Einstein meets Hausdorff, Kantorovich, and Boltmzann

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www.math.toronto.edu/mccann/Talk2.pdf

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Two famous sign laws in physics

- gravity is always attractive, never repulsive
- entropy always goes up, never down

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• Bekenstein '73: 2nd law of black hole dynamics
area of horizons can only increase

• Jacobson '95: Einstein's equation follows from Entropy := Horizon Area

• E Verlinde '11: Gravity as an emergent entropic (i.e. statistical) force

Today I'll describe a connection between gravity and entropy using optimal transport (M. '20) (Mondino-Suhr '22)
which allows one to build a nonsmooth theory of gravity
(Kunzinger-Sämann '18) (Cavalletti-Mondino '20+) (M.-Sämann '21+)

Gravity

Newton (1687)

Non-negativity of classical mass implies gravity acts purely attractively

$$F = m_{inertial} a = -m_{gravitational} DV \quad \text{where} \quad \Delta V = 4\pi\rho \geq 0$$

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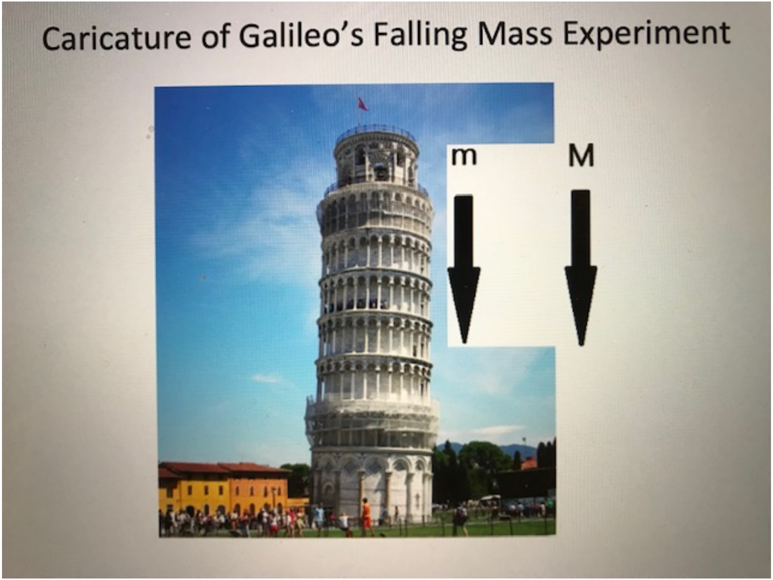
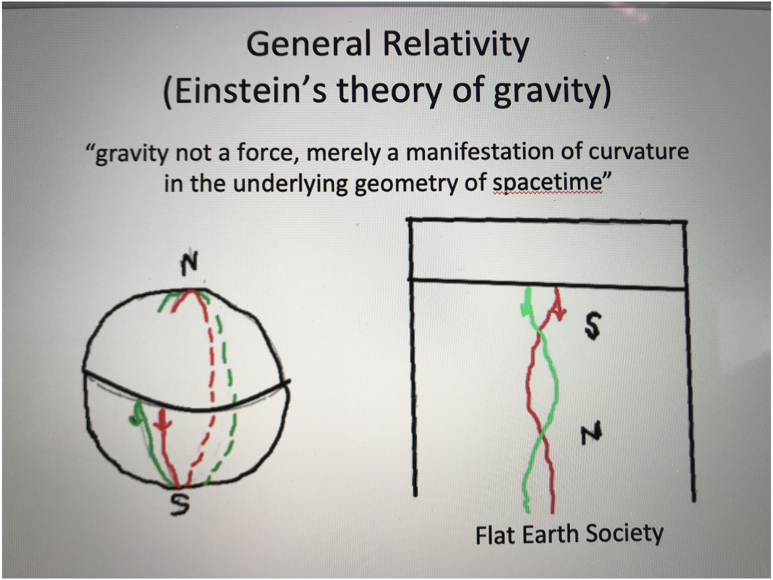
- the perihelion precession of Mercury, and

- $m_{inertial} = m_{gravitational}$


Caricature of Galileo's Falling Mass Experiment
m
M


General Relativity
(Einstein's theory of gravity)
"gravity not a force, merely a manifestation of curvature
in the underlying geometry of spacetime"
N
S
S
N
Flat Earth Society

Einstein's Tensor Equation

"geometry = physics"

