Asymptotics near extinction for nonlinear fast diffusion on a bounded domain

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Outline

Nonlinear diffusion: basic question

Rate (and corrections) at which the nonlinear diffusion equation

$$\frac{\partial \rho}{\partial \tau} = \frac{1}{m}\Delta(\rho^m) \qquad \text{in } \Omega \subset\subset \mathbf{R}^n \text{ open and bounded}$$

$$\rho = 0 \qquad \text{on } (0,\infty) \times \partial\Omega \in C^\infty$$

$$0 \leq \rho = \rho_0 \in L^1(\Omega) \qquad \text{on } \{\tau = 0\} \times \Omega$$

transports heat from Ω to the sink at its boundary $\partial\Omega$?

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Three regimes:

(PM) Porous medium: $m \in]1, \infty[$
(FD) Sobolev subcritical fast diffusion: $0 < m \in]\frac{n-2}{n+2}, 1[$
(FD') Sobolev supercritical fast diffusion: $m \in]-\infty, \frac{n-2}{n+2}[$

Limiting cases: linear heat equation $m = 1$
Sobolev critical diffusion $m = \frac{n-2}{n+2}$

How does this work for the linear heat equation $m = 1$?

Recall: separation of variables yields

$$\rho(\tau, y) = \sum_{i=1}^{\infty} c_i e^{-\lambda_i \tau} \phi_i(y)$$

where

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots,$$

and $c_i = \langle \rho_0, \phi_i \rangle_{L^2}$ where $\{\phi_i\}_{i=1}^{\infty} \subset H_0^1(\Omega)$ for

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$$H_0^1(\Omega) = \{\phi \in L^2(\Omega) \mid D\phi \in L^2(\Omega) \text{ and } \phi = 0 \text{ on } \partial\Omega\}$$

solve

$$-\Delta \phi_i = \lambda_i \phi_i \qquad \text{on } \Omega$$

and form an orthonormal basis for $L^2(\Omega)$

Do the nonlinear dynamics admit a similar description?

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POROUS MEDIUM REGIME ($m > 1$)

- fluid in rock; population spreading; temperature dependent conductivity
- rate of diffusion ρ^{m-1} varies directly with density ρ of diffusing material
- compactly supported ρ_0 remains compactly supported at $\tau > 0$

Motivation: dissipative fluids

$$(\frac{\partial}{\partial t} + u \cdot \nabla)(\rho u) = -\nabla P(\rho) - bu \tag{1}$$

- if drag negligible ($b \ll 1$), (1) couples with continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \tag{2}$$

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- if drag dominates ($b \gg 1$), neglect inertial terms in (1); then (2) yields

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- polytropic equation of state $P(\rho) = \frac{b}{m-1}\rho^m$ gives nonlinear diffusion (3)

Subcritical fast diffusion regime ($\frac{n-2}{n+2} < m < 1$)

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- $m = 1/2$ used to model plasma ions diffusing across a $\vec{B}$ field

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- Rescale (Berryman-Holland '78-'80): if $p = 1/m$ then

$$v(t,x)^p = \frac{\rho(\tau,x)}{\frac{1-m}{m}(T-\tau)^{\frac{1}{1-m}}} \quad \text{and} \quad t = -\frac{1-m}{m}\log|1-\frac{\tau}{T}|$$

satisfy

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$$\frac{\partial}{\partial t}\left(\frac{v^p}{p}\right) = \Delta v + v^p = \frac{\delta E}{\delta v} \qquad \text{in } (t,x) \in (0,\infty) \times \Omega$$

where

$$E(v) := \int_\Omega \left[\frac{1}{2}|\nabla v|^2 - \frac{1}{p+1} v^{p+1}\right] dx$$

is a Lyapunov function for the rescaled dynamics.

