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Independence preservation in expert judgment synthesis

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Abstract. We prove, with a few minor exceptions, that if P_1 and P_2 are probability distributions on the countable set S for which the fixed events E and F are independent, then, both for the standard Euclidean metric and for any metric inducing a topology coarser than the Euclidean topology, there exists a third probability distribution P_3 on S that preserves this independence and is equidistant from P_1 and P_2. We contrast this result with an impossibility theorem from the probability pooling literature, and note its connection with the vigorously debated "epistemic peer problem" in philosophical decision theory.

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1. Introduction

It has often been recommended that the differing probability distributions of a group of experts should be reconciled in such a way as to preserve any agreement on the stochastic independence of events. There appear, however, to be severe limitations on implementing this recommendation. Suppose, for example, that S is a countable set of possible states of the world, assumed to be mutually exclusive and exhaustive. Let Δ denote the set of all probability distributions on S, and Δ^n its n-fold Cartesian product. Each $(P_1, \ldots, P_n) \in \Delta^n$ is regarded as a profile of possible opinions of n experts regarding the correct probability distribution on S. A function $T : \Delta^n \to \Delta$ is called a pooling operator, and represents a method of reconciling disagreement among the experts applicable to every conceivable profile of distributions. Of course, many pooling operators are prima facie unacceptable, among them the constant operators, which map every profile to the same distribution, and the dictatorial operators, which map each profile to its d-th coordinate for some fixed $d \in \{1, \ldots, n\}$. In order to identify reasonable pooling operators it has been the custom to posit certain axiomatic restrictions on pooling, and then to seek to identify the operators, if any, that satisfy those axioms. Typical axioms have included, for example:

Irrelevance of Alternatives (IA): For each $s \in S$, there exists a function $f_s : [0,1]^n \to [0,1]$ such that for all $(P_1, \ldots, P_n) \in \Delta^n$, $T(P_1, \ldots, P_n)(s) = f_s(P_1(s), \ldots, P_n(s))$.

Zero Preservation (ZP): For each $s \in S$ and all $(P_1, \ldots, P_n) \in \Delta^n$, if $P_1(s) = \cdots = P_n(s) = 0$, then $T(P_1, \ldots, P_n)(s) = 0$.

Universal Independence Preservation (UIP): For all $(P_1, \ldots, P_n) \in \Delta^n$, and for all subsets E and F of S, if $P_i(E \cap F) = P_i(E)P_i(F)$ for $i = 1, \ldots, n$, then $T(P_1, \ldots, P_n)(E \cap F) = T(P_1, \ldots, P_n)(E)T(P_1, \ldots, P_n)(F)$.

The pooling operators satisfying these axioms are not particularly attractive:

Theorem 1.1. (Wagner [7]) *If $|S| \geq 3$, a pooling operator T satisfies IA, ZP, and UIP if and only if it is dictatorial.*

Axiom IA is quite restrictive, since it requires that the quantities $f_s(P_1(s), \ldots, P_n(s))$ sum over all $s \in S$ to 1, without any normalization. Indeed, IA and ZP together are satisfied only by weighted arithmetic means, with common weights across all $s \in S$, as shown in Wagner [6]. Yet even deleting ZP and weakening IA to allow for normalization only allows for non-dictatorial accommodation of UIP when $|S| = 4$, as shown in Genest and Wagner [3].

These limitative results have loomed large among decision theorists and philosophers interested in what is termed the epistemic peer problem, that is, the problem of how to reconcile the differing probability distributions, P_1 and P_2, of two individuals with equal expertise. In light of their disagreement, should the individuals suspend judgment, hold fast to their own probabilistic judgments, or revise their original assessments to some sort of average, say, $1/2(P_1 + P_2)$? The latter response, termed the equal weight view, has been propounded by Elga [2] and Christensen [1], yet appears impossible to implement in view of the aforementioned theorems. But this pessimistic judgment is unwarranted, for two reasons. First, all of these theorems are conceptualized in terms of a pooling operator T which is required to reconcile the distributions in every logically conceivable profile. Second, they all require that every single instance of independence common to the distributions of the experts in question should be preserved. But in the epistemic peer problem the issue is how to reconcile one particular pair of probability distributions. Moreover, there are clearly cases of independence having no theoretical significance (as, for example, the independence of the events "fair die comes up even" and "fair die comes up a multiple of 3"), where independence simply emerges as an incidental feature of the distribution at hand. Epistemic peers are unlikely to be interested in preserving common cases of independence of this type. On the other hand, there are cases of independence (such as physical independence, independent testimony, etc.) having genuine theoretical significance, and common cases of such independence have a strong claim to be preserved.

