Research Statement

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My research lies in the area of analysis theory for partial differential equations and its applications. I study various problems related to existence, uniqueness, and regularity estimates of solutions for elliptic and parabolic equations. Besides, I am also interested in studying nonlinear dynamics of solutions of nonlinear dispersive equations, and optimal control problems in mathematical biology. Below, I summarize the two topics that I have been focusing on in the last 5 years. The details of the topics are also subsequently discussed.

Elliptic and parabolic equations: I am interested studying regularity estimates of solutions and then establishing well-posedness of solutions. My works cover two classes of equations: Equations with prescribed degenerate singular coefficients, and quasilinear equations with cross-diffusion features.

Stokes and Navier-Stokes equations: I study a class of Stokes system with measurable singular coefficients. Local interior and boundary regularity estimates of Calderón-Zygmund type for this class of equations are established for the first time in this paper. As an application, we introduced *a new ϵ-regularity criterion for Leray-Hopf's weak solutions of Navier-Stokes equations.* My work also proves existence, uniqueness and stability of singular solutions for stationary Navier-Stokes equations with singular forces.

1. Regularity theory for elliptic and parabolic equations and applications

1.1. Equations with singular-degenerate coefficients. There are 3 classes of equations that I study: Singular-degenerate equations of Fabes-Kenig-Sarapioni type; Singular-degenerate equations of extensional operator types; Degenerate viscous Hamilton-Jacobi equations.

a) *Singular-degenerate equations of Fabes-Kenig-Sarapioni type.* I investigate the class of elliptic equations of the form

$$\operatorname{div}(\mathbb{A}_0(x)\nabla u) = \operatorname{div}(F) \quad \text{in} \quad \Omega \tag{1}$$

with suitable boundary condition on $\partial\Omega$, where $\mathbb{A}_0 = (a_{ij})$ is a matrix which is singular-degenerate

$$\Lambda^{-1}\mu(x)|\xi|^2 \leq \langle \mathbb{A}_0(x)\xi, \xi\rangle \leq \Lambda\mu(x)|\xi|^2, \quad \text{for a.e. } x \in \Omega, \quad \text{for all } \xi \in \mathbb{R}^n.$$

where $\mu \in A_2$ as considered in [22]. In my paper [6, 7], $W^{1,p}$-theory was established when (a_{ij}) satisfies some weighted VMO-condition. This is the first time such type of the results are studied, and the results can be considered as a counterpart of the Hölder's regularity theory established in [22].

In [50], I extended the work [6, 7] and studied the following inhomogeneous equation with singular-degenerate coefficients

$$\begin{cases} \operatorname{div}[\mathbf{A}(x,u,\nabla u)] = \operatorname{div}F, & \text{in} \quad \Omega, \\ u = g, & \text{on} \quad \partial\Omega. \end{cases}$$

We are interested in the case $\mathbf{A}$ is asymptotically Uhlenbeck, i.e.

$$\lim_{|\xi|\to\infty} |\mathbf{A}(x,\tau,\xi) - \mathbb{A}_0(x)\xi| = 0, \quad \text{uniformly in } x \in \Omega, \text{ and } \tau \in \mathbb{R}.$$

The matrix $\mathbb{A}_0$ is allowed be singular or degenerate as before. Under some suitable assumption the oscillation of $\mathbb{A}_0$ with respect to the weight μ, the following two-weighted estimate is proved

$$\|\nabla u\|_{L^q(\Omega,\omega)} \leq C\left[\|F/\mu\|_{L^q(\Omega,\omega)} + \|\nabla g\|_{L^q(\Omega,\omega)}\right],$$

where $2 \leq q < \infty$, $\omega \in A_{q/2}$ is another weight which satisfies some kind of Sawyer's condition [60].

