EXISTENCE UNIQUENESS AND REGULARITY THEORY FOR ELLIPTIC EQUATIONS WITH COMPLEX-VALUED POTENTIALS

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Abstract. This paper studies second order elliptic equations in both divergence and non-divergence forms with measurable complex valued principle coefficients and measurable complex valued potentials. The PDE operators can be considered as generalized Schrödinger operators. Under some sufficient conditions, we establish existence, uniqueness, and regularity estimates in Sobolev spaces for solutions to the equations. We particularly show that the non-zero imaginary parts of the potentials are the main mechanisms that control slowly decaying solutions. Our results can be considered as limiting absorption principle for Schrödinger operators with measurable coefficients and they could be useful in other areas such as dispersive equations. The approach is based on the perturbation technique that freezes the potentials. The results of the paper not only generalize many known results but also provide a key ingredient for the study of L^p-diffusion phenomena for dissipative wave equations.

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1. Introduction and main results

Let $(a_{kl})_{n\times n} : \mathbb{R}^n \to \mathbb{C}^{n\times n}$ be a matrix of measurable complex valued functions that is uniformly elliptic and bounded. In particular, we assume that there exists a constant $\Lambda > 0$ such that

$$\begin{cases} \Lambda|\xi|^2 \leq \operatorname{Re} \displaystyle\sum_{k,l=1}^n a_{kl}(x)\xi_k\xi_l, \quad \forall\, \xi = (\xi_1, \xi_2, \ldots, \xi_k) \in \mathbb{R}^n \text{ and for a.e. } x \in \mathbb{R}^n, \\ |a_{kl}(x)| \leq \Lambda^{-1} \quad \text{and} \quad \operatorname{Im} a_{kl}(x) = \operatorname{Im} a_{lk}(x) \quad \text{for all } k, l = \{1, 2, \ldots, n\} \text{ and for a.e. } x \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

Also let $c : \mathbb{R}^n \to \mathbb{C}$ be a given measurable complex valued potential function. Motivated by the study of L^p-diffusion phenomena for dissipative wave equations in [30] and by the study of other related topics such as [1–3, 5, 7, 9, 11, 13–18, 21–23, 34], we are interested in studying the uniqueness solvability and regularity estimates in $W^{2,p}(\mathbb{R}^n, \mathbb{C})$ for solutions u of the equations

$$\mathcal{L}_\lambda u(x) = f(x) \quad \text{in} \quad \mathbb{R}^n, \tag{1.2}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a given measurable function, and $\mathcal{L}_\lambda$ is the generalized Schrödinger operator in non-divergence form with measurable coefficients defined by

$$\mathcal{L}_\lambda u(x) = -\sum_{k,l=1}^n a_{kl}(x) D_{kl} u(x) + \lambda c(x) u(x)$$

in which $\lambda > 0$ is a scaling parameter. Throughout the paper, for each $k, l = 1, 2, \ldots, n$, we denote $D_k = \partial_{x_k}$ the first order partial derivative operator with respect to the k^{th} -variable and $D_{kl} = \partial_{x_k x_l}$ the second order partial derivative operator with respect to the k^{th}-variable and l^{th} -variable.

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In this paper, we also investigate the uniqueness solvability and regularity in $W^{1,p}(\mathbb{R}^n, \mathbb{C})$ for the generalized divergence form Schrödinger equations

$$Q_\lambda u(x) = \lambda f(x) + \operatorname{div}[g(x)] \quad \text{in} \quad \mathbb{R}^n \tag{1.3}$$

for some given measurable function $f : \mathbb{R}^n \to \mathbb{C}$ and some given measurable vector field $g : \mathbb{R}^n \to \mathbb{C}^n$, where Q_λ is the generalized Schrödinger operator in divergence form with measurable coefficients defined by

$$Q_\lambda u(x) = -\sum_{k,l=1}^{n} D_l[a_{lk}(x) D_k u(x)] + \lambda c(x) u(x).$$

Different from other available work in the theory of $W^{2,p}$ and $W^{1,p}$-regularity estimates such as [6, 9, 10, 21–25] and also motivated by [2, 13–15, 17, 18, 30], this paper investigates the case in which the potential c is a measurable complex valued function. Throughout the paper, we write $c(x) = c_1(x) + ic_2(x)$ where $c_1, c_2 : \mathbb{R}^n \to \mathbb{R}$ are measurable functions. We assume that

$$c_1(x) \geq 0, \quad c_1(x) + c_2(x) \geq \alpha_0, \quad \text{and} \quad c_1(x) + |c_2(x)| \leq \Lambda^{-1}\alpha_0 \quad \text{for a.e.} \quad x \in \mathbb{R}^n, \tag{1.4}$$

and for a fixed number $\alpha_0 > 0$.