The foregoing observations lead us to focus attention on a much more modest problem: Let P_1 and P_2 be probability distributions on the countable set S, and suppose that the events E and F are independent with respect to each of these distributions. Can we find a distribution P_3 which gives equal weight, in some sense, to P_1 and P_2, and for which E and F are independent? Normalized geometric means will do the trick here when $|S| = 4$ (see Sundberg and Wagner [5]). However, for general values of $|S|$, no quasi-arithmetic mean can be counted on to preserve independence. In response to this dilemma, Jehle [4] suggested an alternative conception of equal weighting, based on some metric structure on the space of probability distributions on S. In Jehle's conception, a distribution P_3 is regarded as giving equal weight to P_1 and P_2 if it is equidistant from these distributions. This gives rise to the following question, interesting in its own right: given probability distributions P_1 and P_2 on a countable set S with respect to each of which E and F are independent, does there always exist a distribution P_3 that preserves this independence and is equidistant, say, for the Euclidean metric, from P_1 and P_2? In what follows we answer this question, with a few exceptions, in the affirmative.

2. Preliminaries

Underlying the mathematical results in the sections that follow is an elementary theorem about metric spaces.

Theorem 2.1. *Let C be a connected subset of the metric space (X, d). Then, for each pair of distinct elements x_1 and x_2 in C, there exists an x_3 in C such that $d(x_1, x_3) = d(x_2, x_3)$.*

Proof. Define a function $f : X \longrightarrow \mathbb{R}$ by $f(x) = d(x_1, x) - d(x_2, x)$. It is straightforward to show that f is continuous, from which it follows that $f(C)$ is connected, and consequently an interval in $\mathbb{R}$. Since $f(x_1) < 0$ and $f(x_2) > 0$, the interval $f(C)$ contains 0 as an element. □

Remark 2.1. If x_1 and x_2 are elements of the metric space (X, d), we say that there is a continuous path from x_1 to x_2 if there is a continuous function $\alpha : [0, 1] \longrightarrow X$ such that $\alpha(0) = x_1$ and $\alpha(1) = x_2$. A subset A of X is arcwise connected if there is a continuous path between any two elements of A. It is a basic result of topology that arcwise connectedness implies connectedness, though not conversely. In what follows we will find it convenient to establish the connectedness of various sets of probability distributions by establishing the stronger property of arcwise connectedness.

Recall that subsets E and F of S are independent with respect to the probability distribution P on S ("P-independent") whenever

$$P(E \cap F) = P(E)P(F). \tag{2.1}$$

Substituting $P(E \cap F) + P(E \cap F^c)$ for $P(E)$ and $P(E \cap F) + P(E^c \cap F)$ for $P(F)$ in (2.1), and expanding, yields an alternative, and here more convenient, formulation of independence, namely,

$$P(E \cap F)P(E^c \cap F^c) = P(E \cap F^c)P(E^c \cap F). \tag{2.2}$$

Our aim in what follows is to show, with a few minor exceptions, that if events E and F are P_1-independent and P_2-independent, then there exists a probability distribution P_3 preserving this independence and equidistant from P_1 and P_2. In what follows, the distance $d(P_1, P_2)$ is always the Euclidean distance between P_1 and P_2, i.e., if $S = \{s_i\}$, then

$$d(P_1, P_2) = \left(\sum_i (P_1(s_i) - P_2(s_i))^2 \right)^{1/2}. \tag{2.3}$$

It is easy to show that this sum converges when S is countably infinite. Indeed, $d(P_1, P_2)$ is just the "l_2 -norm" of $P_1 - P_2$, regarded as a vector in $\mathbb{R}^\infty$. Other metrics are discussed in Remark 4.1 below.

3. When S is finite and $E^c \cap F^c = \emptyset$

Note first that unless the sets $E \cap F$, $E \cap F^c$, and $E^c \cap F$ are all nonempty, there may be no probability distribution P_3 equidistant from P_1 and P_2 that preserves the independence of E and F. (Suppose, for example, that the probability distributions P_1 and P_2 are given by $(P_1(s_1), \ldots, P_1(s_n)) = (\frac{1}{n-1}, \ldots, \frac{1}{n-1}, 0)$ and $(P_2(s_1), \ldots, P_2(s_n)) = (0, \ldots, 0, 1)$. Then if (1) $E = \{s_1, \ldots, s_{n-1}\}$ and $F = \{s_n\}$, (2) $E = F = \{s_n\}$, or (3) $E = F = \{s_1, \ldots, s_{n-1}\}$, there exists no probability distribution P_3 on $S = \{s_1, \ldots, s_n\}$ equidistant from P_1 and P_2 and preserving the independence of E and F.) In what follows, therefore, we shall assume that each of the sets $E \cap F$, $E \cap F^c$, and $E^c \cap F$ is nonempty, and hence that $|S| \geq 3$. We begin with the case $|S| = 3$, where we can prove a little more.