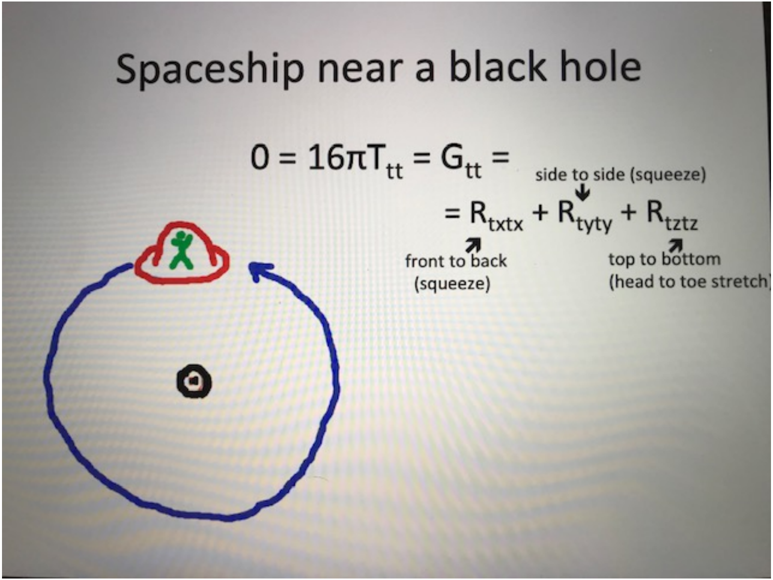
$$G_{ab} = 8\pi T_{ab}$$

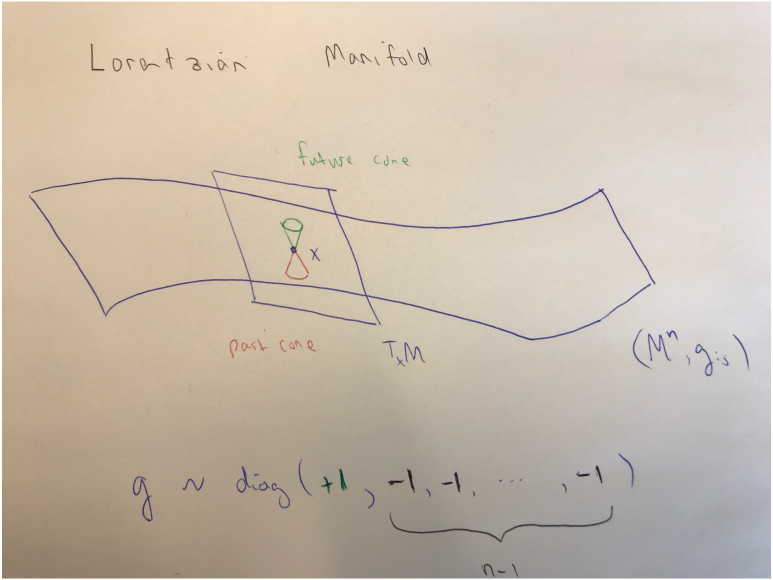
average sectional curvature in a given direction MINUS (a multiple of) the same quantity averaged over all directions

energy and momentum fluxes of matter in system

a,b = t, x, y, z

Signature (i.e. dimensions) of space+time = 3+1


Spaceship near a black hole
$0 = 16\pi T_{tt} = G_{tt} =$
$= R_{txtx} + R_{tyty} + R_{tztz}$
side to side (squeeze)
front to back
(squeeze)
top to bottom
(head to toe stretch)


Lorentzian Manifold
future cone
x
past cone
T_xM
(M^n, g_{ij})
$g \sim \mathrm{diag}(+1, \underbrace{-1, -1, \cdots, -1}_{n-1})$

Terminology and conventions

$0 \neq v \in T_x M$ is

(a) timelike if $g(v, v) > 0$
(b) lightlike (or *null*) if " $= 0$
(c) spacelike if " < 0
(d) causal if (a) or (b) hold, in which case
(e) *future-directed* if it lies in the green cone
(f) *past-directed* if it lies in the red cone

A C^1 curve $\sigma : (a, b) \to M$ is said to have the property (a-f) if each of its tangent vectors does.

Particles with mass follow timelike future-directed curves on M.

Positive Energy Conditions of Hawking and Penrose '70

Weak energy condition: $G_{ij}v^i v^j \geq 0$ for all timelike $(v, x) \in TM$ (believed to be satisfied in all physical geometries)

Strong energy condition: $R_{ij}v^i v^j \geq 0$ for all timelike $(v, x) \in TM$, where

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$$

here R_{ij} is the *Ricci curvature* tensor and $R = g^{ij}R_{ij}$ is its *trace*.