To state the results, let us introduce some notation. For a measurable function f defined in $\mathbb{R}^n$ and for any $x_0 \in \mathbb{R}^n, \rho > 0$, the mean of f in the ball $B_\rho(x_0)$ is written as

$$(f)_{B_\rho(x_0)} = \fint_{B_\rho(x_0)} f(x)dx = \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} f(x)dx, \tag{1.5}$$

and the mean oscillation of f in $B_\rho(x_0)$ is denoted by

$$f_\rho^{\#}(x_0) = \fint_{B_\rho(x_0)} |f(x) - (f)_{B_\rho(x_0)}| dx,$$

where $B_\rho(x_0)$ is the ball in $\mathbb{R}^n$ with radius ρ and centered at x_0. In particular, with the matrix $(a_{kl}(x))_{n\times n}$ and the potential function $c(x) = c_1(x) + ic_2(x)$, we write

$$a_\rho^{\#}(x_0) = \max_{k,l=1,2,\cdots,n} a_{kl,\rho}^{\#}(x_0), \quad c_\rho^{\#}(x_0) = \max_{k=1,2} c_{k,\rho}^{\#}(x_0).$$

Our first main result is the following existence, uniqueness and regularity estimate for strong solutions of the non-divergence form Schrödinger equation (1.2).

Theorem 1.1. *Let $\Lambda > 0, \alpha_0 > 0, R_0 > 0$, and $p \in (1, \infty)$ be given numbers. Then there exist a sufficiently small number $\delta = \delta(\Lambda, n, p) > 0$ and a sufficiently large number $N_0 = N_0(\Lambda, p, n) \geq 1$ such that the following statement holds true. Assume that* (1.1) *and* (1.4) *hold, and assume that*

$$\sup_{x\in\mathbb{R}^n} \sup_{\rho\in(0,R_0)} a_\rho^{\#}(x) \leq \delta \quad \text{and} \quad \sup_{x\in\mathbb{R}^n} \sup_{\rho\in(0,R_0)} c_\rho^{\#}(x) \leq \delta\alpha_0. \tag{1.6}$$

Then, for every $f \in L^p(\mathbb{R}^n, \mathbb{C})$ and $\lambda > \frac{N_0}{\alpha_0 R_0^2}$, there exists a unique strong solution $u \in W^{2,p}(\mathbb{R}^n, \mathbb{C})$ of (1.2). *Moreover, it holds that*

$$\left\|D^2 u\right\|_{L^p(\mathbb{R}^n)} + \sqrt{\lambda\alpha_0}\, \|Du\|_{L^p(\mathbb{R}^n)} + \lambda\alpha_0 \|u\|_{L^p(\mathbb{R}^n)} \leq C(\Lambda, p, n) \|f\|_{L^p(\mathbb{R}^n)}. \tag{1.7}$$

Similarly, we also prove the following result on existence, uniqueness and regularity estimates for weak solutions of the divergence form Schrödinger equation (1.3).

Theorem 1.2. *Let $\Lambda > 0, \alpha_0 > 0, R_0 > 0$, and $p \in (1,\infty)$ be fixed numbers. Then there exist a sufficiently small number $\delta = \delta(\Lambda, n, p) > 0$ and a sufficiently large number $N_0 = N_0(\Lambda, p, n) \geq 1$ such that the following statement holds true. Suppose that* (1.1) *and* (1.4) *hold and suppose also that*

$$\sup_{x\in\mathbb{R}^n}\sup_{\rho\in(0,R_0)} a^{\#}_{\rho}(x) \leq \delta \quad \text{and} \quad \sup_{x\in\mathbb{R}^n}\sup_{\rho\in(0,R_0)} c^{\#}_{\rho}(x) \leq \delta\alpha_0. \tag{1.8}$$

Then, for every $f \in L^p(\mathbb{R}^n,\mathbb{C})$, $g \in L^p(\mathbb{R}^n,\mathbb{C})^n$, and for $\lambda > \frac{N_0}{\alpha_0 R_0^2}$, there exists a unique weak solution $u \in W^{1,p}(\mathbb{R}^n,\mathbb{C})$ of (1.3)*. Moreover, it holds that*

$$\|Du\|_{L^p(\mathbb{R}^n)} + \sqrt{\lambda\alpha_0}\,\|u\|_{L^p(\mathbb{R}^n)} \leq C(\Lambda, p, n)\left[\sqrt{\frac{\lambda}{\alpha_0}}\,\|f\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)}\right]. \tag{1.9}$$

A few remarks regarding Theorems 1.1-1.2 are worth pointing out. Note that the novelty in the estimates (1.7) and (1.9) is that they do not contain any norms of the solutions on the right hand sides. This fact together with $p > 1$ imply that we can control slowly decaying solutions as $|x| \to \infty$. More importantly, this kind of estimates is useful in applications such as in [30] regarding the L^p-diffusion phenomena in dissipative wave equations. When the potential $c(x) = 1 \pm i\epsilon$ with some sufficiently small $\epsilon > 0$, we can take $\alpha_0 = \frac{1}{2}$ and (1.4) still holds. In this case, Theorems 1.1-1.2 can be considered as limiting absorption principle for Schrödinger operators with measurable coefficients, which are new. As such, Theorems 1.1-1.2 could be useful in other areas such as dispersive equations with measurable coefficients. Regarding this, ones may also see Theorem 2.2 and Theorem 3.1 for the special case of Theorem 1.1 and Theorem 1.2 which deal with constant coefficient equations. Note also that our Theorems 1.1-1.2 are still valid when taking $c_1 = 0$, i.e. the potential c is purely imaginary. In comparison with the known work, we would like to note that similar estimates as (1.7) and (1.9) are established in many papers for both linear and nonlinear equations, see [6, 8–10, 20–22, 24–26, 31, 32]. However, in the available mentioned work, the potentials are assumed to be real valued functions. Moreover, in [6,8,21,22,24–27], to obtain the estimates (1.7) and (1.9), the purely real potentials are assumed to be sufficiently large to insure that eigenvalues of the PDE operators $\mathcal{L}_\lambda$ and Q_λ are positive. In our case, due to the non-vanishing of the imaginary parts of the potentials, we can take the real parts of the potentials to be identical to zero. To the best of our knowledge, the case with complex valued potentials have not been investigated before and our estimates (1.7) and (1.9) are new. Also, we would like to note that the estimates (1.7) and (1.9) imply the resolvent estimates of the considered Schrödinger operators, which may have some interesting applications, see [16–18,34]. Moreover, note that both of the equations (1.2) and (1.3) can be rewritten into systems of equations by taking the real and imaginary parts of the equations. Therefore, in some sense, Theorems 1.1-1.2 can be considered as an extension of the results in [6, 8–10, 20–22, 24–26, 31, 32] for systems of equations.