Theorem 3.1. *Let S be a 3-element set containing subsets E and F such that $E \cap F$, $E \cap F^c$, and $E^c \cap F$ are non-empty. Let P_1 and P_2 be distinct probability distributions on the 3-element set S, and suppose that E and F are P_i-independent, $i = 1, 2$. Then there are either one or two probability distributions on S that preserve this independence and are equidistant from P_1 and P_2. Furthermore, the common distance between at least one of the former distributions and P_1 and P_2 is no larger than $d(P_1, P_2)$.*

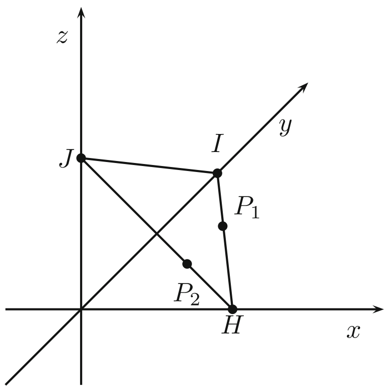


FIGURE 1. Graphical depiction of i-tuples

Proof. Since $E \cap F$, $E \cap F^c$, and $E^c \cap F$ are all nonempty and $|S| = 3$, the set $E^c \cap F^c$ must be empty. Thus, by (2.2), if P is any probability distribution on S, the events E and F are P-independent if and only if at least one of the probabilities $P(E \cap F^c)$ and $P(E^c \cap F)$ is equal to zero. Since $P(E \cap F) + P(E \cap F^c) + P(E^c \cap F) + P(E^c \cap F^c) = 1$, the only probability distributions on S for which E and F are independent are of the form $(u, 1-u, 0)$ or $(v, 0, 1-v)$, where the first, second, and third coordinates give the probabilities of the events $E \cap F$, $E \cap F^c$, and $E^c \cap F$, respectively. Note that the singleton sets $E \cap F$, $E \cap F^c$, and $E^c \cap F$ comprise the three points of S.

Let us call a 3-tuple of the form $(u, 1-u, 0)$ or $(v, 0, 1-v)$, where $0 \leq u, v \leq 1$, an independence triple (henceforth, i-triple). For fixed subsets E and F of the 3-point set S for which $E \cap F$, $E \cap F^c$, and $E^c \cap F$ are nonempty, there is a one-to-one correspondence between the set of i-triples and the set of probability distributions P on S for which E and F are P-independent, and we will henceforth make no distinction between such probability distributions and their corresponding i-triples. Observe that the set of i-triples consists of the union of the line segments HI and HJ in Fig. 1 above. Since this set of points in $\mathbb{R}^3$ is clearly arcwise connected, it follows from Theorem 2.1 and Remark 2.1 that there exists a probability distribution P_3 on S such that E and F are P_3-independent and $d(P_1, P_3) = d(P_2, P_3)$.

In this case, however, more can be said. If both P_1 and P_2 have the form $(u, 1-u, 0)$, i.e., lie on the line segment HI, then their midpoint is an i-triple equidistant from P_1 and P_2. If the perpendicular bisector of the line segment P_1P_2 intersects the line segment HJ, that point of intersection furnishes another i-triple equidistant from P_1 and P_2. Similar remarks apply when both P_1 and P_2 have the form $(v, 0, 1-v)$. Suppose then that P_1 and P_2 have different forms, say $P_1 = (u, 1-u, 0)$ and $P_2 = (v, 0, 1-v)$. The question then becomes:

> *Suppose* $P_1 = (u, 1-u, 0)$ *and* $P_2 = (v, 0, 1-v)$, *where* $u, v \in [0, 1]$. *Does there always exist* $p, q \in [0, 1]$ *such that at least one of the following holds:*

$$(i) \quad d(P_1, P_3) = d(P_2, P_3) \leq d(P_1, P_2), \quad \textit{where} \quad P_3 = (p, 1-p, 0); \tag{3.1}$$

or

$$(ii) \quad d(P_1, Q_3) = d(P_2, Q_3) \leq d(P_1, P_2), \quad \textit{where} \quad Q_3 = (q, 0, 1-q)? \tag{3.2}$$

The following lemma answers the above question in the affirmative, which completes the proof of Theorem 3.1. □

Lemma 3.1. *Suppose* $P_1 = (u, 1-u, 0)$ *and* $P_2 = (v, 0, 1-v)$, *where* $u, v \in [0, 1]$. *Then there are one or two points having either the form* $(p, 1-p, 0)$ *or* $(q, 0, 1-q)$ *that are equidistant from* P_1 *and* P_2, *where* $p, q \in [0, 1]$. *Furthermore, the common distance between at least one of these points and* P_1 *and* P_2 *is no larger than* $d(P_1, P_2)$.