- less universally satisfied
- does not imply weak energy condition
- implies gravity is attractive
- was used by Hawking and Penrose to show "trapped" spacelike surfaces (whose areas decrease instantaneously in all possible futures) imply singularities

We'll assume *global hyperbolicity* of (M, g), meaning

- M is smooth, connected, Hausdorff, and g is time-orientable
- has no closed future-directed curves (i.e. no 'back to the future')
- $J^+(x) \cap J^-(y)$ is compact for all $x, y \in M$, where

$J^+(x)$ is the set of points reached from x along future-directed curves
$J^-(y)$ is the set of points reached from y along past-directed curves

On $(M, \tilde{g})$ Riemannian, for $p > 1$

$$d(x,y)^p := \inf_{\sigma(0)=x, \sigma(1)=y} \int_0^1 (\tilde{g}_{ij} \dot{\sigma}^i \dot{\sigma}^j)^{p/2} dt$$

is attained if M is complete, and σ attains it iff $\sigma \in Geo_d(M)$, where

$$Geo_d(M) := \{\sigma : [0,1] \longrightarrow M \mid d(\sigma(s), \sigma(t)) = (t-s) d(\sigma(0), \sigma(1)) \forall s < t\}.$$

RECALL: On (M, g) Lorentzian, for $q < 1$ the Lorentz distance (or time separation)

$$\ell(x, y)^q := \sup_{\substack{\sigma(0)=x,\sigma(1)=y \\ future\ \ directed}} \int_0^1 (g_{ij}\dot{\sigma}^i\dot{\sigma}^j)^{q/2} dt$$

is attained if M is globally hyperbolic and $\ell(x, y) > 0$. In this case σ attains it iff $\sigma \in Geo_\ell(M)$, where

$$Geo_\ell(M) := \{\sigma : [0, 1] \longrightarrow M \mid \ell(\sigma(s), \sigma(t)) = (t-s)\ell(\sigma(0), \sigma(1)) \forall s < t\}.$$

The Lorentz distance is independent of q and satisfies a *backwards* triangle inequality:

$$\ell(x,z) + \ell(z,y) \leq \ell(x,y) \qquad \forall x,y,z \in M.$$

it denotes the maximum a particle can age between x and y (twin paradox!)

Throughout we adopt the conventions

$$(-\infty)^q := -\infty =: (-\infty)^{1/q}$$

and

$$\ell(x,y) = -\infty$$

if no future-directed Lipschitz curve connects x to y.

In the Riemannian case, given unit-length geodesics $\sigma, \tau \in Geo_d(M)$ through a common point $\sigma(0) = \tau(0)$, a local Taylor expansion yields

$$\begin{aligned} d^2(\sigma(s), \tau(t)) &= s^2 + t^2 - 2st\tilde{g}(\dot{\sigma}(0), \dot{\tau}(0)) \\ &\quad - \frac{s^2 t^2}{6} \tilde{R}_{ijkl} \dot{\sigma}^i \dot{\tau}^j \dot{\sigma}^k(0) \dot{\tau}^l(0) + O(|s|^5 + |t|^5) \end{aligned}$$

where $\tilde{R}_{ijkl}$ is the Riemannian curvature tensor. It measures the leading correction to Pythagoras' law, and also

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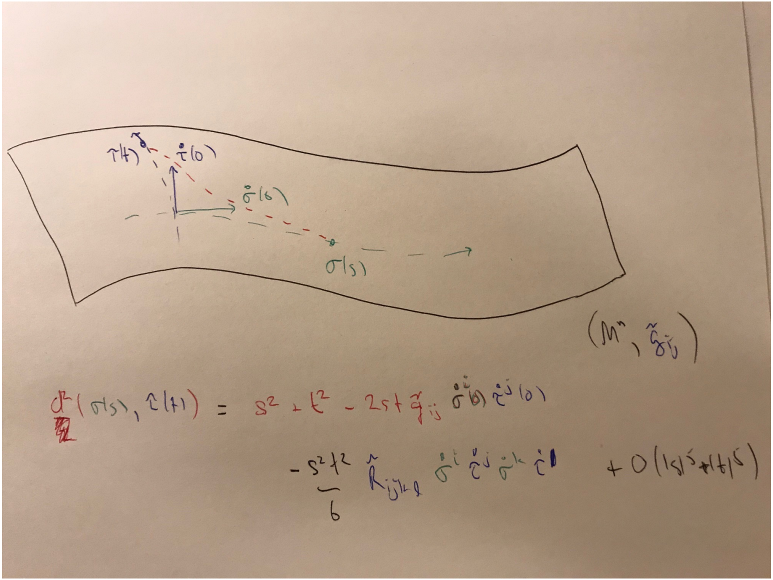
where $\tilde{R}_{ijkl}$ is the Riemannian curvature tensor. It measures the leading correction to Pythagoras' law, and also the failure of covariant derivatives wrt $\tilde{g}$'s Levi-Civita connection ($\tilde{\nabla}_i \tilde{g}_{jk} = 0$) to commute:

$$\tilde{R}_{ijkl} v^k = -[\tilde{\nabla}_i, \tilde{\nabla}_j] v^l$$

Its trace $\tilde{R}_{ik} := \tilde{g}^{jl} \tilde{R}_{ijkl}$ gives the Ricci tensor associated to $\tilde{g}_{ij}$.

The analogous formulas (with ℓ replacing d and the tildes removed) hold in the Lorentzian case.


$\tau(t)$
$\dot\tau(0)$
$\dot\sigma(0)$
$\sigma(s)$
$(M^n, \tilde g_{ij})$
$d^2(\sigma(s), \tau(t)) = s^2 + t^2 - 2st\, \tilde g_{ij}\, \dot\sigma^i(0)\, \dot\tau^j(0) - \frac{s^2 t^2}{6} \tilde R_{ijkl}\, \dot\sigma^i \dot\tau^j \dot\sigma^k \dot\tau^l + O(|s|^5 + |t|^5)$

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In the Riemannian setting, a line of developments starting from
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This inspired Sturm '06, Lott and Villani '09 to adopt such convexity as the *definition* of lower Ricci bounds in a (non-smooth) metric-measure setting, leading to the blossoming study of *curvature-dimension spaces* $(X, d, m) \in CD(K, N)$ developed by Ambrosio, Gigli, Savare, Erbar, Kuwada, Sturm, ...

CD(K,N) and RCD(K,N) spaces

- Highlights of the theory include contraction results for diffusion semigroups (Ambrosio et al, Carrillo-M.-Villani, Otto, Sturm)
- Bonnet-Myers diameter bounds (Lott-V., Sturm)
- Splitting (Gigli) and rigidity (Ketterer) results
- Comparison theorems for isoperimetric profiles (Cavalletti-Mondino, Milman)
- Rectifiability (Mondino-Naber)
- and presumably much more to come

Can something similar be done in the Lorentzian setting?

A function $\ell : X^2 \longrightarrow \{-\infty\} \cup [0, \infty)$ on a (complete, separable) metric space (X, d) is called a time-separation or Lorentz-distance

$$\begin{aligned} \ell(x,y) + \ell(y,z) \leq& \ell(x,z) && \forall x, y, z \in X \\ \ell(x,x) =& 0 && \forall x \in X \\ \ell(x,y) \geq 0 \Rightarrow& \ell(y,x) = -\infty && \text{unless } y = x \end{aligned}$$
$$\ell \text{ is continuous on the closed set} \{\ell \geq 0\}$$

A curve $\sigma : [0,1] \longrightarrow M$ is timelike (respectively causal) if $0 \leq s < t \leq 1$ implies $\ell(\sigma(s), \sigma(t)) > 0$ (respectively ≥ 0)
(future-directed by convention).

(X, d, ℓ) is a *Lorentzian geodesic space* if $\ell(x, y) > 0$ implies the existence of a (Lipschitz) curve $\sigma \in Geo_\ell(X)$ with $\sigma(x) = 0$ and $\sigma(1) = y$ where

$$Geo_\ell(X) = \{\sigma : [0, 1] \longrightarrow M \mid \ell(\sigma(s), \sigma(t)) = (t - s)\ell(\sigma(0), \sigma(1)) \ \forall s < t\}$$