In the near future, I plan to study the class of equations (1) where $\mu(x) = \text{dis}(x, \partial\Omega)^\alpha$ for $\alpha \in (-1,1)$. Applications of the L_p-result estimates to study regularity problems arising in geometric analysis will be also considered.

b) *Singular-degenerate equations with extensional operator types.* In the series of papers [14–17], I study the class of equations of the type

$$x_d^\alpha(u_t + \lambda u) - D_i\big(x_d^\alpha\big[a_{ij}(t,x)D_j u - F_i\big]\big) = x_d^\alpha f \quad \text{in} \quad \Omega_T$$

and its non-divergence counterpart

$$u_t + \lambda u - a_{ij}(t,x)D_{ij}u + \frac{\alpha}{x_d}a_{di}D_i u = f \quad \text{in} \quad \Omega_T \tag{2}$$

with suitable boundary conditions on $(-\infty, T)\times\partial\mathbb{R}^d_+$. Here, $T \in (-\infty, \infty]$ is fixed and $x = (x', x_d) \in \mathbb{R}^{d-1} \times (0,\infty)$, $\alpha \in \mathbb{R}$, and (a_{ij}) is uniformly elliptic and bounded. When $\alpha \in (-1,1)$, the linear operator in the above PDE appears in the study of fractional elliptic/parabolic equations.

Under the homogeneous Dirichlet boundary condition on $\{x_d = 0\}$, and with $\alpha < 1$, the following particlular estimates are proved in [14, 15]

$$\int_{\Omega_T}\big(|u_t|^p + |DD_{x'}u|^p + |D_d^2 u + \frac{\alpha}{x_d}D_d u|^p + |\sqrt{\lambda}Du|^p + |\lambda u|^p\big)x_d^\gamma dz \leq N\int_{\Omega_T}|f(z)|^p x_d^\gamma dz \tag{3}$$

for strong solution of (2), where $\gamma \in (\alpha p - 1, p-1)$ and $N = N(d,p,\alpha,\gamma)$. Meanwhile, with the conormal boundary value problem, we proved in [16, 17] that

$$\int_{\Omega_T}\big(|u_t|^p + |D^2u|^p + |\sqrt{\lambda}Du|^p + |\lambda u|^p\big)x_d^{\beta+\alpha}dz \leq N\int_{\Omega_T}|f(z)|^p x_d^{\beta+\alpha}dz \tag{4}$$

for $\alpha \in (-1,\infty)$ and $\beta \in (-\alpha-1, (\alpha+1)(p-1))$.

We remark that both estimates (3) and (4) appear for the first time. The singularity/degenerate weight x_d^α is not in A_2 as considered in [22]. More comprehensive weighted estimates and wellposedness results are presented in [14–17]. Similar results with an application to the study of nonlocal equations are also obtained in [40].

c) *Degenerate viscous Hamilton-Jacobi equations.* In simplest form, the equations I study reads

$$u_t(z) + \lambda u(z) - x_d\Delta u = \sqrt{\lambda}f(z) \quad \text{in} \quad \Omega_T = (-\infty, T)\times\mathbb{R}^d_+, \tag{5}$$

which resembles well the following degenerate viscous Hamilton-Jacobi equation

$$u_t(z) + \lambda u(z) + H(z, Du) - x_d\Delta u = 0 \quad \text{in} \quad \Omega_T, \tag{6}$$

where $H : \Omega_T \times \mathbb{R}^d \to \mathbb{R}$ is a given Hamiltonian. Here $z = (t,x) \in \Omega_T$ with $x = (x', x_d) \in \mathbb{R}^d_+ = \mathbb{R}^{d-1}\times(0,\infty)$. For typical viscous Hamilton-Jacobi equations with possibly degenerate and bounded diffusions, one often has uniqueness of viscosity solutions, and such solutions are often Lipschitz in z (see [2, 12] and the references therein). Finer regularity of solutions is not very well understood in the literature. In particular, optimal regularity of solutions to (6) near $\{x_d = 0\}$ has not been investigated.

In this line of research, my goal is to understand the regularity of (6), and also similar classes of equations when x_d in (6) is replaced by more general weights $\mu(x_d)$. To pave our way, I first investigate the class of equation

$$x_d^{-1}(u_t + \lambda c_0(z)u) - D_i(a_{ij}(t,x)D_j - F_i) = \sqrt{\lambda}x_d^{-1}f \quad \text{in} \quad \Omega_T \tag{7}$$

with Dirichlet homogeneous boundary condition $u = 0$ on $\{x_d = 0\}$. In [44], suitable class of weighted Sobolev spaces are found to prove well-posedness, and regularity estimates for (7). This paper is the first time that the type of equation (7) is studied. More comprehensive results and more work is needed for various equations of type (6) in my up coming papers.