- Berger '77: $\frac{\delta E}{\delta v} = 0$ has positive solutions in $H^1_0(\Omega)$ 'ground states'
- Berryman-Holland '80: $v(t) \to$ ground state as $t \to \infty$ along a subsequence; conjectured limit unique & higher-order asymptotics
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- Feireisl-Simondon '00: $\lim_{t\to\infty} v(t) = V$, but depends on $v(0)$
- Bonforte-Grillo-Vazquez '12: $\left\|\frac{v(t)}{V} - 1\right\|_\infty \to 0$ (rate if $m \sim 1$)
- Jin-Xiong '20+: $\left\|\frac{v(t)}{V} - 1\right\|_\infty \le \frac{C}{t^\sigma}$ for some $C, \sigma > 0$ if $m \in [\frac{n-2}{n+2}, 1[$
- Bonforte-Figalli '21: $\left\|\frac{v(t)}{V} - 1\right\|_\infty \le Ce^{-\lambda t}$, where the spectral gap $\lambda > 0$ for an open $C^{2,\alpha}$ dense set of domains Ω, including the ball
- Akagi '21+ energetic (rather than entropic) proof

Linearization

relative error

$$h(t) := \frac{v(t)}{V} - 1$$

satisfies

$$\frac{\partial h}{\partial t} + L_V h = N(h) = M_V(h)$$

where

$$\begin{aligned} L_V h &= -\frac{1}{V}\Delta(hV) - ph \\ &= -V^{-p-1}\nabla \cdot (V^2 \nabla h) - (p-1)h \\ &\geq (1-p)h \\ N(h) &= (1+h)^p - 1 - ph - ((1+h)^p - 1)p\frac{\partial h}{\partial t} \\ M_V(h) &= \frac{1}{(1+h)^{p-1}}[(1+h)^p - 1 - ph + ((1+h)^p - 1)L_V h] \end{aligned}$$

Diagonalization (Bonforte-Figalli '21)

$$\|f\|_{L^q_r} := \left(\int_\Omega |f(x)|^q V(x)^r dx\right)^{1/q}$$

$$\langle f, g\rangle_r := \langle f, g\rangle_{L^2_r} = \int_\Omega fg V(x)^r dx$$

implies L_V is self-adjoint on L^2_{p+1} and has a complete basis of eigenvectors, which are critical points for the restriction of the weighted Dirichlet energy

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$$Q_V(\phi) := \|D\phi\|^2_{L^2_2} = \int_\Omega |D\phi|^2 V^2 dx$$

to the L^2_{p+1} unit-sphere, with the boundary trace of ϕV vanishing.

Proof: $\tilde{L} := V \circ (L_V + pI) \circ V^{-1} \geq I$ has compact inverse on L^2_{p-1} ... □

Denote the eigenvalues by

$$\lambda_{-I} < \lambda_{-I+1} \leq \cdots \leq \lambda_{-3} \leq \lambda_{-2} \leq \cdots \leq \lambda_{K-1} \leq 0 < \lambda_K \leq \lambda_{K+1} \leq \cdots$$

where $I \geq 1$ counts the number of unstable modes, $K \geq 0$ the zero modes (if any), and $\lambda_K > 0$ is the first positive eigenvalue or 'spectral gap'

Note $L_V + (p-1) \geq 0$ and $L_V 1 = (1-p)1$ imply $\lambda_{-I} = 1-p$ and simple; this corresponds to time translation symmetry in the original variables.

Bonforte-Figalli '21's exponential convergence rate $\lambda = \lambda_K$ follows from the fact that the unstable modes are suppressed (Feireisl-Simondon '00), while $C^{2,\alpha}$-generic domains admit no zero modes (Saut-Teman '79).

Can we (a) close the gap between Bonforte-Figalli '21's exponential and Jin-Xiong '20's algebraic rate of convergence, and/or
(b) access higher asymptotics conjectured (for $n = 1$) by Berryman-Holland '80?
(c) When do zero modes spoil exponential convergence?

First dichotomy

Theorem (Choi-M.-Seis)

Fix $\Omega \subset\subset \mathbf{R}^n$ *bounded with* $\partial\Omega \in C^\infty$ *and* $0 < m \in]\frac{n-2}{n+2}, 1[$.
If $0 \le v \in L^\infty([0,\infty[\times\Omega)$ *solves dynamics and* $h(t) := \frac{v(t)}{V} - 1 \to 0$ *uniformly, there exist* ϵ, $C(p, V)$ *and* $\lambda \ge \lambda_K$ *such that* $\|h\|_{L^\infty(\mathbf{R}_+\times\Omega)} \le \epsilon$ *implies either*

$$C\|h(t)\|_{L^\infty} \ge \|h(t)\|_{L^2_{p+1}} \ge \frac{1}{Ct} \qquad \forall t \gg 1 \tag{4}$$

$$\text{or} \quad \frac{1}{C}\|h(t)\|_{L^2_{p+1}} \le \|h(t)\|_{L^\infty} \le Ce^{-\lambda t}\|h(0)\|_{L^2_{p+1}} \qquad \forall t \ge 1. \tag{5}$$

