \mu_M \|_{L^2}^2 + \frac{1}{\gamma} \left( \| \nabla \mathbf{u}_M \|_{L^2}^2 + \kappa \| \mathbf{u}_M \|_{L^2}^2 \right) = 0. \tag{3.15}
\]

In integral form, for any \( 0 \leq t \leq T \),

\[
E(\phi_M(t)) + \varepsilon \int_0^t \| \nabla \mu_M(s) \|_{L^2}^2 \, ds
+ \gamma^{-1} \int_0^t \left( \sigma \| \nabla \mathbf{u}_M(s) \|_{L^2}^2 + \kappa \| \mathbf{u}_M(s) \|_{L^2}^2 \right) \, ds = E(\phi_M(0)), \tag{3.16}
\]

from which it follows that

\[
\| \phi_M \|_{L^\infty(0,T;H^1)} + \| \nabla \mu_M \|_{L^\infty(0,T;L^2)}^2 + \| \mathbf{u}_M \|_{L^2(0,T;H^1)}^2 \leq C_1, \tag{3.17}
\]

for some \( C_1 > 0 \) that depends upon \( \| u_0 \|_{H^1} \) but is independent of \( T \) and \( M \). Here we have made use of the stability (2.2). As a consequence of this energy bound, a global-in-time solution for (3.7) – (3.9) is assured, for any fixed \( M \geq 1 \). In other words, \( T > 0 \) may be arbitrarily large. This follows by appealing to [38, Lemma 2.4], after observing that

\[
\max_{0 \leq t \leq T} \| \mathbf{a}_M(t) \|_{L^2}^2 = \| \phi_M(t) \|_{L^2}^2 \leq C_1. \tag{3.18}
\]

Now, fixing \( T > 0 \), more detailed Sobolev analyses indicate the following linear-in-time bounds [12, 24]:

\[
\| \phi_M \|_{L^2(0,T;H^3)}^2 + \| \mu_M \|_{L^2(0,T;H^1)}^2 + \| \partial_t \phi_M \|_{L^2(0,T;H^{-1})}^2 \leq C_2(T + 1), \tag{3.19}
\]

where \( C_2 > 0 \) is independent of \( T \). Since, for fixed \( T > 0 \), estimates (3.17) and (3.19) are uniform in \( M \), there exist subsequences \( \phi_{M_k}, \mu_{M_k}, \mathbf{u}_{M_k}, \) and \( \partial_t \phi_{M_k} \) and limit functions \( \phi \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^3(\Omega)), \mu \in L^2(0,T;H^1(\Omega)), \mathbf{u} \in L^2(0,T;\mathcal{V}), \) and \( \dot{\phi} \in L^2(0,T;H^{-1}(\Omega)) \) such that

\[
\phi_{M_k} \to \phi \text{ in } L^2(0,T;H^3(\Omega)), \quad \phi_{M_k} \rightharpoonup^\ast \phi \text{ in } L^\infty(0,T;H^1(\Omega)), \tag{3.20}
\]

\[
\mathbf{u}_{M_k} \to \mathbf{u} \text{ in } L^2(0,T;\mathcal{V}), \quad \partial_t \phi_{M_k} \to \dot{\phi} \text{ in } L^2(0,T;H^{-1}(\Omega)), \tag{3.21}
\]

\[
\mu_{M_k} \to \mu \text{ in } L^2(0,T;H^3(\Omega)). \tag{3.22}
\]
One can show that \( \dot{\phi} = \partial_t \phi \in L^2(0,T;H^{-1}) \) [18, 38]. Furthermore, by applying an improved compactness result [44, Theorem 2.3, Ch. 3] – see also [38, Theorem 8.1] – we also have

\[
\phi_{M\ell} \to \phi \text{ strongly in } L^2(0,T;H^2(\Omega)). \tag{3.23}
\]

Now we may pass to the limit showing that the limit function \((\phi,\mu,u)\) is indeed a weak solution in the sense of (3.1) – (3.3). One can also prove that \( \phi \in C^0([0,T],L^2(\Omega)) \), with \( \phi(0) = \phi_0 \in H^1(\Omega) \subset L^2(\Omega) \) [18, 38]. Furthermore, the estimates (3.4) and (3.5) hold for the limit functions. The details are omitted for brevity. \( \square \)

4. Existence and uniqueness of a global-in-time strong solution. Next we establish the existence and uniqueness of a global-in-time strong solution to the CHS equation (1.3) – (1.6). Namely, for any positive final time \( T > 0 \), we seek a weak solution with the additional regularities

\[
\phi \in L^\infty(0,T;H^2(\Omega)) \cap L^2(0,T;H^4(\Omega)), \quad \mu \in L^2(0,T;H^2(\Omega)),
\]

\[
u \in L^2(0,T;H^2(\Omega)), \quad \partial_t \phi \in L^2(0,T;L^2(\Omega)), \tag{4.1}
\]

such that (3.1) – (3.3) holds in the strong sense.

4.1. Existence of a global-in-time strong solution. We prove the existence of strong solutions first.

**Theorem 4.1.** Let \( \phi_0 \in H^2_N(\Omega) \). Then there exists a global-in-time strong solution (4.1) for the CHS equation (1.3) – (1.6), with \( \phi(0) = \phi_0 \), such that for any \( T > 0 \)

\[
\|\phi\|_{L^\infty(0,T;H^2)}^2 + \|\mu\|_{L^2(0,T;H^2)}^2 \leq \tilde{C}_3, \tag{4.2}
\]

\[
\|\phi\|_{L^2(0,T;H^4)}^2 + \|\mu\|_{L^2(0,T;H^2)}^2 + \|\partial_t \phi\|_{L^2(0,T;L^2)}^2 \leq \tilde{C}_4(T + 1), \tag{4.3}
\]

where \( \tilde{C}_3, \tilde{C}_4 > 0 \) are constants that are independent of the final time, \( T \).