1.2. Equations with cross-diffusion features. This section outlines and describes results in my papers [6, 20, 25, 26, 40, 42, 45, 50–54]. Essentially, this line of research establishes regularity theory estimates and studies problems on existence and uniqueness of solutions of equations and systems of equations of the form

$$\begin{cases} \partial_t \vec{u} & = & \nabla \cdot [\mathbf{A}(x,t,\vec{u},\nabla\vec{u})] + \mathbf{F}(x,t,\vec{u}), & \text{in} \quad \Omega \times (0,\infty), \\ \frac{\partial \vec{u}}{\partial \vec{n}} & = & 0, & \text{on} \quad \partial\Omega \times (0,\infty), \\ \vec{u}(x,0) & = & \vec{u}_0(x), & \text{for} \quad x \in \Omega, \end{cases} \tag{8}$$

where the principal term $\mathbf{A}$ is a nonlinear matrix of diffusion, $\vec{u} = (u_1, u_2, \cdots, u_d)$ is a vector of unknown physical quantities, $\mathbf{F}(x,t,\vec{u})$ is a vector function of reaction terms, and $\Omega \subset \mathbb{R}^d$ is an open bounded domain with unit outward vector $\vec{n}$ on the sufficiently smooth boundary $\partial\Omega$.

An interesting feature in the classes of nonlinear equations (8) is that their principal $\mathbf{A}$ depend on u. Because of this, (8) is not invariant under usual scaling and dilation. This lack of this homogeneity creates serious challenge to establish Calderón-Zygmund regularity theory for (8), see [68] for the geometric intuition.

A new technique called "double-scaling parameter" is invented in my paper [25] to treat the lack of homogeneity in Calderón-Zygmund theory related to scalings and dilations for the general class of quasi-linear equations (10). This technique is then extensively developed in my series of papers [42, 50–54] for many different classes of equations including nonlinear degenerate p-Laplacian equations. These papers are the first ones the establish the Calderón-Zygmund regularity theory for such kind of general classes of equations. Below are some details.

a) *Calderón-Zygmund regularity theory and SKT system.* Starting with [25], we established and proved the following nonlinear Calderón-Zygmund estimate

$$\|\nabla u\|_{L^q((\Omega)\times(\bar{t},T))} \leq C \left[1 + \|c\|_{L^q(\Omega\times(0,T))}\right],$$

where $q \geq 2$, $0 < \bar{t} < T$, and u is a non-negative bounded weak solution of

$$u_t = \operatorname{div}[(1+u)\mathbb{A}(x,t)\nabla u] + u(1 - c(x,t) - u), \quad \text{in} \quad \Omega \times (0,T), \tag{9}$$

with homogeneous Neumann boundary condition on the boundary $\partial\Omega$, and with uniformly elliptic matrix $\mathbb{A}$ whose BMO-norm is sufficiently small. This is the first time the Calderón-Zygmund theory estimate is known for quasi-linear equations in which the principal term, i.e. $\mathbf{A}(x,t,u,\nabla u) = (1+u)\mathbb{A}(x,t)\nabla u$, depends on the unknown solution u as its variable. We then applied our theory to the SKT system to derive *a priori* estimates and to prove the existence of global-time smooth solutions for a class of SKT system in spatial domains of any dimensions. This result in [25] gives the answer to an open question posed in [38, 43, 69], and studied in [10, 58, 59].

From the pioneer paper [25], we developed in [26, 40, 42, 50–54] a more advanced and complete theory on nonlinear Calderón-Zygmund regularity estimates for weak solutions of general class of nonlinear equations (both elliptic and parabolic):

$$u_t - \operatorname{div}[\mathbf{A}(x,t,u,\nabla u)] = \operatorname{div}[F] + f. \tag{10}$$

Our class of equations includes general quasi-linear p-Laplacian type equations, equations with prescribed singular-degenerate coefficients, and equations with singular divergence-free drifts. Our developed theory provides the Sobolev-analogue regularity theory of the classical Schauder's regularity theory for (10), see [13, 23, 34, 35, 37].