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If (5) *holds and* $2\lambda \in]\lambda_J, \lambda_{J+1}]$, *then there exist* $c_i = c_i(h(0)) \in \mathbf{R}$ *and* $\tilde{C} = \tilde{C}(V,p,\lambda,C)$ *such that the eigenfunctions* $L_V\phi_i = \lambda_i\phi_i \in L^2_{p+1}(\Omega)$ *yield*

$$\left\| h(t) - \sum_{i=K}^{J} c_i e^{-\lambda_i t}\phi_i \right\|_{L^2_{p+1}} \le \tilde{C}\|h(0)\|_{L^2_{p+1}} t e^{-2\lambda t}$$

Let $S \subset H_0^1(\Omega)$ denote the set of fixed points $V \geq 0$ of the rescaled dynamics. Then $S \subset C^{3,\alpha}(\Omega)$ and $V, W \in S$ implies $V/W \in L^\infty$. Topologize S using the relatively uniform 'balls'

$$B_r(V) := \{W \in S \mid \|\frac{W}{V} - 1\|_\infty < r\}$$

as a base. Call $V \in S$ an *ordinary limit* iff S forms a *manifold* of dimension $dim(S) = K := dim(KerL_V)$ near V, which the error relative to V embeds *differentiably* into $L^2_{p+1}(\Omega)$.

Theorem (Second dichotomy)

Under the hypotheses of the preceding theorem, convergence is exponentially fast if V is an ordinary limit.

Remark: All tangent vectors to the embedding of S at V lie in $KerL_V$. Conversely, if V is an ordinary limit, then each $u \in KerL_V$ is tangent to the embedding of S. In the latter case the kernel is said to be *integrable*, a notion exploited by Allard-Almgren '81 and Simon '85 for related purposes in the context of minimal surfaces and geometric evolution equations.

We expect ordinary limits to be in some sense generic in S.

Some history for $\Omega = \mathbf{R}^n$ with $m_q = 1 - \frac{2}{n+q}$

For compactly supported non-negative initial data with $\|\rho_0\|_1 = \|\tilde{\rho}_0\|_1$

Friedman-Kamin '80 $m > m_0 \Rightarrow \|\rho(\tau,\cdot) - \tilde{\rho}(\tau,\cdot)\|_{L^1} = o(1)$ as $\tau \to \infty$

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Carrillo-Toscani '99, Dolbeault-del Pino '00, Otto '01

$m = m_q > m_n$ implies $\|\rho(\tau,\cdot) - \tilde{\rho}(\tau,\cdot)\|_{L^1} = O(\tau^{-\frac{1}{2}(1+\frac{n}{q})})$ as $\tau \to \infty$

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for some translate of $\tilde\rho_0$, and higher-order asymptotics

Some history for $\Omega = \mathbf{R}^n$ with $m_q = 1 - \frac{2}{n+q}$

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Angenent '88 invariant manifolds $m \geq 1 = n$; Koch '99, Seis '15+ $n \geq 1$

A (finite dimensional) dynamical systems approach

$$x'(t) = -F(x(t)) \in \mathbf{R}^n \quad \text{with} \quad x(0) = x_0$$

LINEARIZE around fixed point $F(x_\infty) = 0$ to get:

$$(x(t) - x_\infty)' = -DF(x_\infty)(x(t) - x_\infty)' + O(x(t) - x_\infty)^2$$

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If $\sigma(DF(x_\infty)) = \{0 < \lambda_1 \leq \cdots \leq \lambda_n\}$ with eigenvectors $\hat{\phi}_i$, as $t \to \infty$ expect

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- to simplify, only strive for asymptotics to order $O(e^{-2\lambda_1 t})$
- *differentiability* of $F(x_0)$ or $x(t) = X(t, x_0)$ wrt $x_0 \in \mathbf{R}^n$ was crucial

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using $|N(v(t))| \leq |v(t)| \|\frac{\partial}{\partial t} v(t))\|_{L^\infty}$ to reduce to one of the model cases

$$\dot{a}(t) = -Ca^2 \quad \text{so that} \quad a(t) = \frac{1}{Ct + a(0)^{-1}}$$