Next, we remark that the first smallness condition in both of (1.6) and (1.8) on the BMO-semi norm (bounded mean oscillation) of the coefficient matrix $(a_{kl})_{n\times n}$ is necessary. This is known in [26] by a well-known example, see also [12] for a recent development and discussion. However, in our case, it is not clear that if the same kind of the smallness condition on the mean oscillation of the potential c is necessary. Nevertheless, it could be possible to relax these smallness conditions in the BMO-semi norms and replace them by the corresponding smallness conditions in the partial BMO-semi norms as in [9, 23]. However, we do not pursue this direction to avoid more technical complications. Similarly, it could be also possible to improve Theorems 1.1-1.2 to more general functional settings such as weighted spaces, mixed-norm spaces, and Lorentz spaces as in [10,11,32].

We also would like to mention that the scaling parameter λ plays an essential role in the analysis of our paper. Essentially, both of the estimates (1.7) and (1.9) and both of the classes of the equations

(1.2) and (1.3) are invariant under the natural scaling and dilation

$$u(x) \mapsto \frac{u(x)}{\gamma}, \quad \text{and} \quad u(x) \mapsto u_\gamma(x) := u(\gamma x), \quad \text{with} \quad \gamma > 0. \tag{1.10}$$

In particular, it is not too hard to check that if u is a solution of (1.2), then u_γ defined above with $\gamma > 0$ is also a solution of the equation

$$-\sum_{k,l=1}^{n} a_{kl}^{\gamma}(x) D_{kl} u_\gamma(x) + \lambda c^{\gamma}(x) u_\gamma(x) = f_\gamma(x) \quad \text{in} \quad \mathbb{R}^n,$$

where

$$a_{kl}^{\gamma}(x) = a_{kl}(\gamma x), \quad c^{\gamma}(x) = \gamma^2 c(\gamma x), \quad \text{and} \quad f_\gamma(x) = \gamma^2 f(\gamma x).$$

Note that in this case, the constant α_0 in (1.4), (1.6), and (1.7) is replaced by $\alpha_0\gamma^2$ and the constant R_0 is replaced by $\frac{R_0}{\gamma}$. As a result, the constant $\frac{N_0}{\alpha_0 R_0^2}$ in Theorem 1.1 is invariant. Moreover, the estimate (1.7) in Theorem 1.1 is also invariant with respect the scalings and dilations (1.10). Similar homogeneous properties also hold for the class of divergence form equations (1.3) and Theorem 1.2. Ones may find in the recent work [20, 28, 29, 31, 32] further applications and developments of the scaling parameter technique in studying regularity theory in Sobolev spaces for solutions of nonlinear elliptic and parabolic equations.

We apply the perturbation method using equations with freezing coefficients to prove our results, see [3–6, 8–10, 20–23, 28, 29, 31–33]. We follow the method introduced in [21], which uses the Fefferman-Stein sharp functions. See also [8–10,22,23] for further implementation and development of the method. However, unlike the work [3–6,9,10,21–23] that freeze the principle coefficient matrix $(a_{kl})_{n\times n}$ and treat the lower order terms as forcing terms, we freeze the coefficient matrix $(a_{kl})_{n\times n}$ and also the potentials as well to take advantage of the non-zero imaginary parts of the potentials. In this way, we are not only able to gain the control of the L^p-norms of the derivatives of solutions, but also gain the control of the L^p-norms of the solutions even when the real parts of the potential are identically zero. To achieve this, the role of the scaling parameter λ becomes essential in our approach, see the recent work [20,28,29,31,32] for further intuition and applications. To implement the above perturbation technique, we derive several interesting estimates utilizing the structure of the equations with complex potentials.

The paper is organized as follows. In the next section, Section 2, we prove Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2.

2. Schrödinger equations in non-divergence form

This section proves Theorem 1.1. Our method is based on the perturbation approach using equations with frozen coefficients. See [3–6,8–10,20,20–23,28,29,31–33], for instance. However, unlike in [3, 6, 8–10, 20, 20–23, 33] which consider lower order terms as forcing terms and therefore move them to the right hand sides of the equations, in our approach, we take advantage of the imaginary part of the potentials. Therefore, we freeze the spatial variable in our potentials and establish several estimates for equations with complex constant coefficients. We start the section with the following simple lemma that is useful in our paper.