Proof. We proceed geometrically. Note first that the points P_1 and P_2 lie, respectively, on the sides HI and HJ of the equilateral triangle HIJ with vertices $H = (1, 0, 0)$, $I = (0, 1, 0)$, and $J = (0, 0, 1)$. Suppose, without loss of generality, that $HP_1 \geq HP_2$ as shown below (Fig. 2).

Take P_3 on side HP_1 of triangle HP_1P_2 such that $\angle P_3P_2P_1 = \angle P_3P_1P_2$, i.e., $P_3P_1 = P_3P_2$. Then P_3 is of the form required in (3.1) and thus represents a distribution on S equidistant from the distributions P_1 and P_2 for which the events E and F are independent. In addition, there is only one distribution of this form since there is only one point on segment HI of equal distance from P_1 and P_2 when $HP_1 \geq HP_2$. Notice further that $P_3P_1 = P_3P_2 \leq P_1P_2$ in the isosceles triangle $P_3P_1P_2$ since $\angle P_1P_3P_2 \geq \angle P_2HP_3 = 60°$. Thus, the distance

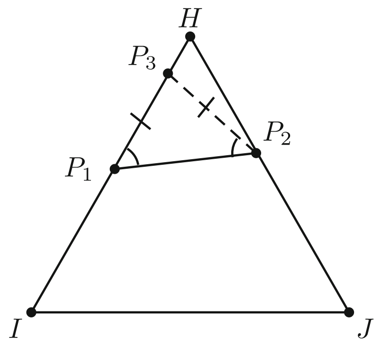


FIGURE 2. Finding an i-tuple of equal distance

between the aggregate distribution P_3 and those of the two peers, P_1 and P_2, is at most the distance between the two peers' distributions.

By similar reasoning, there is at most one point on side HJ of triangle HIJ of equal distance from points P_1 and P_2, and hence at most one point of the form in (3.2). Altogether, there are at most two points equidistant from P_1 and P_2 and lying on segments HI or HJ. □

Next we extend Theorem 3.1 to arbitrary finite sets.

Theorem 3.2. *Let S be a finite set containing subsets E and F such that $E \cap F$, $E \cap F^c$, and $E^c \cap F$ are nonempty, and $E^c \cap F^c = \emptyset$. If P_1 and P_2 are probability distributions on S for which E and F are independent, then there exists at least one probability distribution P_3 that preserves this independence and is equidistant from P_1 and P_2.*

Proof. Let $|E \cap F| = n_1$, $|E \cap F^c| = n_2$, and $|E^c \cap F| = n_3$, where the n_i are positive integers summing to $n = |S|$. Call an n-tuple

$$(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}, z_1, \dots, z_{n_3})$$

with nonnegative coordinates summing to 1 an extended i-triple of type (n_1, n_2, n_3) if

$$(x_1 + \cdots + x_{n_1}, y_1 + \cdots + y_{n_2}, z_1 + \cdots + z_{n_3}) \tag{3.3}$$

is an i-triple. Given fixed E and F as above, there is a one-to-one correspondence between extended i-triples of type (n_1, n_2, n_3) and probability distributions P on S for which E and F are independent: first list the probabilities of the n_1 points of $E \cap F$ in some order, then the probabilities of the n_2 points of $E \cap F^c$ in some order, and finally the probabilities of the n_3 points of $E^c \cap F$ in some order. Lemma 6.3 in the Appendix establishes the arcwise connectedness of the set of extended i-triples of type (n_1, n_2, n_3) in $\mathbb{R}^n$, from which Theorem 3.2 follows from Theorem 2.1 and Remark 2.1. □

4. When S is finite and $E^c \cap F^c \neq \emptyset$

To continue our analysis of independence preservation, we now consider the case in which the sets $E \cap F$, $E \cap F^c$, $E^c \cap F$, and $E^c \cap F^c$ are all nonempty, whence $|S| \geq 4$. We start with the case $|S| = 4$. In what follows, we will call a 4-tuple (x, y, z, w) with nonnegative coordinates summing to 1 an independence quadruple (henceforth, i-quadruple) if $xw = yz$.

Theorem 4.1. *Let P_1 and P_2 be distinct probability distributions on the 4-element set S, and suppose that E and F are P_i-independent, $i = 1, 2$, and that $E \cap F$, $E \cap F^c$, $E^c \cap F$, and $E^c \cap F^c$ are nonempty. Then there exists at least one probability distribution P_3 that preserves this independence and is equidistant from P_1 and P_2.*

Proof. By (2.2), given fixed subsets E and F of the 4-point set S for which $E \cap F$, $E \cap F^c$, $E^c \cap F$, and $E^c \cap F^c$ are all nonempty, there is a one-to-one correspondence between i-quadruples and probability distributions P on S for which E and F are independent (here, $x = P(E \cap F)$, etc.). Theorem 4.1 now follows from Theorem 2.1, Remark 2.1, and the arcwise connectedness of the set of i-quadruples in $\mathbb{R}^4$, which is proved as Lemma 6.1 in the Appendix. □

We now extend Theorem 4.1 to arbitrary finite sets.