**Proof.** Again, we construct approximate solutions using (3.7) – (3.9). Using the previous uniform estimates for the approximate solutions, we aim to prove that, for fixed \( M \geq 1 \),

\[
\|\phi_M\|_{L^\infty(0,T;H^2)}^2 + \|\mu_M\|_{L^2(0,T;H^2)}^2 \leq C_3, \tag{4.4}
\]

\[
\|\phi_M\|_{L^2(0,T;H^4)}^2 + \|\mu_M\|_{L^2(0,T;H^2)}^2 + \|\partial_t \phi_M\|_{L^2(0,T;L^2)}^2 \leq C_4(T + 1), \tag{4.5}
\]

where \( C_3, C_4 > 0 \) are independent of \( T \) and \( M \). To this end, setting \( \nu = \Delta^2 \phi_M \) in (3.7) and \( \psi = -\varepsilon \Delta^3 \phi_M \) in (3.8), using Lemma 2.2, and adding equations, we find, for
any $0 \leq t \leq T$,
\[
\frac{1}{2} \frac{d}{dt} \| \Delta \phi_M \|^2_{L^2} + \varepsilon^2 \| \Delta^2 \phi_M \|^2_{L^2} = (\Delta(\phi_M^3), \Delta^2 \phi_M) + \| \nabla \Delta \phi_M \|^2_{L^2} - (\bar{u}_M \cdot \nabla \phi_M, \Delta^2 \phi_M).
\]

The term associated with the concave diffusion can be treated in a straightforward way, with the help of Sobolev inequality (2.14):
\[
\| \nabla \Delta \phi_M \|^2_{L^2} \leq \| \nabla \phi_M \|^2_{L^2} \| \Delta^2 \phi_M \|^{\frac{4}{5}}_{L^2} \leq C_7^\frac{4}{5} \| \Delta^2 \phi_M \|^{\frac{4}{5}}_{L^2}.
\]

For the term associated with the nonlinear convection, we start with an application of Hölder’s inequality:
\[
\Delta(\phi_M^3) = 3\phi_M^2 \Delta \phi_M + 6 \phi_M \nabla \phi_M \cdot \nabla M
\]

and the 3D Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we have
\[
(\Delta(\phi_M^3), \Delta^2 \phi_M) \leq \| \Delta(\phi_M^3) \|_{L^2} \| \Delta^2 \phi_M \|_{L^2}
\]
\[
\leq \left( 3 \| \phi_M^3 \Delta \phi_M \|_{L^2} + 6 \| \phi_M |\nabla \phi_M|^2 \|_{L^2} \right) \| \Delta^2 \phi_M \|_{L^2}
\]
\[
\leq \left( 3 \| \phi_M \|^2_{L^6} \| \Delta \phi_M \|_{L^6} + 6 \| \phi_M \|_{L^6} \| \nabla \phi_M \|^2_{L^6} \right) \| \Delta^2 \phi_M \|_{L^2}
\]
\[
\leq C \left( \| \phi_M \|^2_{H^1} \| \nabla \Delta \phi_M \|_{L^2} + \| \phi_M \|_{H^1} \| \nabla \phi_M \|^2_{H^1} \right) \| \Delta^2 \phi_M \|_{L^2}.
\]

Using Theorem 2.5, Lemma 2.6, the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we find
\[
\| \nabla \phi_M \|_{H^1} \leq C \| \phi_M \|_{H^2} \leq C \left( \| \phi_M \|_{L^6}^{\frac{4}{3}} \| \phi_M \|_{H^2}^{\frac{2}{3}} + \| \phi_M \|_{L^6} \right)
\]
\[
\leq C \left( C_7^\frac{4}{5} C \left( \| \Delta^2 \phi_M \|_{L^2}^{\frac{4}{5}} + C_7^\frac{4}{5} \right)^{\frac{3}{5}} + C_7^\frac{4}{5} \right)
\]
\[
\leq C_5 \| \Delta^2 \phi_M \|_{L^2}^{\frac{4}{5}} + C_5,
\]

\[
\| \nabla \Delta \phi_M \|_{L^2} \leq C \| \phi_M \|_{H^1} \leq C \left( \| \phi_M \|_{L^6}^{\frac{4}{3}} \| \phi_M \|_{H^2}^{\frac{2}{3}} + \| \phi_M \|_{L^6} \right)
\]
\[
\leq C_6 \| \Delta^2 \phi_M \|_{L^2}^{\frac{4}{5}} + C_6,
\]

where $C_5, C_6 > 0$ are independent of $M, t,$ and $T$. Combining estimate (4.9) and the last two estimates gives
\[
(\Delta(\phi_M^3), \Delta^2 \phi_M) \leq C_7 \| \Delta^2 \phi_M \|_{L^2}^{\frac{4}{5}} + C_7
\]

where $C_7 > 0$ is independent of $M, t,$ and the final time $T$.

For the term associated with the nonlinear convection, we start with an application of Hölder’s inequality:
\[
(\bar{u}_M \cdot \nabla \phi_M, \Delta^2 \phi_M) \leq \| \bar{u}_M \|_{L^\infty} \| \nabla \phi_M \|_{L^2} \| \Delta^2 \phi_M \|_{L^2}
\]
\[
\leq C_1 \| \bar{u}_M \|_{L^\infty} \| \Delta^2 \phi_M \|_{L^2}.
\]
Now, since $\tilde{u}_M \in [C^\infty(\Omega)]^3 \cap \mathcal{V}$,
\[
\|\Delta \tilde{u}_M\|_{L^2}^2 + \|\kappa \tilde{u}_M\|^2_{L^2} \leq -\Delta u_M + \kappa \tilde{u}_M \|^2_{L^2} \leq \gamma^2 \|\phi_M \nabla \mu_M\|^2_{L^2}.
\]

Appealing to a standard elliptic regularity result and the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ in 3D, gives
\[
\|\tilde{u}_M\|_{L^\infty} \leq C \|\tilde{u}_M\|_{H^2} \leq C \|\phi_M\|_{L^\infty} \|\nabla \mu_M\|_{L^2}
\leq C \|\phi_M\|_{L^\infty} \|\nabla (\mathcal{P}_M(\phi^3_M) - \phi_M - \varepsilon^2 \Delta \phi_M)\|_{L^2}
\leq C \|\phi_M\|_{L^\infty} (\|\nabla \mathcal{P}_M(\phi^3_M)\|_{L^2} + \|\phi_M\|_{L^2} + \varepsilon^2 \|\nabla \Delta \phi_M\|_{L^2})
\leq C \|\phi_M\|_{L^\infty} \left(\|\nabla (\phi^3_M)\|_{L^2} + C_1^2 + \varepsilon^2 \|\nabla \Delta \phi_M\|_{L^2}\right).
\] (4.14)

Using the Gagliardo-Nirenberg inequality (2.21), $\|\phi_M\|_{L^\infty}$ can be bounded by
\[
\|\phi_M\|_{L^\infty} \leq C \left(\|\phi_M\|_{H^1}^\frac{2}{3} \|\nabla \phi_M\|_{L^2} + \|\phi_M\|_{H^2}\right)
\leq C \left(\|\phi_M\|_{H^1}^\frac{2}{3} \left(\|\nabla \phi_M\|_{L^2} \|\Delta \phi_M\|_{L^2} \right) + \|\phi_M\|_{H^2}\right)
\leq C \left(C_1^{\frac{2}{3}} \|\Delta \phi_M\|_{L^2} + C_1^2\right).
\] (4.15)