(X, d, ℓ) to be $\mathcal{K}$-globally hyperbolic, meaning no closed causal loops and compactness of $A, B \subset X$ implies compactness of $J(A, B) := J^+(A) \cap J^-(B)$ where

$$\begin{aligned} J^+(A) &= \bigcup_{a \in A} \ell(a, \cdot)^{-1}(\mathbb{R}) \\ J^-(B) &= \bigcup_{b \in B} \ell(\cdot, b)^{-1}(\mathbb{R}) \end{aligned}$$

are the causal future of A and past of B

Use Lorentz distance $\ell(x, y)$ to lift the geometry from M to the set $\mathcal{P}_c(M)$ of (compactly supported for simplicity) Borel probability measures on M: Given $0 < q \leq 1$ and $\mu_0, \mu_1 \in \mathcal{P}_c(M)$ define

$$\ell_q(\mu_0, \mu_1) := \left(\sup_{\gamma \in \Gamma(\mu_0, \mu_1)} \int_{M \times M} \ell(x, y)^q d\gamma(x, y) \right)^{1/q},$$

where the supremum is over joint measures $\gamma \geq 0$ on $M \times M$ having μ_0 and μ_1 as left and right marginals

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- this is a (Kantorovich '42) optimal transport problem with lower semicontinuous cost $-\ell^q$ whose gradient diverges as the boundary of the causal set $J^+ = \ell^{-1}([0, \infty))$ is approached, and which jumps to $+\infty$ outside J^+.

- still the supremum is attained by some γ which will be called ℓ^q-optimal (unless $\ell_q(\mu_0, \mu_1) = -\infty$).

A close variant of $\ell_q(\mu_0, \mu_1)$ was defined in Eckstein & Miller '17, who show ℓ_q inherits the reverse triangle inequality from $\ell(x, y)$:

$$\ell_q(\mu_0, \mu_1) \geq \ell_q(\mu_0, \nu) + \ell_q(\nu, \mu_1).$$

DEFN: We say $(\mu_s)_{s\in[0,1]}$ is an ℓ_q-geodesic in $\mathcal{P}_c(M)$ iff

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DEFN: We say $(\mu_s)_{s\in[0,1]}$ is an ℓ_q-geodesic in $\mathcal{P}_c(M)$ iff

$$\ell_q(\mu_s, \mu_t) = |t - s|\ell_q(\mu_0, \mu_1) > 0 \qquad \forall \quad 0 \leq s < t \leq 1.$$

- ℓ_q-geodesics exist, if $\ell > 0$ a.e. wrt a Kantorovich maximizer $\gamma \in \Gamma(\mu_0, \mu_1)$

- When $\ell > 0$ on $\mathrm{spt}[\mu_0 \times \mu_1]$ ($:=$ smallest closed set of full mass), so that μ_1 lies entirely in the timelike future of μ_0, one can characterize the q-geodesic joining them. In the smooth setting, its unique provided $\mu_0 \in \mathcal{P}_c^{ac}(M)$, meaning μ_0 is absolutely continuous wrt the Lorentzian volume

$$d\mathrm{vol}_g(x) \quad (:= |\det g_{ij}(x)|^{1/2} d^n x \text{ in coordinates}).$$

To prove this, used linear programming duality to analyze the optimal transportation problem defining

$$(*) \qquad \frac{1}{q}\ell_q(\mu,\nu)^q \leq \inf_{u \oplus v \geq \frac{1}{q}\ell^q} \int_M u\, d\mu + \int_M v\, d\nu,$$

Unfortunately, the singularities of ℓ may prevent attainment of this Kantorovich dual infimum by (lsc) potentials (u, v) satisfying

$$u(x) + v(y) \geq \frac{1}{q}\ell(x,y)^q \qquad \forall \quad (x,y) \in \mathrm{spt}[\mu \times \nu] =: X \times Y$$

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DEFN: Fix $q \in (0, 1]$. We say $(\mu, \nu) \in \mathcal{P}_c(M)$ are timelike q-dualizable if the infimum is finite (in which case equality holds in $*$) and there exists ℓ^q-maximizing $\gamma \in \Gamma(\mu, \nu)$ such that $\gamma[\{\ell > 0\}] = 1$.
This timelike q-dualizability is strong if there exists an ℓ^q_+-cyclically monotone $S \subset \{\ell > 0\} \cap \mathrm{spt}[\mu \times \nu]$ outside of which all ℓ^q-maximers $\gamma \in \Gamma(\mu, \nu)$ vanish. Here $\ell_+ = \max\{\ell, 0\}$ and cyclical monotonicity means

$$\sum_{i=1}^{k} \ell^q(x_i, y_i) \geq \sum_{i=1}^{k} \ell^q(x_i, y_{i+1(mod k)}) \qquad \forall k \in \mathbf{N} \text{ and } \{(x_i, y_i)\}_{i=1}^{k} \in S$$