In terms of technique, we enlarged class of equations called equations with scaling parameter

$$u_t - \mathrm{div}[\mathbf{A}(x, t, \lambda u, \nabla u)] = \mathrm{div}[\mathbf{F}] + f$$

where $\lambda \geq 0$ is a parameter. Certainly, λ can not be random, and *quantifiers to control the solutions and the parameters are introduced.* For example, For example, the $\|\lambda u\|_{L^\infty}$ is invariant under the scalings and dilations. This is very important in securing all the intermediate steps so that the outcome results must be independent on λ. Combining this technique and the perturbation method (see [1, 4, 32]), we establish in in [25, 42, 50–53], for the first time, an advance and complete theory on nonlinear Calderón-Zygmund regularity estimates for general class of nonlinear equations (10).

b) *General quasi-linear equations of p-Laplacian type.* In the series of paper [42, 51, 52], we established, for the first time, the Calderón-Zygmund regularity theory for a general class quasi-linear and nonlinear degenerate equations. In the simplest setting, we studied the following general class of nonlinear p-Laplacian type equations

$$\mathrm{div}[\mathbf{A}(x, u, \nabla u)] = \mathrm{div}[F], \quad \text{in} \quad B_2,$$

where $\mathbf{A}$ behaves like $|\nabla u|^{p-2}\nabla u$ for some $1 < p < \infty$. As we already discussed, the novel in the work [42, 51, 52] is that the vector field $\mathbf{A}$ depends on u-variable. Due to this, the homogeneity under the scalings and dilation is no longer available. This creates one of the most challenges in the study of Calderón-Zygmund theory for this class of equations.

Under some suitable conditions on the oscillation of $\mathbf{A}$ with respect to x and the continuity of $\mathbf{A}$ in u, it is proved in [42] that the following estimate holds

$$\|\nabla u\|_{L^q(B_1)} \leq C(M_0, n, p, q)\Big[\, \|u\|_{L^p(B_2)} + \Big\| |F|^{\frac{1}{p-1}} \Big\|_{L^q(B_2)} \Big], \tag{11}$$

where u is a weak solution with $\|u\|_{L^\infty(B_1)} \leq M_0$, and $p \leq q < \infty$. Note that this is one of the first results where regularity estimates in Sobolev spaces are established for weak solutions of the nonlinear equations (11). Note also that this result complements well the classical Schauder's regularity estimates (see [23, 34, 37]) for bounded weak solutions of (11). Moreover, this result also recovers all available results in which $\mathbf{A}$ is independent of u (see [3, 29, 33]), and it is already new when $p = 2$.

In the recent papers [51, 52], more complete and further developments are established. In particular, results in [51, 52] relax and replace the boundedness condition $\|u\|_{L^\infty(B_1)} \leq M_0$ by the borderline case $\|u\|_{\mathrm{BMO}(B_1)} \leq M_0$. These results in [51, 52] therefore catch the critical case, and hence can be useful in many applications. For example, in n-Laplacian equations, i.e. when $p = n$, the $W^{1,n}$-weak solutions are already in BMO. Moreover, in [51, 52], the estimates (11) are also extended to weighted Lebesgue spaces with suitable weights. Global estimates for solutions of equations with inhomogeneous boundary conditions are also established.

c) *Quasi-linear equations with singular coefficients.* In the papers [20, 54], I studied the class of singular parabolic equations

$$u_t - \mathrm{div}[\mathbf{A}(x, t, u, \nabla u)] = \mathrm{div}[\mathbf{F}] \tag{12}$$

where $\mathbf{A} = \mathbf{A}(x, t, \tau, \xi)$ is a vector field measurable and singular in (x, t), continuous in $\xi \in \mathbb{R}^n$ and could be continuous in $\tau \in \mathbb{R}$. Moreover, $\mathbf{A}$ is asymptotically Uhlenbeck. This means that as

$|\xi| \to \infty$, the vector field $\mathbf{A}$ behaves like linearly in ξ, i.e. there is a matrix $\mathbb{A}_0$ such that

$$\lim_{|\xi|\to\infty} |\mathbf{A}(x,t,\tau,\xi) - \mathbb{A}_0(x,t)\xi| = 0, \quad \text{uniformly in } (x,t), \text{ and } \tau.$$

The most important feature in the paper [20,54] is that $\mathbb{A}_0$ is not required to be bounded. Precisely, we write $\mathbb{A}_0 = \hat{\mathbb{A}} + \mathbb{D}$, where $\hat{\mathbb{A}}$ is symmetric, uniformly elliptic, and $\mathbb{D}$ is skew-symmetric and in $L^\infty(\text{BMO})$. In particular, $\mathbb{A}_0$ is not bounded, hence (12) is singular.