The two other terms in the bound for $\|\nabla \mu_M\|_{L^2}$ can be analyzed as follows: using estimate (4.10),
\[
\|\nabla (\phi^3_M)\|_{L^2} = \|3\phi_M^2 \nabla \phi_M\|_{L^2} \leq 3 \|\phi_M\|_{L^6}^2 \|\nabla \phi_M\|_{L^6} \leq C \|\phi_M\|_{H^1}^2 \|\nabla \phi_M\|_{H^1},
\]
\[
\leq CC_1 \|\phi_M\|_{H^2} + C_8 \|\Delta \phi_M\|_{L^2}^\frac{1}{2} + C_8,
\] (4.16)

where $C_8 > 0$ is independent of $M$, $t$, and $T$; and using (2.14),
\[
\|\nabla \Delta \phi_M\|_{L^2} \leq \|\nabla \phi_M\|_{L^2} \|\Delta \phi_M\|_{L^2} \leq C_1^2 \|\Delta \phi_M\|_{L^2}.
\] (4.17)

A substitution of (4.15) – (4.17) into (4.14) yields
\[
\|\tilde{u}_M\|_{L^\infty} \leq C_9 \|\Delta \phi_M\|_{L^2}^\frac{5}{2} + C_9,
\] (4.18)

where $C_9 > 0$ is independent of $M$, $t$, and $T$. Going back to (4.13), we arrive at
\[
- (\tilde{u}_M \cdot \nabla \phi_M, \Delta^2 \phi_M) \leq C_{10} \|\Delta^2 \phi_M\|_{L^2}^\frac{11}{2} + C_{10},
\] (4.19)

where $C_{10} > 0$ is independent of $M$, $t$, and $T$.

A combination of (4.6), (4.7), (4.12) and (4.19) results in
\[
\frac{1}{2} \frac{d}{dt} \|\Delta \phi_M\|^2_{L^2} + \varepsilon^2 \|\Delta^2 \phi_M\|^2_{L^2} \leq C_{11} \|\Delta^2 \phi_M\|_{L^2}^\frac{11}{2} + C_{11},
\] (4.20)
where $C_{11} > 0$ is independent of $M$, $t$, and $T$. To be precise, $C_{11}$ depends upon the equation parameters, some embedding and regularity constants, the domain $\Omega$, and $C_1$. Since the leading exponent $\frac{11}{6}$ is strictly less than 2, an application of Young’s inequality gives

$$\frac{1}{2} \frac{d}{dt} \| \Delta \phi_M \|_{L^2}^2 + \varepsilon^2 \| \Delta^2 \phi_M \|_{L^2}^2 \leq \frac{\varepsilon^2}{2} \| \Delta^2 \phi_M \|_{L^2}^2 + \frac{C_{12}}{2}, \quad (4.21)$$

for some constant $C_{12} > 0$ that is independent of $M$, $t$, and $T$. Therefore,

$$\frac{d}{dt} \| \Delta \phi_M \|_{L^2}^2 + \varepsilon^2 \| \Delta^2 \phi_M \|_{L^2}^2 \leq C_{12}. \quad (4.22)$$

Now, by (2.14) and Young’s inequality, we find

$$\| \Delta \phi_M \|_{L^2}^2 \leq C_1^2 \| \Delta^2 \phi_M \|_{L^2}^2 \leq \frac{1}{3} \| \Delta^2 \phi_M \|_{L^2}^2 + \frac{2}{3} C_1, \quad (4.23)$$

and it follows that

$$\frac{d}{dt} \| \Delta \phi_M \|_{L^2}^2 + 3 \varepsilon^2 \| \Delta \phi_M \|_{L^2}^2 \leq C_{12} + 2C_1 \varepsilon^2 =: C_{13}. \quad (4.24)$$

This leads to the following exponential decay: for any $t \in (0, \infty)$,

$$\| \Delta \phi_M (t) \|_{L^2}^2 \leq e^{-3 \varepsilon^2 t} \| \Delta (P_M \phi_0) \|_{L^2}^2 + \frac{C_{13} \varepsilon}{3 \varepsilon^2} \leq \| \Delta \phi_0 \|_{L^2}^2 + \frac{C_{13}}{3 \varepsilon^2}, \quad (4.25)$$

where we have used (2.2). By elliptic regularity, for any $t \in (0, \infty)$,

$$\| \phi_M (t) \|_{H^2} \leq C \left( \| \phi_M (t) \|_{L^2} + \| \Delta \phi_M (t) \|_{L^2} \right) \leq C \left( C_1^4 + \sqrt{\| \Delta \phi_0 \|_{L^2}^2 + \frac{C_{13}}{3 \varepsilon^2}} \right) =: C_{14}, \quad (4.26)$$

where $C_{14} > 0$ is independent of $M$, $t$, and $T$.

Now, the following estimate can also be derived, based on (4.22), for any finite final time $T > 0$:

$$\int_0^T \| \Delta^2 \phi_M (t) \|_{L^2}^2 dt \leq \frac{C_{12} T + \| \Delta \phi_0 \|_{L^2}^2 \varepsilon^2}{\varepsilon^2}. \quad (4.27)$$

Due to the elliptic regularity,

$$\| \phi_M (t) \|_{H^4} \leq C \left( \| \phi_M (t) \|_{L^2} + \| \Delta^2 \phi_M (t) \|_{L^2} \right),$$

for any $t > 0$, we have

$$\int_0^T \| \phi_M (t) \|_{H^4}^2 dt \leq C_{15} (T + 1), \quad (4.28)$$

for some constant $C_{15} > 0$ that is independent of $M$, and $T$. 

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For $\bar{u}_M$, we have the following uniform estimate with the help of (4.14) and embedding $H^2(\Omega) \hookrightarrow L^{\infty}(\Omega)$:

$$\|\bar{u}_M(t)\|_{H^2}^2 \leq C \|\phi(t)\|_{H^2}^2 \|\nabla \mu(t)\| \leq CC_{14} \|\nabla \mu(t)\|^2,$$

(4.29)

for any $t > 0$. This in turn implies that

$$\|\bar{u}_M\|_{L^2(0,T;H^2)}^2 \leq CC_{14} \|\nabla \mu\|_{L^2(0,T;H^2)}^2 \leq CC_{14} C_{16}.$$

(4.30)

Finally, one can show, using similar techniques as above, that

$$\|\partial_t \phi\|_{L^2(0,T;L^2)} + \|\mu\|_{L^2(0,T;H^2)} \leq C_{16}(T + 1),$$

(4.31)

for some constant $C_{16} > 0$ that is independent of $M$ and $T$. We suppress the details for the sake of brevity. Therefore, the preliminary uniform estimates (4.4) and (4.5) are valid. Extracting weakly convergent subsequences, and passing to the appropriate limits, the estimates (4.2) and (4.3) are likewise valid, and the proof is concluded. □

### 4.2. Uniqueness of the strong solution

Now, we prove the uniqueness of the strong solutions. To this end, assume that $(\phi_1, \mu_1, u_1)$ and $(\phi_2, \mu_2, u_2)$ are two strong solutions to (1.3) – (1.6), with the same initial data: $\phi_i(0) = \phi_0 \in H^2_N(\Omega), \quad i = 1, 2$. Because of the regularities of the strong solution, the following homogeneous Neumann boundary conditions hold: for a.e. $t \in (0, T)$,

$$\partial_n \phi_i(t) = \partial_n \mu_i(t) = \partial_n \Delta \phi_i = 0 \quad \text{on} \quad \partial \Omega.$$