Lemma 2.1. *Let $(a_{kl})_{n\times n}$ be a matrix such that* $\operatorname{Im} a_{kl} = \operatorname{Im} a_{lk}$ *for all $k, l = 1, 2, \ldots, n$. Assume that the ellipticity condition in* (1.1) *holds. Then*

$$\operatorname{Re} \sum_{k,l=1}^{n} a_{kl} \xi_k \bar{\xi}_l \geq \Lambda |\xi|^2,$$

for all $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{C}^n$.

Proof. Let us denote

$$a_{kl} = a_{1,kl} + i a_{2,kl} \quad \text{and} \quad \xi_k = \xi_{1,k} + i\xi_{2,k},$$

where $a_{j,kl}, \xi_{j,l}$ are in $\mathbb{R}^n$ for $k, l = 1, 2, \ldots, n$ and for $j = 1, 2$. Then, we see that

$$\sum_{k,l=1}^{n} a_{kl}\xi_k\bar{\xi}_l = \sum_{k,l=1}^{n} a_{kl}(\xi_{1,k}\xi_{1,l} + \xi_{2,k}\xi_{2,l}) + i\sum_{k,l=1}^{n} a_{kl}(\xi_{2,k}\xi_{1,l} - \xi_{1,k}\xi_{2,l}).$$

From this it follows that

$$\operatorname{Re}\sum_{k,l=1}^{n} a_{kl}\xi_k\bar{\xi}_l = \sum_{k,l=1}^{n} a_{1,kl}(\xi_{1,k}\xi_{1,l} + \xi_{2,k}\xi_{2,l}) - \sum_{k,l=1}^{n} a_{2,kl}(\xi_{2,k}\xi_{1,l} - \xi_{1,k}\xi_{2,l})$$

As $\operatorname{Im} a_{kl} = \operatorname{Im} a_{lk}$, we see that $a_{2,kl} = a_{2,lk}$ for all $k, l = 1, 2, \ldots, n$ and consequently,

$$\sum_{k,l=1}^{n} a_{2,kl}(\xi_{2,k}\xi_{1,l} - \xi_{1,k}\xi_{2,l}) = 0.$$

Then, it follows from (1.1) that

$$\begin{aligned}\operatorname{Re}\sum_{k,l=1}^{n} a_{kl}\xi_k\bar{\xi}_l &= \sum_{k,l=1}^{n} a_{1,kl}(\xi_{1,k}\xi_{1,l} + \xi_{2,k}\xi_{2,l}) = \operatorname{Re}\sum_{k,l=1}^{n} a_{kl}\xi_{1,k}\xi_{1,l} + \operatorname{Re}\sum_{k,l=1}^{n} a_{kl}\xi_{2,k}\xi_{2,l}\\ &\geq \Lambda\Big(|\xi_{,1}|^2 + |\xi_{,2}|^2\Big) = \Lambda|\xi|^2.\end{aligned}$$

where we have used the notation $\xi_{,l} = (\xi_{l,1}, \xi_{l,2}, \ldots, \xi_{l,n}) \in \mathbb{R}^n$ with $l = 1, 2$. The assertion is then proved. □

2.1. **Equations with constant coefficients.** This section derives important estimates for solutions of second order elliptic equations with complex constant coefficients. We consider the following equation

$$-\sum_{j,l=1}^{n} a_{jl}D_{jl}u + \lambda[c_1 + ic_2]u = f \quad \text{in} \quad \mathbb{R}^n, \tag{2.1}$$

where $\lambda > 0$ is a scaling parameter constant, $f : \mathbb{R}^n \to \mathbb{C}$ is a given measurable function and $u : \mathbb{R}^n \to \mathbb{C}$ is an unknown solution. Moreover, $(a_{jl})_{n\times n}$ is a given $n \times n$ matrix of complex numbers, and $c_1 \in \mathbb{R}, c_2 \in \mathbb{R}$ are given numbers satisfying

$$c_1 \geq 0, \quad c_1 + c_2 \geq \alpha_0, \quad \text{and} \quad c_1 + |c_2| \leq \Lambda^{-1}\alpha_0. \tag{2.2}$$

The following theorem is a special case of Theorem 1.1 in which the coefficients are constants.

Theorem 2.2. *Let* $\Lambda > 0, \alpha_0 > 0, p \in (1, \infty)$*, and let* $(a_{jl})_{n\times n}$ *be a matrix of complex numbers satisfying* (1.1)*. Then, for every real numbers* c_1, c_2 *satisfying* (2.2) *and for* $\lambda > 0$*,* $f \in L^p(\mathbb{R}^n, \mathbb{C})$*, there exists a unique strong solution* $u \in W^{2,p}(\mathbb{R}^n, \mathbb{C})$ *of* (2.1)*. Moreover,*

$$\left\|D^2u\right\|_{L^p(\mathbb{R}^n)} + \sqrt{\lambda\alpha_0}\,\|Du\|_{L^p(\mathbb{R}^n)} + \lambda\alpha_0\,\|u\|_{L^p(\mathbb{R}^n)} \leq C\,\|f\|_{L^p(\mathbb{R}^n)} \tag{2.3}$$

where $C = C(\Lambda, p, n)$*.*

Theorem 2.2 plays an important role in the proof of our Theorem 1.1. Even though Theorem 2.2 is for equations with constant coefficients, it is new, and we prove it. The first step in the proof of Theorem 2.2 is to prove it for $p = 2$. This step is carried out in the following lemma.