In [54], the existence and uniqueness of weak solutions are established. Sufficient conditions on $\mathbb{A}_0$ are given in [54] so that the regularity of ∇u in weighted $L^q(\Omega,\omega)$-spaces are obtained for some class of weights ω. Moreover, higher regularity theory estimates of Meyers's type are proved in [20] and a counter-example is also provided in this paper to demonstrate the optimality of the result.

Observe that the work [20, 54] provide existence uniqueness and regularity results in Sobolev spaces, for the first time, for this type of unbounded measurable coefficients equations are established. In the linear setting, our results are stronger than the classical one. One key ingredient in this work is the curl-div theorem discovered and proved in [11] by R. Coifman, P.-L. Lions, Y. Meyer, S. Semmes in 1993. Another key ingredient is the result of C. Fefferman and E. M. Stein [21] proving that the dual of Hardy space is the John-Nirenberg BMO-space.

d) *Quasi-linear equations with singular divergence-free drifts.* Several other important results are also obtained in my work [53,55] on equations with singular divergence-free drifts

$$u_t - \text{div}[\mathbf{A}(x,t,u,\nabla u) + \mathbf{b}u] = \text{div}[\mathbf{F}] + f, \quad \text{in} \quad \Omega \times (0,T), \tag{13}$$

where $\mathbf{b}$ is divergence-free, and could be singular. These papers are closely related to [54]. Moreover, $\mathbf{A}(x,t,u,\nabla u) \sim \nabla u$. However, the papers [53,55] establish for the first time the regularity theory in L^q-spaces and Lorentz spaces for ∇u of weak solutions of (13) with singular drifts $\mathbf{b}$. In the simplest setting, the following estimate in Lorentz space is proved

$$\|\nabla u\|_{L^{q,r}(\Omega\times(\bar{t},T))} \le C \left[\|F\|_{L^{p,r}(\Omega\times(0,T))} + \|f\|_{L^{\frac{p(n+2)}{n+4},\frac{r(n+2)}{n+4}}(\Omega\times(0,T))} + [[u]]_{\text{BMO}} \|\mathbf{b}\|_{L^{p,r}(\Omega\times(0,T))} \right],$$

for $0 < \bar{t} < T$, $2 \le q < \infty$, and $1 < r \le \infty$. Observe that in the linear setting, i.e. $\mathbf{A}(x,t,u,\nabla u) = \mathbb{A}(x,t)\nabla u$ and $f = 0$, the classical Calderón-Zygmund theory gives

$$\|\nabla u\|_{L^q(\Omega\times(\bar{t},T))} \le C \left[\|F\|_{L^p(\Omega\times(0,T))} + [[u]]_{L^\infty(\Omega\times(0,T))} \|\mathbf{b}\|_{L^p(\Omega\times(0,T))} \right].$$

Our result is therefore stronger than the classical one since the L^∞-norm of u can be replaced by the BMO-norm. Note also that in the linear equation setting, the papers [53,55] can be considered as the Sobolev- analogue of the Hölder's regularity results studied by many mathematicians such as [62,67].

2. Stokes and Navier-Stokes equations

We outline some results obtained in [18,27,46–49,56]. These papers investigate the existence, uniqueness and regularity estimates for solution of Stokes systems and Navier-Stokes equations in various critical spaces.