(4.32)

The velocities $u_i$ satisfy the no-penetration, free-slip boundary conditions (1.9). Of course, both solutions satisfy the estimates (4.2) – (4.3). We denote the difference functions as

$$\tilde{\phi} := \phi_1 - \phi_2, \quad \tilde{\mu} := \mu_1 - \mu_2, \quad \tilde{u} := u_1 - u_2.$$

(4.33)

**Lemma 4.2.** Assume that $(\phi_i, \mu_i, u_i), \ i = 1, 2,$ are two strong solutions to (1.3) – (1.6), with $\phi_i(0) = \phi_0 \in H^2_N(\Omega), \ i = 1, 2$. The following identities and estimates
are valid:

\[
\left( \hat{\phi}(t), 1 \right)_{L^2(O)} = 0, \quad \text{a.e. } t \in (0, \infty), \quad (4.34)
\]

\[
\left\| \hat{\phi} \right\|_{L^p} \leq C \left\| \nabla \hat{\phi} \right\|_{L^2}, \quad \forall \ p \in [1, 6], \quad \exists \ C = C(p) > 0, \quad (4.35)
\]

\[
\left\| \nabla \left\{ \chi(\phi_1, \phi_2) \hat{\phi} \right\} \right\|_{L^2} \leq C_{17} \left\| \nabla \hat{\phi} \right\|_{L^2}, \quad \chi(\phi_1, \phi_2) := \phi_1^2 + \phi_1 \phi_2 + \phi_2^2, \quad (4.36)
\]

\[
\int_0^T M_1^2(t) \, dt \leq C_{18}, \quad M_1(t) := \left\| u_1(t) \right\|_{L^\infty}, \quad (4.37)
\]

\[
\int_0^T M_2^2(t) \, dt \leq C_{19}(T + 1), \quad M_2(t) := \left\| \nabla \phi_1(t) \right\|_{L^2}, \quad (4.38)
\]

\[
\left\| \hat{\phi} \nabla \mu_1 \right\|_{L^2} \leq C_{20}(1 + M_2) \left\| \nabla \hat{\phi} \right\|_{L^2}^2 \left\| \nabla \Delta \hat{\phi} \right\|_{L^2}^{\frac{1}{2}}, \quad (4.39)
\]

where \( C_{17}, C_{18}, C_{19}, C_{20} > 0 \) are independent of \( t \in (0, T] \) and the final time \( T \).

**Proof.** The identity (4.34) and estimate (4.35) are straightforward, and the proofs are omitted. For estimate (4.36), we start from the expansion \( \nabla \left( \phi_1^2 \hat{\phi} \right) = \phi_1^2 \nabla \hat{\phi} + 2\phi_1 \hat{\phi} \nabla \phi_1 \), obtaining

\[
\left\| \nabla \left( \phi_1^2 \hat{\phi} \right) \right\|_{L^2} \leq \left\| \phi_1 \right\|_{L^\infty}^2 \left\| \nabla \hat{\phi} \right\|_{L^2} + 2 \left\| \phi_1 \right\|_{L^\infty} \left\| \nabla \phi_1 \right\|_{L^6} \left\| \hat{\phi} \right\|_{L^3} \leq C \bar{C}_3 \left\| \nabla \phi_1 \right\|_{L^2}. \quad (4.40)
\]

Similar bounds may be derived for \( \left\| \nabla (\phi_1 \phi_2 \hat{\phi}) \right\|_{L^2} \) and \( \left\| \nabla (\phi_1^2 \hat{\phi}) \right\|_{L^2} \), and (4.36) is proven by the triangle inequality. Estimate (4.37) is a direct consequence of

\[
\left\| u \right\|_{L^2(0,T;L^\infty)}^2 \leq C \left\| u \right\|_{L^2(0,T;H^4)}^2 \leq C \bar{C}_3. \quad (4.41)
\]

To get estimate (4.38), using \( \partial_n \Delta \phi_1 |_{\partial \Omega} = 0 \) in (4.32), we first obtain

\[
\left\| \nabla \Delta \phi_1 \right\|_{L^2}^2 = - \left( \Delta \phi_1, \Delta^2 \phi_1 \right) \leq \left\| \Delta \phi_1 \right\|_{L^2} \left\| \Delta^2 \phi_1 \right\|_{L^2} \leq \bar{C}_3^2 \left\| \Delta^2 \phi_1 \right\|_{L^2}. \quad (4.42)
\]

Applying the \( L^2(0,T;H^4) \) estimate for \( \phi_1 \), we arrive at

\[
\int_0^T M_2^2(t) \, dt \leq \bar{C}_3 \int_0^T \left\| \Delta^2 \phi_1(t) \right\|^2 \, dt \leq \bar{C}_3 \bar{C}_4(T + 1). \quad (4.43)
\]

Finally, note that, because of the regularities of the variables, for a.e. \( t \in (0, T] \),

\[
\nabla \mu_1(t) = 3\varepsilon^{-1} \phi_1^2(t) \nabla \phi_1(t) - \varepsilon^{-1} \nabla \phi_1(t) - \varepsilon \nabla \Delta \phi_1(t), \quad (4.44)
\]

and to prove estimate (4.39), we observe that

\[
\left\| \hat{\phi} \nabla \mu_1 \right\|_{L^2} \leq \left\| \nabla \mu_1 \right\|_{L^2} \left\| \hat{\phi} \right\|_{L^\infty} \leq \left( 3\varepsilon^{-1} \bar{C}_3^2 \bar{C}_3 + \varepsilon^{-1} \bar{C}_1^2 \hat{\phi} + \varepsilon M_2 \right) \left\| \hat{\phi} \right\|_{L^\infty} \leq C_{20}(1 + M_2) \left\| \nabla \hat{\phi} \right\|_{L^2}^2 \left\| \nabla \Delta \hat{\phi} \right\|_{L^2}^{\frac{1}{2}}, \quad (4.45)
\]
where we used Gagliardo-Nirenberg inequality (2.21) and the fact that \((\phi, 1) = 0\) in the last step. The constant \(C_{20} > 0\) is clearly independent of \(t\) and \(T\).