Theorem 4.2. *Let S be a finite set containing subsets E and F such that $E \cap F$, $E \cap F^c$, $E^c \cap F$, and $E^c \cap F^c$ are nonempty. If P_1 and P_2 are probability distributions on S for which E and F are independent, then there exists at least one probability distribution P_3 on S that preserves this independence and is equidistant from P_1 and P_2.*

Proof. Let $|E \cap F| = n_1$, $|E \cap F^c| = n_2$, $|E^c \cap F| = n_3$, and $|E^c \cap F^c| = n_4$, where the n_i are positive integers summing to $n = |S|$. Call an n-tuple

$$(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}, z_1, \dots, z_{n_3}, w_1, \dots, w_{n_4})$$

with nonnegative coordinates summing to 1 an extended i-quadruple of type (n_1, n_2, n_3, n_4) if

$$(x_1 + \cdots + x_{n_1}, y_1 + \cdots + y_{n_2}, z_1 + \cdots + z_{n_3}, w_1 + \cdots + w_{n_4}) \tag{4.1}$$

is an i-quadruple. Given fixed E and F as above, there is a one-to-one correspondence between extended i-quadruples of type (n_1, n_2, n_3, n_4) and probability distributions P on S for which E and F are independent: first list the probabilities of the n_1 points of $E \cap F$ in some order, then the probabilities of the n_2 points of $E \cap F^c$ in some order, then the probabilities of the n_3 points of $E^c \cap F$ in some order and finally the probabilities of the n_4 points of $E^c \cap F^c$ in some order. Theorem 4.2 now follows from the arcwise connectedness of the set of extended i-quadruples of type (n_1, n_2, n_3, n_4) in $\mathbb{R}^n$, which is proved as Lemma 6.2 in the Appendix. □

Remark 4.1. If a set C is connected for a topology T on X, it is also connected for any topology on X that is coarser than T. Hence our proofs of the foregoing theorems work equally well with the Euclidean distance d replaced by any metric m whose induced topology on $\mathbb{R}^n$ is coarser than the normal Euclidean topology on $\mathbb{R}^n$. This includes many metrics on $\mathbb{R}^n$ arising in applications, including most norms, since the majority of these metrics are themselves continuous functions from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ in the usual topologies. A useful criterion for determining when a metric induces a coarser topology on $\mathbb{R}^n$ than the normal one which we could not find is as follows: a metric d on $\mathbb{R}^n$ induces a coarser topology on $\mathbb{R}^n$ than the Euclidean metric if and only if the function $d(x, q)$, $x \in \mathbb{R}^n$, is continuous (in the usual topologies on $\mathbb{R}^n$ and $\mathbb{R}$) for all $q \in \mathbb{R}^n$.

Remark 4.2. The arcwise connectedness of the set of i-triples (x, y, z) does not, as might appear at first glance, follow from our proof of the arcwise connectedness of the set of i-quadruples (x, y, z, w) merely by restricting attention to i-quadruples in which $w = 0$. For, as one sees in the proof of Lemma 6.1 in the Appendix, the continuous path between any two i-quadruples constructed in that proof passes through $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. Hence there is no guarantee that an i-quadruple equidistant from two i-quadruples, each with fourth coordinate equal to zero, also has fourth coordinate equal to zero.

5. When S is infinite

The arguments presented above also carry over without too much difficulty to the case in which S is countably infinite, which we now assume. If P_1 and P_2 are probability distributions on $S = \{s_1, s_2, \dots\}$, recall that the distance $d(P_1, P_2)$ is defined as the Euclidean distance in (3.1) from the sequence $(P_1(s_1), P_1(s_2), \dots)$ to the sequence $(P_2(s_1), P_2(s_2), \dots)$. We conclude by extending Theorems 3.2 and 4.2 to countably infinite sets.

Theorem 5.1. *Let S be a countably infinite set containing subsets E and F such that $E \cap F$, $E \cap F^c$, and $E^c \cap F$ are nonempty. If P_1 and P_2 are probability distributions on S for which E and F are independent, then there exists at least one probability distribution P_3 on S that preserves this independence and is equidistant from P_1 and P_2.*