3.1) *Local L_p estimates for Stokes systems and applications.* In my papers [18,27], we prove local interior and local boundary regularity estimates in mixed norm for systems of Stokes equations. Consequently, the results are used to study regularity for Leray-Hopf's weak solutions of Navier-Stokes equations. The paper [18] considers equations in both divergence and non-divergence form, however, I only outline here the results in divergence form. Because of this, we essentially study the following time-dependent Stokes system with general coefficients:

$$u_t - D_i(a_{ij}D_j u) + \nabla p = \mathrm{div} f, \quad \mathrm{div}(u) = g, \tag{14}$$

where $u = u(t,x) \in \mathbb{R}^d$ is an unknown vector solutions representing the velocity of the considered fluid, $p = p(t,x)$ is an unknown fluid pressure. Moreover, $f(t,x) = (f_{ij}(t,x))$ is a given measurable matrix of external forces, $g = g(t,x)$ is a given measurable function, and $a_{ij} = b_{ij}(t,x) + d_{ij}(t,x)$ is a given measurable matrix of viscosity coefficients that satisfies the following boundedness and ellipticity conditions with ellipticity constant $\nu \in (0,1)$:

$$\nu|\xi|^2 \le a_{ij}\xi_i\xi_j, \quad |b_{ij}| \le \nu^{-1}, \tag{15}$$

and

$$b_{ij} = b_{ji}, \quad d_{ij} \in L_{1,\mathrm{loc}}, \quad d_{ij} = -d_{ji}, \quad \forall\ i,j \in \{1,2,\ldots,d\}. \tag{16}$$

The following result is roughly one of the results that we obtained in [18].

Theorem 1. *Let $s,q \in (1,\infty)$, $\nu \in (0,1)$, and $\alpha_0 \in (\min(s,q)/(\min(s,q)-1),\infty)$. Under some standard assumption, the following estimate hold for solution u of* (14)

$$\begin{aligned}\|Du\|_{L_{s,q}(Q_{1/2})} \le N(d,\nu,s,q,\alpha_0)\Big[\|f\|_{L_{s,q}(Q_1)} + \|g\|_{L_{s,q}(Q_1)}\Big] \\ + N(d,\nu,s,q,R_0,\alpha_0)\|u\|_{L_{s,q}(Q_1)}.\end{aligned} \tag{17}$$

Similar results on local boundary estimates are proved in [27].

Note that there is no required regularity assumption on the pressure p. Moreover, the coefficients a_{ij} can be unbounded. This is the first time that this kind of regularity estimate is established for the time-dependent Stokes system with measurable coefficient. Even in the case $p = s = 2$, this result is already new and it is considered as Caccioppoli's type estimates.

The method for proving Theorem (1) is based on perturbation using equations with coefficients frozen in the spatial variable and sharp function technique introduced in [32]. As we already mentioned, even when $s = q = 2$, the estimates (17) is not available to start the perturbation process. Our main idea to overcome this is to use the equations of vorticity, which is in the spirit of Serrin [61]. Therefore, we need to derive several necessary estimates for the vorticity, and then, use the divergence equation and these estimates to derive desired estimates for the solutions.

The application of Theorem 1 is even more fascinating than the theorem itself. To see this, let us consider the Navier-Stokes equations

$$u_t - \Delta u + (u\cdot\nabla)u + \nabla p = 0, \quad \mathrm{div} u = 0 \quad \text{in} \quad Q_1. \tag{18}$$

Let $u = (u_1, u_2, \cdots, u_d)$ be a Leray-Hopf weak solution of (18) in Q_1. For each $i,j = 1,2,\ldots,d$, let d_{ij} be the solution of the equation

$$\left\{\begin{array}{rcll} \Delta d_{ij} & = & D_j u_i - D_i u_j & \text{in} \quad B_1 \\ d_{ij} & = & 0 & \text{on} \quad \partial B_1. \end{array}\right. \tag{19}$$

Observe that for a.e. $t \in (-1,0)$, we have $u(t,\cdot) \in L^2(B_1)$. Therefore, the existence and uniqueness of $d_{ij}(t,\cdot) \in W_0^{1,2}(B_1)$ follows, and the solution $d_{ij}(t,\cdot)$ satisfies the standard energy estimate.

Let $[d_{ij}]_{B_\rho(x_0)}(t)$ be the average of d_{ij} with respect to x on $B_\rho(x_0)$. As a corollary of Theorem 1, we obtain the following new ϵ-regularity criterion for the Navier-Stokes equation (18).