**Theorem 4.3.** With the same assumptions as in Lemma 4.2, there exists at most one global-in-time strong solution.

**Proof.** The difference functions satisfy the following: in the sense of \(L^2(0, T; L^2(\Omega))\),
\[
\partial_t \hat{\phi} + u_1 \cdot \nabla \hat{\phi} + \hat{u} \cdot \nabla \phi_2 = \varepsilon \Delta \hat{\mu},
\]
(4.46)
with \(\hat{\phi}(0) = 0\), where
\[
-\Delta \hat{u} + \kappa \hat{u} = -\gamma \mathcal{P}_H \left( \phi_2 \nabla \hat{\mu} + \hat{\phi} \nabla \mu_1 \right) = -\gamma \left( \phi_2 \nabla \hat{\mu} + \hat{\phi} \nabla \mu_1 \right) - \nabla \hat{p},
\]
(4.47)
\[
\nabla \hat{\mu} = \varepsilon^{-1} \nabla \left\{ (\chi(\phi_1, \phi_2) - 1) \hat{\phi} \right\} - \varepsilon \Delta \hat{\phi},
\]
(4.48)
\(\mathcal{P}_H\) is the Helmholtz projection, and \(\hat{p} \in H^1(\Omega)\) is a pressure. Taking the inner product of (4.46) with \(-\Delta \hat{\phi}\) results in
\[
\frac{1}{2} \frac{d}{dt} \left\| \nabla \hat{\phi} \right\|_{L^2}^2 + \varepsilon^2 \left\| \nabla \Delta \hat{\phi} \right\|_{L^2}^2 = - \left( u_1 \hat{\phi} + \hat{u} \phi_2, \nabla \Delta \hat{\phi} \right) + \left( \nabla \hat{\phi}, \nabla \Delta \hat{\phi} \right),
\]
(4.49)
after integration-by-parts and using the boundary conditions (4.32). The term associated with the negative diffusion can be controlled in a standard way:
\[
- \left( \nabla \hat{\phi}, \nabla \Delta \hat{\phi} \right) \leq \left\| \nabla \hat{\phi} \right\|_{L^2} \left\| \nabla \Delta \hat{\phi} \right\|_{L^2} \leq C \left\| \nabla \hat{\phi} \right\|_{L^2}^2 + \frac{\varepsilon^2}{8} \left\| \nabla \Delta \hat{\phi} \right\|_{L^2}^2.
\]
(4.50)
The last nonlinear term in (4.49) can be analyzed with the help of (4.36):
\[
\left( \nabla \left\{ \chi(\phi_1, \phi_2) \hat{\phi} \right\}, \nabla \Delta \hat{\phi} \right) \leq C_{17} \left\| \nabla \hat{\phi} \right\|_{L^2} \left\| \nabla \Delta \hat{\phi} \right\|_{L^2}
\leq C \left\| \nabla \hat{\phi} \right\|_{L^2}^2 + \frac{\varepsilon^2}{8} \left\| \nabla \Delta \hat{\phi} \right\|_{L^2}^2.
\]
(4.51)
Now, we consider the two nonlinear convection terms, the first of which may be estimated as
\[
- \left( u_1 \hat{\phi}, \nabla \Delta \hat{\phi} \right) \leq \left\| u_1 \right\|_{L^\infty} \left\| \hat{\phi} \right\|_{L^2} \left\| \nabla \Delta \hat{\phi} \right\|_{L^2} = M_1 \left\| \hat{\phi} \right\|_{L^2} \left\| \nabla \Delta \hat{\phi} \right\|_{L^2}
\leq CM_1^2 \left\| \nabla \hat{\phi} \right\|_{L^2}^2 + \frac{\varepsilon^2}{8} \left\| \nabla \Delta \hat{\phi} \right\|_{L^2}^2.
\]
(4.52)
For the other, using the error equations (4.47) and (4.48), we rewrite
\[
\phi_2 \nabla \Delta \hat{\phi} = \frac{1}{\varepsilon \gamma} (-\Delta \hat{u} + \kappa \hat{u} + \nabla \hat{p}) + \mathcal{N},
\]
(4.53)
where
\[
\mathcal{N} := \frac{1}{\varepsilon^2} \phi_2 \nabla \left\{ (\chi(\phi_1, \phi_2) - 1) \hat{\phi} \right\} + \frac{1}{\varepsilon} \hat{\phi} \nabla \mu_1.
\]
(4.54)
In turn, the following estimate is valid:

\[-(\tilde{u}\phi_2, \nabla \Delta \tilde{\phi}) = -(\tilde{u}, \phi_2 \nabla \Delta \tilde{\phi}) = -\frac{1}{\gamma \varepsilon} (\tilde{u}, -\Delta \tilde{u} + \kappa \tilde{u} + \nabla \tilde{p}) + \tilde{u}, \mathcal{N})\]

\[= -\frac{1}{\gamma \varepsilon} \left( \|\nabla \tilde{u}\|_{L^2}^2 + \kappa \|\tilde{u}\|_{L^2}^2 \right) + \tilde{u}, \mathcal{N} \]

\[\leq (\tilde{u}, \mathcal{N}) \leq \|\tilde{u}\|_{L^2} \|\mathcal{N}\|_{L^2}, \quad (4.55)\]

where we have used the no-penetration, free-slip boundary conditions for \(\tilde{u}\). On the other hand, we have

\[-\Delta \tilde{u} + \kappa \tilde{u} = -\gamma \varepsilon \mathcal{P}_H \left( \mathcal{N} - \phi_2 \nabla \Delta \tilde{\phi} \right), \quad (4.56)\]

and applying the \(L^2\) stability of \(\mathcal{P}_H\) and an elliptic regularity result, we obtain

\[\|\tilde{u}\|_{H^2} \leq C \|\Delta \tilde{u} + \kappa \tilde{u}\|_{L^2} = C \|\mathcal{P}_H \left( \mathcal{N} - \phi_2 \nabla \Delta \tilde{\phi} \right)\|_{L^2} \leq C \|\mathcal{N} - \phi_2 \nabla \Delta \tilde{\phi}\|_{L^2}\]

\[\leq C \left( \|\mathcal{N}\|_{L^2} + \|\phi_2\|_{L^\infty} \right) \|\nabla \Delta \tilde{\phi}\|_{L^2} \leq C \left( \|\mathcal{N}\|_{L^2} + C \bar{C}_3^{\frac{1}{3}} \|\nabla \Delta \tilde{\phi}\|_{L^2} \right). \quad (4.57)\]

Substitution into (4.55) yields

\[-(\tilde{u}\phi_2, \nabla \Delta \tilde{\phi}) \leq C \|\mathcal{N}\|_{L^2}^2 + C \|\nabla \Delta \tilde{\phi}\|_{L^2} \|\mathcal{N}\|_{L^2}. \quad (4.58)\]