Proof. Relabel the points of S using the positive integers. Under this relabelling, let $E \cap F = X$, $E \cap F^c = Y$, $E^c \cap F = Z$, and $E^c \cap F^c = W$. Then X, Y, Z, and W are pairwise disjoint sets whose union is the set of positive integers, where W is allowed to be empty. Call a sequence $(t_1, t_2, \dots)$ with nonnegative components summing to 1 an extended i-quadruple of type (X, Y, Z, W) if

$$\left(\sum_{i \in X} t_i, \sum_{i \in Y} t_i, \sum_{i \in Z} t_i, \sum_{i \in W} t_i\right) \tag{5.1}$$

is an i-quadruple (Note that if W is empty, the final coordinate is zero, being an empty sum.) Given fixed E and F as above, there is a one-to-one correspondence between extended i-quadruples of type (X, Y, Z, W) and probability distributions P on S for which E and F are independent, by (2.2). Theorem 5.1 now follows from Theorem 2.1, Remark 2.1, and the arcwise connectedness of the set of extended i-quadruples of type (X, Y, Z, W) in $\mathbb{R}^\infty$, which is proved as Lemma 6.4 in the Appendix. □

Remark 5.1. It seems unlikely that the foregoing results can be extended to preserving multiple cases of independence. Suppose, for example, that the events E_k and F_k are both P_1- and P_2-independent for $k = 1, \dots, m$. For each such k, we know that the family Δ_k of probability distributions equidistant from P_1 and P_2 and preserving the independence of E_k and F_k is nonempty. But it seems unrealistic to expect in general that the intersection of the families Δ_k will be nonempty, for this would necessitate the simultaneous satisfaction of what would appear to be simply too many geometric constraints. Similar geometric problems arise if we are trying to preserve one or more cases of independence common to the distributions of more than two epistemic peers.

6. Appendix

Lemma 6.1. *The set of i-quadruples in* $\mathbb{R}^4$ *is arcwise connected.*

Proof. We start with an observation:

> *If x and w are nonnegative real numbers with $s := x + w \leq \frac{1}{2}$, then there exists an i-quadruple whose first and last coordinates are x and w.*

To see this, note first that the product of x and w necessarily satisfies $xw \leq \frac{s^2}{4}$. Thus, there exist y and z with $y+z = 1-s$ such that $yz = xw$ since the product $yz = y(1-s-y)$ ranges continuously from 0 to $\frac{(1-s)^2}{4} \geq \frac{s^2}{4}$ as y ranges form 0 to $\frac{1-s}{2}$. Note that the observation above applies equally well with "first and last" replaced by "middle two".

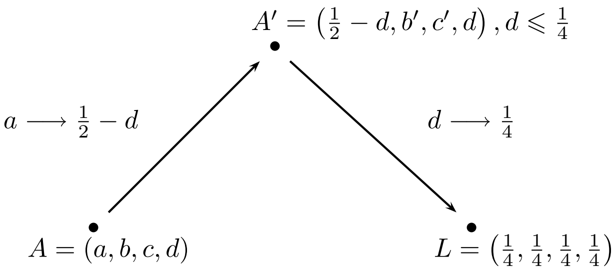
We now prove Lemma 6.1. Let A and B be distinct i-quadruples with $A = (a, b, c, d)$ and let L be the i-quadruple $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. It suffices to construct a continuous path consisting of i-quadruples from A to L, as then we may simply join a path going from A to L with a path going from L to B to obtain a path going from A to B. By symmetry, we may assume $s := a+d \leq \frac{1}{2}$ with $a \geq d$ (if $s > \frac{1}{2}$, then consider b and c, as $b + c < \frac{1}{2}$).

By the observation, we can increase a continuously to $\frac{1}{2} - d$, keeping d fixed and changing the middle two coordinates continuously so as to remain an i-quadruple. For if a and d are given, then b and c must be the roots of the quadratic equation $t^2 - (1-a-d)t + ad = 0$, i.e.,

$$b, c = \frac{1 - a - d \pm \sqrt{(1-a-d)^2 - 4ad}}{2}.$$

Substituting $a = \frac{1}{2} - d$ into this implies that the resulting i-quadruple A' must be either of the form $(\frac{1}{2} - d, d, \frac{1}{2} - d, d)$ or of the form $(\frac{1}{2} - d, \frac{1}{2} - d, d, d)$, where $d \leq \frac{1}{4}$.

Now join A' to L, using either the line $(\frac{1}{2} - t, t, \frac{1}{2} - t, t)$ or the line $(\frac{1}{2} - t, \frac{1}{2} - t, t, t)$, where $d \leq t \leq \frac{1}{4}$. Combining the path from A' to L with the path from A to A' results in a continuous path from A to L consisting of i-quadruples, as illustrated below.



□

Lemma 6.2. *The set of extended i-quadruples of type* (n_1, n_2, n_3, n_4) *in* $\mathbb{R}^n$ *is arcwise connected.*

Proof. Let A and B be distinct extended i-quadruples of type (n_1, n_2, n_3, n_4) and let $C = (a, b, c, d)$ and $D = (e, f, g, h)$ be the i-quadruples obtained by summing the coordinates of A and B, respectively, as in (4.1). By the prior lemma, there is a continuous path in $\mathbb{R}^4$ consisting of i-quadruples joining C and D, which we parametrize as

$$(g_1(t), g_2(t), g_3(t), g_4(t)), \quad t \in [0, 1], \tag{6.1}$$

where the g_i are continuous and the coordinates of C and D correspond to putting $t = 0$ and $t = 1$, respectively, in (6.1).