Lemma 2.3. *Let $\Lambda > 0, \alpha_0 > 0$ and assume that $(a_{kl})_{n\times n}$ is a matrix of complex numbers satisfying* (1.1). *Also, let c_1, c_2 be real numbers satisfying* (2.2) *. Then, for every $\lambda > 0$ and for $f \in L^2(\mathbb{R}^n, \mathbb{C})$, there exists a unique solution $u \in W^{2,2}(\mathbb{R}^n, \mathbb{C})$ of* (2.1). *Moreover,*

$$\left\|D^2 u(x)\right\|_{L^2(\mathbb{R}^n)} + \sqrt{\lambda\alpha_0}\,\|Du\|_{L^2(\mathbb{R}^n)} + \lambda\alpha_0\,\|u\|_{L^2(\mathbb{R}^n)} \leq C(\Lambda)\,\|f\|_{L^2(\mathbb{R}^n)}.$$

Proof. Observe also that since $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ is dense in $W^{2,2}(\mathbb{R}^n, \mathbb{C})$, we can find a sequence of functions $u_k \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ such that $u_k \to u$ in $W^{2,2}(\mathbb{R}^n, \mathbb{C})$. Moreover, u_k is a solution of

$$-\sum_{j,l=1}^n a_{jl} D_{jl} u_k + \lambda[c_1 + ic_2]u_k = -\sum_{j,l=1}^n a_{jl} D_{jl}(u - u_k) + \lambda[c_1 + ic_2](u_k - u) + f, \quad \text{in} \quad \mathbb{R}^n.$$

As the right hand side of the above equation converges to f in $L^2(\mathbb{R}^n, \mathbb{C})$, it is sufficient to prove the estimate in the lemma with the assumption that $u \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$.

We use standard energy estimate taking advantage of the fact that $c_1 + |c_2| \geq \alpha_0$. By multiplying the equation (2.1) with $\lambda\bar{u}$, and using the integration by parts, we obtain

$$\lambda \sum_{j,l=1}^n \int_{\mathbb{R}^n} a_{kl} D_l u D_k \bar{u} dx + \lambda^2 [c_1 + ic_2] \int_{\mathbb{R}^n} |u|^2 dx = \lambda \int_{\mathbb{R}^n} f(x)\bar{u}(x) dx. \tag{2.4}$$

From this and by taking the real part of (2.4) and using Lemma 2.1, the boundedness of c_1, c_2 in (1.4) and the Young's inequality, we see that

$$\lambda\alpha_0 \int_{\mathbb{R}^n} |Du(x)|^2 dx \leq C(\Lambda)\left[\lambda^2\alpha_0^2 \int_{\mathbb{R}^n} |u(x)|^2 dx + \int_{\mathbb{R}^n} |f(x)|^2 dx\right]. \tag{2.5}$$

Now, let $\epsilon > 0$ be sufficiently small which will be determined. By taking the real part of (2.4) and using Lemma 2.1, we see that

$$\begin{aligned}\lambda\Lambda \int_{\mathbb{R}^n} |Du(x)|^2 dx + \lambda^2 c_1 \int_{\mathbb{R}^n} |u(x)|^2 dx &\leq \lambda \int_{\mathbb{R}^n} |f(x)||u(x)| dx \\ &\leq \frac{\epsilon\lambda^2}{2} \int_{\mathbb{R}^n} |u(x)|^2 dx + \frac{1}{2\epsilon} \int_{\mathbb{R}^n} |f(x)|^2 dx.\end{aligned}$$

This estimate and since $c_1 \geq 0$, we particularly infer that

$$\begin{aligned}&\lambda \int_{\mathbb{R}^n} |Du(x)|^2 dx \leq C(\Lambda)\left[\epsilon\lambda^2 \int_{\mathbb{R}^n} |u(x)|^2 dx + \frac{1}{\epsilon} \int_{\mathbb{R}^n} |f(x)|^2 dx\right], \quad \text{and} \\ &\lambda^2 c_1 \int_{\mathbb{R}^n} |u(x)|^2 dx \leq \frac{\epsilon\lambda^2}{2} \int_{\mathbb{R}^n} |u(x)|^2 dx + \frac{1}{2\epsilon} \int_{\mathbb{R}^n} |f(x)|^2 dx.\end{aligned} \tag{2.6}$$

On the other hand, by taking the imaginary part of (2.4) and by using boundedness condition of the coefficients a_{kl} in (1.1) and Young's inequality, we obtain

$$\begin{aligned}\lambda^2 c_2 \int_{\mathbb{R}^n} |u|^2 dx &\leq \lambda\Lambda^{-1} \int_{\mathbb{R}^n} |\nabla u|^2 dx + \lambda \int_{\mathbb{R}^n} |f(x)||u(x)| dx \\ &\leq \lambda\Lambda^{-1} \int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{\epsilon\lambda^2}{2} \int_{\mathbb{R}^n} |u(x)|^2 dx + \frac{1}{2\epsilon} \int_{\mathbb{R}^n} |f(x)|^2 dx \\ &\leq C(\Lambda)\left[\epsilon\lambda^2 \int_{\mathbb{R}^n} |u(x)|^2 dx + \frac{1}{\epsilon} \int_{\mathbb{R}^n} |f(x)|^2 dx\right],\end{aligned} \tag{2.7}$$

where we have used the first estimate of (2.6) in the last step. Now, we combined this last estimate with the second estimate in (2.6) to imply that

$$\lambda^2(c_1+c_2)\int_{\mathbb{R}^n}|u(x)|^2dx \leq C(\Lambda)\left[\epsilon\lambda^2\int_{\mathbb{R}^n}|u(x)|^2dx + \frac{1}{\epsilon}\int_{\mathbb{R}^n}|f(x)|^2dx\right].$$