Theorem 2. *Let $\alpha_0 \in (2(d+2)/(d+4), \infty)$. There exists $\epsilon \in (0,1)$ sufficiently small depending only on the dimension d and α_0 such that, if u is a Leray-Hopf weak solution of* (18) *in Q_1 and*

$$\sup_{z_0 \in Q_{2/3}} \sup_{\rho \in (0,R_0)} \left(\frac{1}{|Q_\rho(z_0)|} \int_{Q_\rho(z_0)} |d_{ij}(t,x) - [d_{ij}]_{B_\rho(x_0)}(t)|^{\alpha_0} \, dx \, dt \right)^{1/\alpha_0} \le \epsilon, \tag{20}$$

for every $i, j = 1, 2, \ldots, d$ and for some $R_0 \in (0, 1/2)$ and with d_{ij} defined in (19)*, then u is smooth in $Q_{1/2}$.*

We note that the parameter α_0 in the above theorem can be less than 2, which might be useful in applications. We would like to note that many other ϵ-regularity criteria for solutions to the Navier-Stokes equations were established, for instance, in [5,24]. See also [63, Chapter 6] for further discussion on this. To the best of our knowledge, compared to these known criteria, our result in Theorem 2 is completely new.

A key ingredient in the proof of Theorem 2 is our novel *solution decomposition technique*. Essentially, with a simple calculation, the Navier-Stokes system of equation (18) can be written as the following Stokes system of equations with singular coefficient matrix

$$u_t - D_i[(I_d + d_{ij})D_j u] + \nabla p = \operatorname{div} f \quad \text{in} \quad Q_1, \tag{21}$$

where I_d is the $d \times d$ identity matrix, and $f_{jk} = h_j u_k$ with some harmonic bounded function $h_j(\cdot, t)$. From this, Theorem 2 follows from our newly developed regularity Theorem 1 and a bootstrap argument.

I now conclude the statement with a discussion on an immediate consequence of Theorem 2. The following regularity criteria for weak solutions to the Navier-Stokes equations are proved in my paper [18], which implies Serrin's regularity criterion in the borderline case established by Fabes-Jones-Rivière [19] and by Struwe [64].

Corollary 3. *Assume that u is a Leray-Hopf weak solution of* (18) *in Q_1.*

(i) Let $s, q \in (1, \infty]$ be such that $2/s + d/q = 1$. Suppose that $u \in L_s((-1,0); L_q^w(B_1))$ when $s < \infty$, or the $L_\infty((-1,0); L_d^w(B_1))$ norm of u is sufficiently small. Then, u is smooth in $Q_{1/2}$.

(ii) Let $s, q \in (1, \infty]$ be such that $2/s + d/q = 1$. Suppose that $u \in L_s^w((-1,0); L_q^w(B_1))$ with a sufficiently small norm. Then, u is smooth in $Q_{1/2}$.

(iii) Let $\alpha \in [0,1)$, $\beta \in [0,d)$, and $s, q \in (1, \infty)$ be constants satisfying

$$\frac{2\alpha}{s} + \frac{\beta}{q} = \frac{2}{s} + \frac{d}{q} - 1 (> 0), \quad \frac{1}{s} < \frac{1}{2} + \frac{1}{d+2}, \quad \textit{and} \quad \frac{1}{q} < \frac{1}{2} + \frac{1}{d+2} + \frac{1}{d}. \tag{22}$$

Suppose that $u \in \mathscr{M}_{s,\alpha}((-1,0); \mathscr{M}_{q,\beta}(B_1))$ with a sufficiently small norm. Then, u is smooth in $Q_{1/2}$.

Here L_q^w denotes the weak-L_s space, and $\mathscr{M}_{q,\beta}$ denotes the Morrey space

$$\|f\|_{\mathscr{M}_{q,\beta}(B_1)} := \left(\sup_{x_0 \in \overline{B_1},\, r > 0} r^{-\beta} \int_{B_r(x_0) \cap B_1} |f|^q \, dx \right)^{1/q}.$$

Notice that in particular, when $d = 3$, Corollary 3 (i) recovers a result by Kozono [31]. When $d = 3$ and $q < \infty$, Corollary 3 (ii) was obtained in [28]. Our approach only uses linear estimates and is very different from these in [28,31]. It is also worth mentioning that we can take $q > 1$ and $s > 10/7$ in Corollary 3 (iii) in the case when $d = 3$.

c) *Other results.*

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