With the help of (4.56) and (4.59), \(\|\mathcal{N}\|_{L^2}\) may be analyzed as

\[\|\mathcal{N}\|_{L^2} \leq \frac{1}{\varepsilon^2} \left( \|\phi_2\|_{L^2} \right) \|\mathcal{N}\|_{L^2} + \frac{1}{\varepsilon^2} \|\nabla \mathcal{N}\|_{L^2} + \frac{1}{\varepsilon^2} \|\nabla \Delta \mathcal{N}\|_{L^2}\]

\[\leq C \bar{C}_3^{\frac{1}{3}} C_{17} \|\nabla \tilde{\phi}\|_{L^2} + C_{20} (1 + M_2) \|\nabla \tilde{\phi}\|_{L^2} \|\nabla \Delta \tilde{\phi}\|_{L^2} + C \bar{C}_3^{\frac{1}{3}} \|\nabla \tilde{\phi}\|_{L^2} \|\nabla \Delta \tilde{\phi}\|_{L^2} \|\mathcal{N}\|_{L^2} \|\mathcal{N}\|_{L^2}\]

\[\leq C \|\nabla \tilde{\phi}\|_{L^2} + C (1 + M_2) \|\nabla \tilde{\phi}\|_{L^2} \|\nabla \Delta \tilde{\phi}\|_{L^2}. \quad (4.59)\]

Using Young's inequality,

\[\|\mathcal{N}\|_{L^2}^2 \leq C \|\nabla \tilde{\phi}\|_{L^2}^2 + (1 + M_2^2) \|\nabla \tilde{\phi}\|_{L^2} \|\nabla \Delta \tilde{\phi}\|_{L^2}^2 \]

\[\leq C \|\nabla \tilde{\phi}\|_{L^2}^2 + (1 + M_2^{\frac{1}{3}}) \|\nabla \tilde{\phi}\|_{L^2}^2 + \frac{\varepsilon^2}{16} \|\nabla \Delta \tilde{\phi}\|_{L^2}^2 \]

\[\leq C \left( 1 + M_2^{8/3} \right) \|\nabla \tilde{\phi}\|_{L^2}^2 + \frac{\varepsilon^2}{16} \|\nabla \Delta \tilde{\phi}\|_{L^2}^2. \quad (4.60)\]

Likewise,

\[\|\nabla \Delta \tilde{\phi}\|_{L^2} \|\mathcal{N}\|_{L^2} \leq C \|\nabla \tilde{\phi}\|_{L^2} \|\nabla \Delta \tilde{\phi}\|_{L^2} \|\nabla \Delta \tilde{\phi}\|_{L^2} + C (1 + M_2) \|\nabla \tilde{\phi}\|_{L^2} \|\nabla \Delta \tilde{\phi}\|_{L^2}^2 \]

\[\leq C \left( 1 + M_2^{8/3} \right) \|\nabla \tilde{\phi}\|_{L^2}^2 + \frac{\varepsilon^2}{16} \|\nabla \Delta \tilde{\phi}\|_{L^2}^2. \quad (4.61)\]

Going back to (4.58), we get

\[-(\tilde{u}\phi_2, \nabla \Delta \tilde{\phi}) \leq C \left( 1 + M_2^{8/3} \right) \|\nabla \tilde{\phi}\|_{L^2}^2 + \frac{\varepsilon^2}{8} \|\nabla \Delta \tilde{\phi}\|_{L^2}^2. \quad (4.62)\]
Consequently, a combination of (4.49), (4.50), (4.51), (4.52), and (4.62) leads to
\[
\frac{d}{dt} \| \nabla \bar{\phi} \|_{L^2}^2 + \varepsilon^2 \| \nabla \Delta \bar{\phi} \|_{L^2}^2 \leq C \left( 1 + M_1^2 + M_2^{8/3} \right) \| \nabla \bar{\phi} \|_{L^2}^2. \tag{4.63}
\]
Using the preliminary estimates (4.37), (4.38), we obtain the integrability of the coefficient:
\[
C \int_0^T \left( 1 + M_1^2(t) + M_2^{8/3} \right) dt \leq C (T + C_{18} + C_{19}(T + 1)) \leq C_{21}(T + 1), \tag{4.64}
\]
for some \( C_{21} > 0 \) that is independent of \( T \). Therefore, an application of Gronwall inequality to (4.63) implies that
\[
\| \nabla \bar{\phi}(t) \|_{L^2}^2 \leq \exp \left( C_{21}(T + 1) \right) \| \nabla \bar{\phi}(0) \|_{L^2}^2 = 0, \tag{4.65}
\]
with the trivial initial data (4.48) applied in the last step. Finally, the uniqueness of the strong solution is assured by (4.35). \( \Box \)

5. Global-in-time smooth solution. Similar techniques can be applied to analyze the higher order derivatives of the CHS solution.

5.1. Existence of a solution with intermediate regularity. First, we prove an intermediate result, namely the existence of a solution with \( \phi \) in \( L^\infty(0, T; H^3(\Omega)) \cap L^2(0, T; H^5(\Omega)) \).

**Theorem 5.1.** Let \( \phi_0 \in H_N^2 \cap H^3(\Omega) \). Then there exists a global-in-time weak solution with the additional regularities

\[
\phi \in L^\infty(0, T; H^3(\Omega)) \cap L^2(0, T; H^5(\Omega)), \quad \mu \in L^2(0, T; H^3(\Omega)),
\]
\[
\mathbf{u} \in L^2(0, T; H^3(\Omega)), \quad \partial_t \phi \in L^2(0, T; H^1(\Omega)), \tag{5.1}
\]

with \( \phi(0) = \phi_0 \), such that, for any \( T > 0 \),

\[
\| \phi \|_{L^\infty(0, T; H^3)}^2 \leq \tilde{C}_1^{(3)}, \tag{5.2}
\]
\[
\| \phi \|_{L^2(0, T; H^5)}^2 + \| \mu \|_{L^2(0, T; H^3)}^2 + \| \mathbf{u} \|_{L^2(0, T; H^3)}^2 + \| \partial_t \phi \|_{L^2(0, T; H^1)}^2 \leq \tilde{C}_2^{(3)}(T + 1), \tag{5.3}
\]

where \( \tilde{C}_1^{(3)}, \tilde{C}_2^{(3)} > 0 \) are constants that are independent of the final time, \( T \).

**Proof.** Setting \( \nu = -\Delta^3 \phi_M \) in (3.7) and \( \psi = \varepsilon \Delta^4 \phi_M \) in (3.8), integrating by parts, using Lemma 2.2, and adding the results, we have
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \Delta \phi_M \|_{L^2}^2 + \varepsilon^2 \| \nabla \Delta^2 \phi_M \|_{L^2}^2 = \langle \nabla \Delta (\phi_M^3), \nabla \Delta^2 \phi_M \rangle + \langle \Delta^2 \phi_M, \Delta^2 \phi_M \rangle
\]
\[
- \langle \nabla (\bar{u}_M \cdot \nabla \phi_M), \nabla \Delta^2 \phi_M \rangle. \tag{5.4}
\]
The term associated with the concave diffusion can be treated in a similar manner as (4.7), with the help of a Sobolev inequality in (2.15):

\[ \| \Delta^2 \phi_M \|^2_{L^2} \leq \| \Delta \phi_M \|^2_{L^2} \| \nabla \Delta^2 \phi_M \|^4_{L^2} \leq C^2_3 \| \nabla \Delta^2 \phi_M \|^4_{L^2}, \]

(5.5)
in which the uniform-in-time \( H^2 \) estimate (4.4) was applied in the last step.