From this and the condition $c_1+c_2 \geq \alpha_0$, we can choose ϵ such that $C(\Lambda)\epsilon = \alpha_0/2$ to obtain

$$\lambda^2\alpha_0^2\int_{\mathbb{R}^n}|u|^2dx \leq C(\Lambda)\int_{\mathbb{R}^n}|f(x)|^2dx. \tag{2.8}$$

It then follows from (2.5) and (2.8) that

$$\lambda\alpha_0\int_{\mathbb{R}^n}|Du(x)|^2dx \leq C(\Lambda)\int_{\mathbb{R}^n}|f(x)|^2dx. \tag{2.9}$$

From (2.8) and (2.9), we see that it remains to control the L^2-norm of the second derivative of u. To this end, for each $k \in \{1,2,\cdots,n\}$, by multiplying the equation (2.1) with $D_{kk}\bar{u}$ and using the integration by parts, we see that

$$\sum_{j,l=1}^{n}\int_{\mathbb{R}^n}a_{jl}D_l(D_ku)Dj(D_k\bar{u})dx = \lambda[c_1+ic_2]\int_{R^n}|D_ku(x)|^2dx + \sum_{j,l=1}^{n}\int_{\mathbb{R}^n}f(x)D_{kk}\bar{u}(x)dx.$$

Then, by taking the real part of the above equation, and using Lemma 2.1, (2.9), and the Young's inequality again, we obtain

$$\int_{\mathbb{R}^n}|D^2u(x)|^2dx \leq C(\Lambda)\int_{\mathbb{R}^n}|f(x)|^2dx.$$

This completes the proof of the estimate in lemma.

Finally, we prove the unique solvability of (2.1) in $W^{2,2}(\mathbb{R}^n,\mathbb{C})$. Observe that the uniqueness follows directly from the estimate that we just proved. Also, the solvability can be obtained by the method of continuity (see [22, Theorem 1.4.4, p. 15 and Theorem 6.4.1 p. 139]) using the solvability of the equation

$$-\Delta u + \lambda\alpha_0 u = f \quad \text{in} \quad \mathbb{R}^n.$$

The proof of the lemma is completed. □

Observe that Lemma 2.3 justifies Theorem 2.2 when $p = 2$. For general $p \in (1,\infty)$, the proof of Theorem 2.2 is more involved and it requires more analytic theory and regularity estimates. Our next lemma gives local regularity estimates for solutions of the homogeneous constant coefficient equations.

Lemma 2.4. *Let $(a_{kl})_{n\times n}$ and c_1, c_2 be as in Lemma 2.3. If $u \in C^\infty(B_1)$ is a solution of*

$$-\sum_{j,l=1}^{n}a_{jl}D_{jl}u + \lambda[c_1+ic_2]u = 0 \quad \textit{in} \quad B_1 \tag{2.10}$$

with some $\lambda > 0$ and $q \in (1,\infty)$, then

$$\|u\|_{C^1(\overline{B}_{1/2})} \leq C(\Lambda,q,n)\,\|u\|_{L^q(B_1)},$$

where $C(\Lambda,q,n)$ is independent on λ and α_0.

Proof. Let $r, R \in (1/2, 1)$ with $r < R$. Let $\phi \in C_0^\infty(B_R)$ be a real valued function satisfying $\phi = 1$ on B_r and $0 \le \phi \le 1$ on B_R. Multiplying the equation (2.10) with $\bar{u}\phi^2$ and using the integration by parts, we obtain

$$\begin{aligned}&\int_{B_R} \phi^2(x) a_{jl} D_l u(x) D_j \bar{u}(x) dx + \lambda[c_1 + ic_2] \int_{B_R} |u(x)|^2 \phi^2(x) dx \\ &= -2 \int_{B_R} a_{jl} D_l u(x) Dj\phi(x)[\bar{u}(x)\phi(x)] dx.\end{aligned} \tag{2.11}$$

Then, by taking the real part of the identity (2.11) and using the Lemma 2.1, and the boundedness of the coefficient a_{kl} in (1.1), we obtain

$$\Lambda \int_{B_R} |Du(x)|^2 \phi^2(x) dx + \lambda c_1 \int_{B_R} |u(x)|^2 \phi^2(x) dx \le 2\Lambda^{-1} \int_{B_R} |Du(x)||D\phi(x)||\bar{u}(x)|\phi(x) dx.$$

Now, using the Hölder's inequality and Young's inequality for the right hand side term of the last estimate, we obtain

$$\Lambda \int_{B_r} |Du(x)|^2 \phi^2(x) dx + \lambda c_1 \int_{B_R} |u(x)|^2 \phi^2(x) dx \le C(\Lambda) \int_{B_R} |u(x)|^2 |D\phi(x)|^2 dx. \tag{2.12}$$