To define a Ricci lower bound requires a Radon measure m on (X, d)

e.g. $dm(x) = e^{-V(x)} d\mathrm{vol}_g(x)$ with $V \in C^2(M)$ on a smooth spacetime

DEFN We define the *relative entropy* by

$$E_m(\mu) := \begin{cases} \int_M \rho \log \rho \, dm & \text{if } \mu \in \mathcal{P}_c^{ac}(M) \text{ and } \rho := \frac{d\mu}{dm}, \\ +\infty & \text{if } \mu \in \mathcal{P}_c(M) \setminus \mathcal{P}^{ac}(M). \end{cases}$$

- our sign convention is opposite to that of the physicists' entropy

Entropic weak timelike curvature-dimension conditions

DEF For $(K, N) \in \mathbb{R} \times [1, \infty]$ write $(X, d, \ell, m) \in wTCD_q^e(K, N)$ if and only if every strongly q-dualizable finite entropy pair $\mu_0, \mu_1 \in \mathcal{P}_c(M)$ admit an ℓ_q-maximizing γ generating an ℓ_q-geodesic $(\mu_t)_{t\in[0,1]}$ along which the entropy $t \in [0, 1] \mapsto e(t) := E_m(\mu_t)$ satisfies the (semi)convexity inequality

$$e''(t) \geq \frac{e'(t)^2}{N} + K\|\ell\|^2_{L^2(\gamma)}$$

distributionally.

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Cavalletti-Mondino '20+ go on to prove the set $wTCD_q^e(K,N)$ is closed in a suitable (pointed measured Gromov-Wasserstein) topology and its elements inherit remarkable similarities to smooth lower Ricci bounded spacetimes (such as an analog of the Hawking singularity theorem)

c.f. Burtscher-Ketterer-M.-Woolgar analogous sharp Riemannian injectivity bound

Positive energy = entropic displacement convexity

DEF (N-Bakry-Emery modified Ricci tensor; cf. Erbar-Kuwada-Sturm'15)
Given $N \in (n, \infty]$ and $V \in C^2(M)$ define

$$R_{ij}^{(N,V)} := R_{ij} + \nabla_i \nabla_j V - \frac{1}{N-n} (\nabla_i V)(\nabla_j V)$$

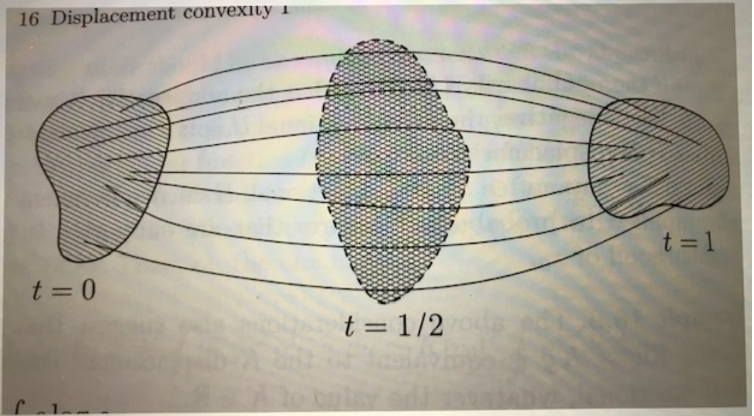
THM 1 (M. '20) Fix $(K, N, q) \in \mathbb{R} \times [1, \infty] \times (0, 1)$ and a globally hyperbolic spacetime (M^n, g) with $dm = e^{-V} d\mathrm{vol}_g$. Then $(M, d_{\tilde{g}}, \ell_g, m) \in wTCD_q^e(K, N)$ if and only if either

(a) $N = n$, $V = const$ and $R_{ij} v^i v^j \geq K$ for all unit timelike $(v, x) \in TM$,