Introduce nonnegative, continuous, real-valued functions defined on $[0, 1]$, which are positive on $(0, 1)$:

$$f_1^{(1)}, \ldots, f_{n_1}^{(1)}, f_1^{(2)}, \ldots, f_{n_2}^{(2)}, f_1^{(3)}, \ldots, f_{n_3}^{(3)}, f_1^{(4)}, \ldots, f_{n_4}^{(4)}. \tag{6.2}$$

Let $f_i^{(1)}(0) = a_i$ and $f_i^{(1)}(1) = e_i$, $1 \leq i \leq n_1$, where the a_i and e_i denote the first n_1 coordinates of A and B, respectively; note that the a_i and e_i sum to a and e, respectively. Similarly, define the values of $f_i^{(k)}$, $k = 2, 3, 4$, at the endpoints $t = 0$ and $t = 1$, using the remaining coordinates of A and B.

As one travels from C to D in $\mathbb{R}^4$, the first coordinate varies according to $g_1(t)$, $t \in [0, 1]$, where $g_1(0) = a$ and $g_1(1) = e$. We decompose this first coordinate into n_1 components which sum to it by taking

$$\frac{1}{\sum_{i=1}^{n_1} f_i^{(1)}(t)} (f_1^{(1)}(t) g_1(t), f_2^{(1)}(t) g_1(t), \ldots, f_{n_1}^{(1)}(t) g_1(t)), \tag{6.3}$$

provided $\sum_{i=1}^{n_1} f_i^{(1)}(t) \neq 0$. (Put 0 for all components in the case when $\sum_{i=1}^{n_1} f_i^{(1)}(t) = 0$, which can occur only when $t = 0$ or $t = 1$.) To verify continuity on $[0, 1]$ of the decomposition in (6.3), one only needs to check the cases $t = 0$ and $t = 1$. The other three coordinates as one goes from C to D may be similarly decomposed using $f_i^{(2)}$, $f_i^{(3)}$ and $f_i^{(4)}$. Putting these four vectors together yields a continuous path joining A and B and consisting of extended i-quadruples of type (n_1, n_2, n_3, n_4). Note that A corresponds to $t = 0$ and B to $t = 1$. □

Remark 6.1. If $C = D$ in the prior proof, then all points along the line joining A and B in $\mathbb{R}^n$ are extended i-quadruples of type (n_1, n_2, n_3, n_4) having associated i-quadruple C, in which case one may simply take the midpoint of A and B to obtain the desired probability distribution in Theorem 4.2.

Lemma 6.3. *The set of extended i-triples of type (n_1, n_2, n_3) in $\mathbb{R}^n$ is arcwise connected.*

Proof. We modify the argument for Lemma 6.2. Let A and B be distinct extended i-triples of type (n_1, n_2, n_3) and let $C = (a, b, c)$ and $D = (e, f, g)$ be the i-triples obtained by summing the coordinates of A and B, respectively, as in (3.3). Since the set of i-triples is arcwise connected (consisting of the union of the segments HI and HJ in Fig. 1), there is a continuous path in $\mathbb{R}^3$ consisting of i-triples joining C and D, which we parametrize as

$$(g_1(t), g_2(t), g_3(t)), \quad t \in [0, 1], \tag{6.4}$$

where the g_i are continuous and the coordinates of C and D correspond to putting $t = 0$ and $t = 1$, respectively, in (6.4). The rest of the proof goes in much the same way as the preceding proof, the main difference being that there should only be three sets of auxiliary functions, instead of four as in (6.2). □

Remark 6.2. If C and D are both of the form $(u, 1-u, 0)$ or $(v, 0, 1-v)$ in the prior proof, then all points along the line joining A and B in $\mathbb{R}^n$ are extended i-triples of type (n_1, n_2, n_3), in which case one may simply take the midpoint of A and B to obtain the desired probability distribution in Theorem 3.2.

Notation. *Let $(x_i)_{i \geq 1}$ denote the sequence $(x_1, x_2, \ldots)$ of real numbers and $\mathbb{R}^\infty$ denote the set of all sequences of real numbers. If $x = (x_i)_{i \geq 1}$, then define the norm of x, denoted $||x||$, by $(\sum_{i \geq 1} x_i^2)^{\frac{1}{2}}$, provided this sum converges, so that the (Euclidean) distance between the sequences y and z is simply $||y - z||$.*

Remark 6.3. In the proof of Lemma 6.2 above, we use, implicitly in establishing the continuity of the vector in (6.3), the fact that an $\mathbb{R}^n$-valued function of $\mathbb{R}$ is continuous if and only if each of its components is continuous (in the usual topologies). This fact, however, does not hold for an $\mathbb{R}^\infty$-valued function of $\mathbb{R}$ (simply take $|t|^{\frac{1}{n}}$, $n \geq 1$, for the component functions and the limit as $t \to 0$). Because of this, extending Lemmas 6.2 and 6.3 to the countably infinite case will require an additional step, wherein we select certain $\mathbb{R}^\infty$-valued functions of $\mathbb{R}$ which are continuous and show that they converge uniformly on $[0, 1]$.