Similarly, by taking the imaginary part of the identity (2.11), and using the boundedness condition in (1.1) and the Young's inequality, we obtain

$$\lambda c_2 \int_{B_r} |u(x)|^2 dx \le \Lambda^{-1} \left[\int_{B_R} |Du|^2 \phi^2(x) dx + \int_{B_R} |u(x)|^2 |D\phi(x)|^2 dx \right]. \tag{2.13}$$

This estimate together with (2.12) imply that

$$\int_{B_r} |Du|^2 dx + \lambda\alpha_0 \int_{B_r} |u(x)|^2 dx \le C(\Lambda, n, R - r) \int_{B_R} |u(x)|^2 dx.$$

Because Du is also a solution of (2.10), we also obtain

$$\int_{B_r} |D^2 u|^2 dx + \lambda\alpha_0 \int_{B_r} |Du(x)|^2 dx \le C(\Lambda, n, R - r) \int_{B_R} |Du(x)|^2 dx.$$

By an iteration, we then see that

$$\int_{B_r} |D^{k+1} u|^2 dx + \lambda\alpha_0 \int_{B_r} |D^k u(x)|^2 dx \le C(\Lambda, n, R - r) \int_{B_R} |u(x)|^2 dx, \quad \forall\, k \in \mathbb{N}. \tag{2.14}$$

Then, by taking $k > n/2$ and using Sobolev's imbedding, we obtain

$$(1 + \lambda\alpha_0) \|u\|_{L^\infty(B_r)} \le C(\Lambda, n) \left(\fint_{B_{2r}} |u(x)|^2 dx \right)^{1/2}, \quad r \in (0, 1/2).$$

From this, and a standard iteration technique (see [19, p. 75]), we obtain

$$(1 + \lambda\alpha_0) \|u\|_{L^\infty(B_{1/2})} \le C(\Lambda, q, n) \left(\fint_{B_1} |u(x)|^q dx \right)^{1/q} \tag{2.15}$$

for $q \in (1, \infty)$. Because $D^k u$ satisfying the same equation as u, we use (2.14) and (2.15) to derive that

$$(1 + \lambda\alpha_0) \left\| D^k u \right\|_{L^\infty(B_{1/2})} \le C(\Lambda, n, k) \left(\fint_{B_1} |u(x)|^q dx \right)^{1/q}.$$

The proof of the lemma is completed. □

Next, we state and prove a corollary of Lemma 2.4, which gives the control of the mean oscillations of the solutions and its derivatives for the equation (2.10).

Lemma 2.5. *Let $\Lambda \in (0,1)$ and $q \in (1,\infty)$ be fixed. Then, there is $C = C(\Lambda, q, n) > 0$ such that the following statement holds true. Suppose that $\rho > 0$ and* (1.1) *holds for some matrix of complex numbers $(a_{kl})_{n\times n}$. Suppose also that $u \in C^\infty(B_\rho)$ is a solution of*

$$-\sum_{j,l=1}^{n} a_{jl}D_{jl}u + \lambda[c_1 + ic_2]u = 0 \quad in \quad B_\rho,$$

with some $\lambda > 0$ and some real numbers c_1, c_2 satisfying (2.2). *Then, for every $\kappa \in (0,1/2)$, the following estimates hold*

$$\fint_{B_{\kappa\rho}} |D^2u - (D^2u)_{B_{\kappa\rho}}|dx \leq \kappa C_0 \left(\fint_{B_\rho} |D^2u(x)|^q dx\right)^{1/q},$$

$$\fint_{B_{\kappa\rho}} |Du - (Du)_{B_{\kappa\rho}}|dx \leq \kappa C_0 \left(\fint_{B_\rho} |Du(x)|^q dx\right)^{1/q},$$

$$\fint_{B_{\kappa\rho}} |u - (u)_{B_{\kappa\rho}}|dx \leq \kappa C_0 \left(\fint_{B_\rho} |u(x)|^q dx\right)^{1/q}.$$

Proof. By scaling, we may assume that $\rho = 1$. Observe that from Lemma 2.4, we see that

$$\fint_{B_\kappa} |u - (u)_{B_\kappa}|dx \leq C(\Lambda, q, n)\kappa\|Du\|_{L^\infty(B_{1/2})} \leq C(\Lambda, n)\kappa \left(\fint_{B_1} |u(x)|^q dx\right)^{1/q}.$$

This proves the last estimate in the lemma. The other first two estimates in the lemma can be proved similarly as Du, D^2u are solutions of the same equations as u. The proof is then completed. ◻

The next lemma provides us the mean oscillation estimates of solutions and their derivatives for the non-homogeneous equation (2.1).

Lemma 2.6. *For a given constant $\Lambda \in (0,1)$, there exists $C = C(\Lambda, n)$ such that the following statement holds. Suppose that $(a_{kl})_{n\times n}$ is a matrix of complex numbers satisfying* (1.1). *Suppose also that $f \in L^2(B_\rho(x_0), \mathbb{C})$. Then, if $u \in W^{2,2}(B_\rho(x_0), \mathbb{C})$ is a solution of*

$$-\sum_{j,l=1}^{n} a_{jl}D_{jl}u + \lambda[c_1 + ic_2]u = f \quad in \quad B_\rho(x_0)$$

for some $x_0 \in \mathbb{R}^n$, some $\rho > 0$, $\lambda > 0$, and for c_1, c_2 satisfying (2.2), *the following estimates hold*

$$\fint_{B_{\kappa\rho}(x_0)} |U - (U)_{B_{\kappa\rho}(x_0)}|dx \leq C\left[\kappa\left(\fint_{B_\rho(x_0)} |U(x)|^2 dx\right)^{1/2} + \kappa^{-\frac{n}{2}}\left(\fint_{B_\rho(x_0)} |f(x)|^2 dx\right)^{1/2}\right],$$

for every $\kappa \in (0,1/4)$ and for $U = D^2u$, $\sqrt{\lambda\alpha_0}Du$, or $\lambda\alpha_0 u$.