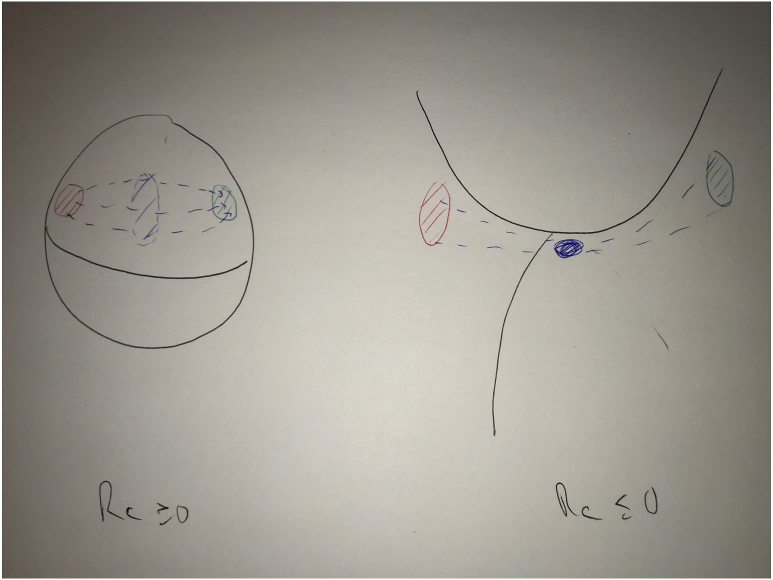
(b) $N > n$ and $R_{ij}^{(N,V)} v^i v^j \geq K$ for all unit timelike vectors $(v, x) \in TM$.

Mondino-Suhr '22 Can also use entropic convexity to say when equality holds, leading to a weak (but unstable) notion of solution to Einstein Field equations.

Lazy Gas Experiment (M. 94, Villani 09)



Action minimizing paths satisfy pressureless Euler equation.


Rc ≥ 0
Rc ≤ 0

A Lorentzian analog for Hausdorff dimension and measure ?

In metric geometry, Hausdorff dimension and measure play a central role:

$$\mathcal{H}^N_\delta(A) := \inf\{c_N \sum (diam A_i)^N \mid A \subset \cup A_i, diam A_i \leq \delta\}$$

makes $\mathcal{H}^N = \sup_{\delta>0} \mathcal{H}^N_\delta$ a Borel measure and

$$\begin{aligned} dim_{\mathcal{H}} A &= \inf\{N \geq 0 \mid \mathcal{H}^N(A) = 0\} \\ &= \sup\{N \geq 0 \mid \mathcal{H}^N(A) = \infty\} \end{aligned}$$

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DePhillipis-Gigli '18 call $(X, d, m) \in CD(K, N)$ non-collapsed if $m = \mathcal{H}^N$ (inspired by Colding-Cheeger's dichotomy for Ricci limit spaces)

Brue-Semola '19: $(X, d, m) \in RCD(K, N)$ implies $\exists k \in \{1, \ldots, N\}$ such that $m|_R \ll \mathcal{H}^k$ and $m(X \setminus R) = 0$.

LEMMA (M.-Sämann) Given (X, d, ℓ) and $A \subset X$ setting

$$\mathcal{V}_\delta^N(A) := \inf\{\omega_N \sum \ell(a_i, b_i)^N \geq 0 \mid A \subset \cup J(a_i, b_i), diamJ(a_i, b_i) \leq \delta\}$$

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$$(= \sup\{N \geq 0 \mid \mathcal{V}^N(A) = \infty\})$$

the geometric dimension of A (assumes $\lim_{d(x,y)\to 0} \ell(x, y) = 0$ locally uniformly; depends mostly on ℓ, mildly on d).

Already in Minkowski space $\mathbb{R}^n_1$, this geometric dimension dintinguishes spacelike from null subspaces $S \subset \mathbb{R}^n_1$.

If S is spacelike (i.e. Euclidean), then $\dim_{\mathcal{H}} S = \dim_\ell S$ and the nontrivial measures $\mathcal{H}^N$ and $\mathcal{V}^N$ are positive multiples of each other;

If the metric degenerates on S then $\dim_{\mathcal{H}} S = N = 1 + \dim_\ell S$ and $\mathcal{V}^{N-1}$ and $\mathcal{H}^0 \times \mathcal{H}^{N-1}$ are positive multiplies of each other.

If $(X, d, \ell, m) \in wTCD^e_q(K, N)$ one expects doubling properties to yield $N \geq \dim_\ell X$ (by analogy with Hausdorff dimension in the $(X, d, m) \in CD(K, N)$ case) but we were not able to prove this in general. However we were able to show this for continuous spacetimes (a case of mathematical physical interest):

THM (M.-SÄMANN) If $(X, d, \ell, \mathrm{vol}_g) \in wTCD^e_q(K, N)$ arises from a smooth spacetime $X = M^n$ with a merely continuous metric tensor g_{ij} (so timelike branching can occur), then at least $N + 1 \geq \dim_\ell X = n$. Moreover, if timelike branching does not occur then $N \geq \dim_\ell X = n$ as expected.