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or

$$\dot{x}_i = -\lambda_i x_i + N_i(x), \qquad x = (x_1, \ldots, x_n) \in \mathbf{R}^n$$

where $N(x) \leq k|x|^2$ if $|x| \leq \epsilon^2$, and $\lambda_i \geq \lambda > 0$. In this case Gronwall implies $f(t) = |x(t)|^2$ satisfies

$$f(T) \leq f(0) \exp[-2(\lambda - k\epsilon) T],$$

and then, if $f(t) \leq C^2 \exp[-2\Lambda t]$ for some $\Lambda > 0$, C and all $t \geq 0$,

$$|x(T)| \leq |x(0)| \exp[-\lambda T + \frac{C}{2\Lambda}]$$

Merle-Zaag '98 (Filippas-Kohn) dominant balance variant

Lemma (K. Choi-Haslhofer-Hershkovits '18+)

Let $X(s)$, $Y(s)$, and $Z(s)$ be non-negative AC functions on $[0, \infty)$ satisfying

$$\begin{aligned} \frac{dX}{ds} - X &\geq -\epsilon(Y + Z), \\ \left|\frac{dY}{ds}\right| &\leq \quad \epsilon(X + Y + Z), \text{and} \\ \frac{dZ}{ds} + Z &\leq \quad \epsilon(X + Y) \end{aligned}$$

for each $\epsilon \in (0, \frac{1}{100})$ and a.e. $s \in [s_0(\epsilon), \infty)$.

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for each $\epsilon \in (0, \frac{1}{100})$ and a.e. $s \in [s_0(\epsilon), \infty)$. If $\lim_{s\to\infty}(X+Y+Z)(s) = 0$ then $X \leq 2\epsilon(Y+Z)$ for $s \geq s_0(\epsilon)$ and either

$$X(s) + Z(s) = o(Y(s)) \text{ as } s \to \infty \tag{6}$$

or

$$X(s) + Y(s) \leq 100\epsilon Z(s) \text{ for } s \geq s_0(\epsilon). \tag{7}$$

Quadratic Hilbert-space estimate for the nonlinearity:

First dichotomy: apply the lemma to $X(t)$, $Y(t)$ and $Z(t)$ defined as the Hilbert norm $L^2_{p+1} := L^2(V^{p+1})$ of the orthogonal projection of $h(t)$ onto the unstable, neutral, and stable modes respectively. To absorb the nonlinearity into the ϵ corrections, apply the following theorem with $t = 1$.

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Theorem (Spatially uniform control of time derivatives)

Let $k \in \{0, 1, 2, \ldots\}$ and $t > 0$ fixed. Then if $\|h\|_{L^\infty} \leq \epsilon$ with ϵ sufficiently small, there exists a constant $C = C(t, k, m, V)$ such that

$$\|\partial_t^k h(t)\|_{L^\infty} \leq C \|h_0\|_{L^2_{p+1}}.$$

Proof: degenerate parabolic smoothing, with delicate control near the boundary of Ω where $V(x) \sim d_{\partial\Omega}(x)$. □

c.f. Jin-Xiong '19+

Among other ingredients, second dichotomy relies on

Lemma (K. Choi-Sun '20+)

Suppose $X(s)$, $Y(s)$, and $Z(s)$ are non-negative absolutely continuous functions on some interval $[-L, L]$ such that $0 < X + Y + Z < \eta$ for some $\eta > 0$. Suppose that there exist two constants $\sigma > 0$ and $\Lambda > 0$ such that

$$\frac{dX}{ds} - \Lambda X \geq -\sigma(Y + Z),$$

$$\left|\frac{dY}{ds}\right| \leq \sigma(X + Y + Z),$$

$$\frac{dZ}{ds} + \Lambda Z \leq \sigma(X + Y),$$

for any $s \in [-L, L]$. Then there exists $\sigma_0 = \sigma_0(\Lambda)$ such that if $0 < \sigma < \sigma_0$ it holds

$$X + Z \leq \frac{8\sigma}{\Lambda} Y + 4\eta e^{-\frac{\Lambda L}{4}} \text{ for any } s \in [-L/2, L/2].$$

Thank you!