For the term associated with the nonlinear convection, we observe

\[ (\nabla \Delta (\phi^3_M), \nabla \Delta^2 \phi_M) \leq \| \nabla \Delta (\phi^3_M) \|_{L^2} \| \nabla \Delta^2 \phi_M \|_{L^2}. \]

(5.6)
The expansion (4.8) implies that

\[ \nabla \Delta (\phi^3_M) = 3 \phi^2_M \nabla \Delta \phi_M + 6 \phi_M \Delta \phi_M \nabla \phi_M \]

\[ + 6 |\nabla \phi_M|^2 \nabla \phi_M + 12 \phi_M H(\phi_M) \nabla \phi_M, \]

(5.7)
where \( H(\phi_M) \) is the Hessian matrix of \( \phi_M \). This in turn leads to

\[ \| \nabla \Delta (\phi^3_M) \| \leq 3 \| \phi_M \|^2_{L^\infty} \| \nabla \Delta \phi_M \|_{L^2} + 6 \| \phi_M \|_{L^\infty} \| \Delta \phi_M \|_{L^2} \| \nabla \phi_M \|_{L^\infty} \]

\[ + 6 \| \nabla \phi_M \|^3_{L^6} + 12 \| \phi_M \|_{L^\infty} \| H(\phi_M) \|_{L^2} \| \nabla \phi_M \|_{L^\infty} \]

\[ \leq C \left( \| \phi_M \|^3_{L^2} \| \nabla \Delta \phi_M \|_{L^2} + \| \phi_M \|^2_{H^2} \| \nabla \phi_M \|_{L^\infty} \right) \]

\[ + \| \phi_M \|^3_{H^2} + \| \phi_M \|^2_{H^2} \| \nabla \phi_M \|_{L^\infty} \]

\[ \leq C \left( 3 \| \phi_M \|^3_{H^2} + 3 \| \phi_M \|^2_{H^2} \| \nabla \Delta \phi_M \|_{L^2} \right) \]

\[ \leq C \left( C^3_3 + C^2_3 \right) \| \nabla \Delta^2 \phi_M \|^4_{L^2}, \]

(5.8)
in which the 3D Sobolev embeddings \( H^1(\Omega) \hookrightarrow L^6(\Omega) \) and \( H^2(\Omega) \hookrightarrow L^{\infty}(\Omega) \) were applied. The last estimate in combination with (5.6) results in

\[ (\nabla \Delta (\phi^3_M), \nabla \Delta^2 \phi_M) \leq C_{22} \left( 1 + \| \nabla \Delta^2 \phi_M \|^\frac{4}{3}_{L^2} \right), \]

(5.9)
for some constant \( C_{22} > 0 \) that is independent of \( M, t, \) and \( T \).

For the term associated with the nonlinear convection, we observe

\[ \nabla (\bar{u}_M \cdot \nabla \phi_M) = \nabla \bar{u}_M \cdot \nabla \phi_M + H(\phi_M) \bar{u}_M, \]

(5.10)
and, therefore,

\[ \| \nabla (\bar{u}_M \cdot \nabla \phi_M) \|_{L^2} \leq \| \nabla \bar{u}_M \|_{L^6} \cdot \| \nabla \phi_M \|_{L^2} + \| \bar{u}_M \|_{L^\infty} \| H(\phi_M) \|_{L^2} \]

\[ \leq C \| \bar{u}_M \|_{H^2} \cdot \| \phi_M \|_{H^2} \leq CC^3_3 \| \bar{u}_M \|_{H^2}, \]

(5.11)
in which the 3D Sobolev embeddings $H^2(\Omega) \hookrightarrow W^{1,6}(\Omega)$, $H^2(\Omega) \hookrightarrow W^{1,3}(\Omega)$, and $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ were applied. On the other hand, estimate (4.14) indicates that

$$\|\nabla M\|_{H^2} \leq CC_3^\frac{1}{2} \left( \|\nabla (\phi_M^3)\|_{L^2} + C_1^\frac{1}{2} + \varepsilon^2 \|\nabla \Delta \phi_M\|_{L^2} \right).$$

(5.12)

The two terms $\|\nabla (\phi_M^3)\|_{L^2}$ and $\|\nabla \Delta \phi_M\|_{L^2}$ can be analyzed with the help of (4.16) and Lemma 2.4, respectively:

$$\|\nabla (\phi_M^3)\|_{L^2} \leq C \|\phi_M\|_{H^\frac{1}{2}}^2 \|\nabla \phi_M\|_{H^1} \leq CC_1C_3^\frac{1}{2},$$

(5.13)

$$\|\nabla \Delta \phi_M\|_{L^2} \leq \|\Delta \phi_M\|_{L^2}^\frac{1}{4} \|\nabla \Delta^2 \phi_M\|_{L^2}^\frac{1}{4} \leq C_3^\frac{1}{2} \|\nabla \Delta^2 \phi_M\|_{L^2}^\frac{1}{2}.$$  

(5.14)

The combination of (5.11) – (5.14) yields

$$\|\nabla (\mathbf{u}_M \cdot \nabla \phi_M)\|_{L^2} \leq CC_3 \left( C_1C_3^\frac{1}{2} + C_3^\frac{1}{2} \|\nabla \Delta^2 \phi_M\|_{L^2}^\frac{1}{2} \right),$$

(5.15)

and this in turn indicates that

$$- (\nabla (\mathbf{u}_M \cdot \nabla \phi_M), \nabla \Delta^2 \phi_M) \leq C_{23} \left( 1 + \|\nabla \Delta^2 \phi_M\|_{L^2}^\frac{1}{2} \right),$$

(5.16)

for some $C_{23} > 0$ that is independent of $M$, $t$, and $T$.