Lemma 6.4. *The set of extended i-quadruples of type (X, Y, Z, W) in $\mathbb{R}^\infty$ is arcwise connected.*

Proof. Let $A = (a_i)_{i\geq 1}$ and $B = (b_i)_{i\geq 1}$ be distinct extended i-quadruples of type (X, Y, Z, W) and let $C = (a, b, c, d)$ and $D = (e, f, g, h)$ be the i-quadruples obtained by summing the entries of A and B, respectively, as in (5.1). Suppose for now that the coordinates of both C and D are all nonzero. Join C and D by a continuous path (g_1, g_2, g_3, g_4) consisting of i-quadruples, as in (6.1). Replace the auxiliary functions in (6.2) with the four sets of functions—$(f_{p_i})_{p_i\in X}, (f_{r_i})_{r_i\in Y}, (f_{s_i})_{s_i\in Z}, (f_{t_i})_{t_i\in W}$—all of which are continuous and nonnegative on $[0, 1]$.

To simplify notation, let $(f_{p_i})_{p_i\in X} = (f_i)_{i\geq 1}$. If $i \geq 1$, then let $f_i(t) = (1 - t)a_{p_i} + tb_{p_i}, 0 \leq t \leq 1$. Decompose $g_1(t)$ into

$$\frac{1}{\sum_{i\geq 1} f_i(t)}(f_1(t)g_1(t), f_2(t)g_1(t), \ldots), \quad 0 \leq t \leq 1. \tag{6.5}$$

We show that this decomposition of $g_1(t)$ is continuous. First note that $0 \leq g_1(t) \leq 1$ for all t in $[0, 1]$, since $g_1(t)$ is a coordinate of an i-quadruple. Next, observe that the sequence of $\mathbb{R}^\infty$-valued functions $h_n(t)$, where the ith coordinate of $h_n(t)$ equals $f_i(t)g_1(t)$ if $1 \leq i \leq n$ and 0 if $i > n$, converges uniformly to $h(t) := (f_i(t)g_1(t))_{i\geq 1}$ on $[0, 1]$ since, for all t,

$$\begin{aligned}
||h_n(t) - h(t)|| = ||(f_i(t)g_1(t))_{i>n}|| &= |g_1(t)| \left(\sum_{i>n} f_i^2(t)\right)^{\frac{1}{2}} \\
&\leq g_1(t) \sum_{i>n} f_i(t) = g_1(t) \left[(1 - t)\sum_{i>n} a_{p_i} + t\sum_{i>n} b_{p_i}\right] \\
&\leq \sum_{i>n}(a_{p_i} + b_{p_i}) \to 0,
\end{aligned}$$

as $n \to \infty$. Thus $h(t)$ is continuous on $[0, 1]$, being the uniform limit of continuous functions, which implies that the decomposition in (6.5) is also continuous since $\sum_{i\geq 1} f_i(t) = (1 - t)\sum_{i\geq 1} a_{p_i} + t\sum_{i\geq 1} b_{p_i} = (1 - t)a + te$ is nonzero on $[0, 1]$. Note that if the set X is finite, then no convergence argument is needed to establish continuity. Similarly, one can decompose the other three coordinates of the path joining C and D. Putting together the four resulting vectors yields a continuous path from A to B consisting of extended i-quadruples of type (X, Y, Z, W).

The same proof also applies if it is the case that C or D contain one or more zero coordinates, but do not share a zero coordinate. If say $a = 0$ and $e \neq 0$, then the decomposition of $g_1(t)$ in (6.5) is not defined at $t = 0$ since $\sum_{i\geq 1} f_i(t) = (1 - t)a + te = 0$, in which case, we simply put zero for all of the components. The function in (6.5) is still continuous on (0,1], as before. It is also continuous at $t = 0$ since $\lim_{t\to 0^+} g_1(t) = a = 0$ implies

$$\lim_{t\to 0^+}\left[(f_i(t)g_1(t))_{i\geq 1}/\sum_{i\geq 1} f_i(t)\right] = (0, 0, \ldots).$$

On the other hand, if C and D share a zero coordinate, then the problem reduces to extending Lemma 6.3 to the case in which S is countably infinite, and this can be done in a similar manner. □

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