Conclusions: optimal transport relates gravity to entropy

1. Fractional powers $0 < q < 1$ of the time-separation $\ell(x, y)$ come from a Lagrangian L, smooth and strictly(!) convex away from the light cone.

2. Optimal transport with respect to this cost lifts the geometry from spacetime events M to probability measures on M.

3. strong timelike q-dualizability of the target and source makes this transportation problem and its dual analytically tractable.

4. Convexity properties of Boltzmann's entropy along timelike geodesics of probability measures provide a robust formulation of the strong energy condition of Hawking and Penrose '70 — and via Mondino & Suhr 18+'s parallel work, of Einstein's field equations.

5. This provides a new approach to gravity without smoothness — much desired in view of the singularity theorems from general relativity.

6. Whereas the second law of thermodynamics is encoded in the first time-derivative of entropy, the Einstein equations of gravity are encoded in its second time-derivative along q-geodesics.

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THANK YOU!

THM (Lagrangian characterization of q-geodesics)
Fix $0 < q < 1$. If $(\mu_0, \mu_1) \in \mathcal{P}_c^2(M)$ is q-separated by (γ, u, v) and $\mu_0 << \mathrm{vol}_g$ then the map $F_s(x) := \exp_x sDH(Du(x); q)$ induces the unique q-geodesic $s \in [0,1] \mapsto \mu_s$ in $\mathcal{P}_c(M)$ linking μ_0 to μ_1.
Moreover, $\mu_s << \mathrm{vol}_g$ if $s < 1$ (by using uniform convexity of L away from light cone to adapt Monge-Mather 'shortening' estimate).

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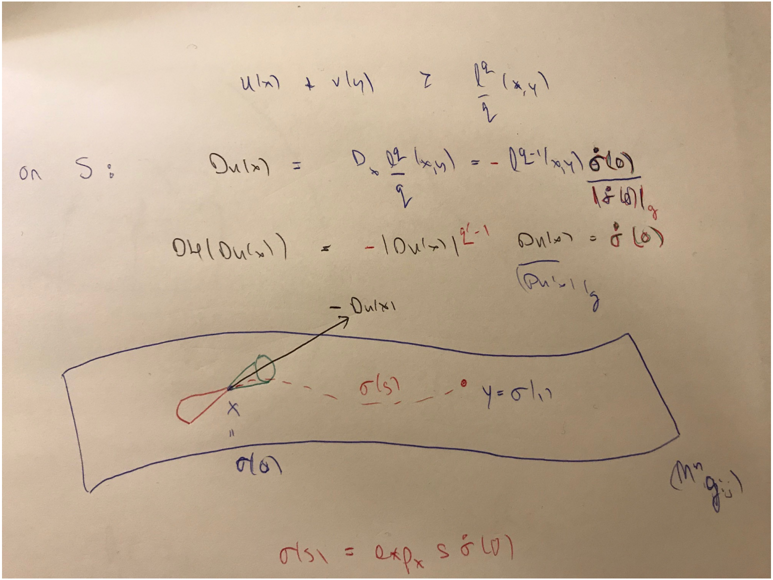
F_s is an optimal (i.e. Monge) map between μ_0 and μ_s, and $\gamma = (id \times F_1)_\# \mu_0$ uniquely maximizes the Kantorovich problem defining $\ell_q(\mu_0, \mu_1)$.

Setting $\rho_s := \frac{d\mu_s}{d\mathrm{vol}_g}$ yields the Monge-Ampère type equation

$$\rho_0(x) = \rho_s(F_s(x))|JF_s(x)| \qquad \rho_0 - a.e.,$$

where $JF_s(x) = \det D\tilde{F}_s(x)$ is the (approximate) Jacobian of F_s and

$$\frac{\partial}{\partial s}\Big|_{s=0}(\tilde{D}F_s) = D^2H|_{Du}\tilde{D}^2u.$$


$u(x) + v(y) \ge \frac{\ell^q}{q}(x,y)$
on S:
$Du(x) = D_x \frac{\ell^q}{q}(x,y) = -\ell^{q-1}(x,y) \frac{\dot\sigma(0)}{|\dot\sigma(0)|_g}$
$DH(Du(x)) = -|Du(x)|^{q'-1} \frac{Du(x)}{|Du(x)|_g} = \dot\sigma(0)$
$-Du(x)$
$\sigma(s)$
$y = \sigma(1)$
x
$=$
$\sigma(0)$
(M^n, g_{ij})
$\sigma(s) = \exp_x s\dot\sigma(0)$