Subsequently, a combination of (5.4), (5.5), (5.9), and (5.16) leads to

$$\frac{1}{2} \frac{d}{dt} \|\nabla \Delta \phi_M\|_{L^2}^2 + \varepsilon^2 \|\nabla \Delta^2 \phi_M\|_{L^2}^2 \leq C_{24} \left( 1 + \|\nabla \Delta^2 \phi_M\|_{L^2}^\frac{1}{2} \right),$$

(5.17)

where $C_{24} = C_{24}(C_1, C_3, \varepsilon, \gamma, \kappa, \Omega) > 0$ is a constant that is independent of $M$, $t$, and the final time $T$. Since the exponent of $\|\nabla \Delta^2 \phi_M\|_{L^2}$ is less than 2, the Young inequality may be used to obtain

$$\frac{d}{dt} \|\nabla \Delta \phi_M\|_{L^2}^2 + \varepsilon^2 \|\nabla \Delta^2 \phi_M\|_{L^2}^2 \leq C_{25},$$

(5.18)

where $C_{25} > 0$ is a constant that is independent of $M$, $t$, and $T$. Moreover, by Lemma 2.4,

$$\|\nabla \Delta \phi_M\|_{L^2}^2 \leq \|\Delta \phi_M\|_{L^4}^4 \|\nabla \Delta^2 \phi_M\|_{L^2}^2 \leq \frac{1}{3} \|\nabla \Delta^2 \phi_M\|_{L^2}^2 + \frac{2}{3}C_3,$$

(5.19)

we get

$$\frac{d}{dt} \|\nabla \Delta \phi_M\|_{L^2}^2 + 3\varepsilon^2 \|\nabla \Delta \phi_M\|_{L^2}^2 \leq C_{26},$$

(5.20)

where $C_{26} > 0$ is some constant that is independent of $M$, $t$, and $T$. This in turn gives an estimate for $\|\nabla \Delta \phi_M\|_{L^2}$, containing an exponential decay:

$$\|\nabla \Delta \phi_M(t)\|_{L^2}^2 \leq \exp \left( -3\varepsilon^2 t \right) \|\nabla \Delta \phi_M(0)\|_{L^2}^2 + \frac{C_{26}}{3\varepsilon^2}.$$

(5.21)
Since, by Lemma 2.3 \((m = 1)\), the stability \(\|\nabla \Delta \phi_M(0)\|_{L^2}^2 \leq \|\nabla \Delta \phi_0\|_{L^2}^2\) holds, there is some constant \(C_{27} > 0\) independent of \(T\) such that, for all \(t \in (0, T]\),

\[
\|\phi_M(t)\|_{H^3}^2 \leq C \left(\|\phi_M(t)\|_{L^2}^2 + \|\nabla \Delta \phi_M(t)\|_{L^2}^2\right) \leq C_{27},
\]

where we have used an elliptic regularity estimate. Then, using (5.18):

\[
\int_0^T \|\nabla \Delta^2 \phi_M\|_{L^2}^2 \, dt \leq \frac{C_{27} T + \|\nabla \Delta \phi_M(0)\|_{L^2}^2}{\varepsilon^2} \leq C_{28}(T + 1),
\]

where \(C_{28} > 0\) is independent of \(T\). In turn, an \(L^2(0, T; H^5(\Omega))\) estimate is available for \(\phi\): there is a constant \(C_{29} > 0\), independent of \(T\), such that

\[
\|\phi\|_{L^2(0, T; H^5)}^2 \leq C \left(\|\phi(t)\|_{L^2}^2 + \|\nabla \Delta^2 \phi_M(t)\|_{L^2}^2\right)
\]

\[
\leq C_{29}(T + 1),
\]

de the elliptic regularity estimate \(\|\phi_M(t)\|_{H^5} \leq C \left(\|\phi_M(t)\|_{L^2} + \|\nabla \Delta^2 \phi_M(t)\|_{L^2}\right),\)

which is valid for any \(t > 0\).

As in (4.14), an \textit{a priori} estimate for \(\bar{u}_M\) is available:

\[
\|\bar{u}_M(t)\|_{H^3} \leq C \|\phi_M(t)\|_{H^3} \|\nabla \mu_M(t)\|_{H^3} \leq C \|\phi_M(t)\|_{H^3} \|\nabla \mu_M(t)\|_{H^3}
\]

\[
\leq C \|\phi_M(t)\|_{H^3} \|\nabla \mu_M(t)\|_{H^3} \leq C C_{27}^2 \|\nabla \mu(t)\|_{H^1},
\]

for any \(t > 0\). This in turn implies that

\[
\|\bar{u}_M\|_{L^2(0, T; H^3)}^2 \leq C C_{27} \|\nabla \mu_M\|_{L^2(0, T; H^1)}^2 \leq C C_{27} C_4(T + 1).
\]

Finally, we conclude with the following \textit{a-priori} estimate for \(\partial_t \phi_M\):

\[
\|\partial_t \phi_M\|_{L^2(0, T; H^1)}^2 \leq C \left(\|\bar{u}_M\|_{L^2(0, T; W^{1, \infty})}^2 \|\nabla \phi_M\|_{L^2(0, T; H^1)}^2 + \|\Delta (\phi_M^3)\|_{L^2(0, T; H^3)}^2 + \|\phi_M\|_{L^2(0, T; H^3)}^2\right)
\]

\[
\leq C \left(C_3 \|\bar{u}_M\|_{L^2(0, T; H^3)}^2 + \|\Delta (\phi_M^3)\|_{L^2(0, T; H^1)}^2 + C_{29}(T + 1)\right)
\]

\[
\leq C \left(C_3 C_4 C_{27}(T + 1) + \|\Delta (\phi_M^3)\|_{L^2(0, T; H^1)}^2 + C_{29}(T + 1)\right)
\]

\[
\leq C_{30}(T + 1),
\]

for some \(C_{30} > 0\) that is independent of \(T\), where a Sobolev analysis has been applied for \(\|\Delta (\phi^3)\|_{H^1}\) in the last step. Passing to the appropriate limits, we obtain the desired estimates (5.2) and (5.3). \(\Box\)
5.2. Solutions with arbitrary spacial regularity. The higher order derivatives (with \( m > 3 \)) can be analyzed in the same fashion as in the last section. Thus we are able to prove the following general result using induction; the details are skipped for brevity.

**Theorem 5.2.** Suppose that \( \phi_0 \in H^m(\Omega) \), \( m \geq 3 \), satisfies the boundary compatibility conditions (2.5). Then there exists a unique global-in-time weak solution for the CHS equation (1.3) – (1.6), with the following additional regularities

\[
\phi \in L^\infty(0, T; H^m(\Omega)) \cap L^2(0, T; H^{m+2}(\Omega)), \quad \mu \in L^2(0, T; H^m(\Omega)), \\
u \in L^2(0, T; H^m(\Omega)), \quad \partial_t \phi \in L^2(0, T; H^{m-2}(\Omega)),
\]

with \( \phi(0) = \phi_0 \), such that, for any \( T > 0 \),

\[
\|\phi\|^2_{L^\infty(0,T;H^m)} \leq \tilde{C}_1^{(m)},
\]

\[
\|\phi\|^2_{L^2(0,T;H^{m+2})} + \|\mu\|^2_{L^2(0,T;H^m)} \\
+ \|\nu\|^2_{L^2(0,T;H^m)} + \|\partial_t \phi\|^2_{L^2(0,T;H^{m-2})} \leq \tilde{C}_2^{(m)}(T + 1),
\]

where \( \tilde{C}_1^{(m)}, \tilde{C}_2^{(m)} > 0 \) are constants that are independent of the final time, \( T \). As a result, a unique global smooth solution exists for any smooth initial data.

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