Proof. By using the translation $x \mapsto x - x_0$, we can assume that $x_0 = 0$. Let $\eta \in C_0^\infty(B_\rho)$ be a standard cut-off function which satisfies

$$\eta = 1 \quad \text{on} \quad B_{\rho/2}.$$

Then, let $w \in W^{2,2}(\mathbb{R}^n, \mathbb{C})$ be the solution of the equation

$$-\sum_{j,l=1}^{n} a_{jl}D_{jl}w + \lambda[c_1 + ic_2]w = \eta(x)f(x) \quad \text{in} \quad \mathbb{R}^n. \tag{2.16}$$

Note that the existence of w is obtained from Lemma 2.3. By writing $W = (D^2 w, \sqrt{\lambda\alpha_0} Dw, \lambda\alpha_0 w)$, we can see that from Lemma 2.3 that

$$\begin{aligned} &\left(\fint_{B_{\kappa\rho}} |W|^2 dx\right)^{1/2} \leq \frac{C(\Lambda, n)}{\kappa^{\frac{n}{2}}} \left(\fint_{B_\rho} |f(x)|^2 dx\right)^{1/2}, \quad \text{and} \\ &\left(\fint_{B_\rho} |W|^2 dx\right)^{1/2} \leq C(\Lambda, n) \left(\fint_{B_\rho} |f(x)|^2 dx\right)^{1/2}. \end{aligned} \tag{2.17}$$

Now, let $v = u - w$, we see that v is a solution of the equation

$$-\sum_{j,l=1}^{n} a_{jl} D_{jl} v + \lambda[c_1 + ic_2] v = 0 \quad \text{in} \quad B_{\rho/2}.$$

Again, by writing $V = (D^2 v, \sqrt{\lambda\alpha_0} Dv, \lambda\alpha_0 v)$, we can apply Lemma 2.5 for V to see that

$$\fint_{B_{\kappa\rho}} |V(x) - (V)_{B_{\kappa\rho}}| dx \leq \kappa C_0 \left(\fint_{B_{\rho/2}} |V(x)|^2 dx\right)^{1/2}, \quad \forall\, \kappa \in (0, 1/4). \tag{2.18}$$

Recall that

$$\fint_{B_{\kappa\rho}} |U - (U)_{B_{\kappa\rho}}| dx \leq 2 \fint_{B_{\kappa\rho}} |U - c| dx, \quad \forall c \in \mathbb{R}.$$

Then, by taking $c = (V)_{B_{\kappa\rho}}$, and using the triangle inequality and Hölder's inequality, we see that

$$\begin{aligned} \fint_{B_{\kappa\rho}} |U - (U)_{B_{\kappa\rho}}| dx &\leq 2 \fint_{B_{\kappa\rho}} |U - (V)_{B_{\kappa\rho}}| dx \\ &\leq 2 \left[\fint_{B_{\kappa\rho}} |V - (V)_{B_{\kappa\rho}}| dx + \left(\fint_{B_{\kappa\rho}} |W|^2 dx\right)^{1/2} \right]. \end{aligned}$$

From this estimate, the first estimate in (2.17), and from (2.18), we see that

$$\begin{aligned} &\fint_{B_{\kappa\rho}} |U - (U)_{B_{\kappa\rho}}| dx \\ &\leq C \left[\kappa \left(\fint_{B_{\rho/2}} |V(x)|^2 dx\right)^{1/2} + \kappa^{-\frac{n}{2}} \left(\fint_{B_\rho} |f(x)|^2 dx\right)^{1/2} \right] \\ &\leq C \left[\kappa \left(\fint_{B_{\rho/2}} |U(x)|^2 dx\right)^{1/2} + \kappa \left(\fint_{B_{\rho/2}} |W(x)|^2 dx\right)^{1/2} + \kappa^{-\frac{n}{2}} \left(\fint_{B_\rho} |f(x)|^2 dx\right)^{1/2} \right]. \end{aligned}$$

Now, using the second estimate in (2.17), we can control the second term on the right hand side of the last estimate and infer that

$$\fint_{B_{\kappa\rho}} |U - (U)_{B_{\kappa\rho}}| dx \leq C \left[\kappa \left(\fint_{B_\rho} |U(x)|^2 dx\right)^{1/2} + \kappa^{-\frac{n}{2}} \left(\fint_{B_\rho} |f(x)|^2 dx\right)^{1/2} \right],$$

where C is a constant depending only on Λ and n. The proof of the lemma is therefore completed. □

Remark 2.7. *We observe that* (2.7) *and* (2.13) *still hold true if the constant* c_2 *in the terms on the left hand sides of* (2.7) *and* (2.13) *is replaced by* $|c_2|$. *As such, Lemmas 2.3, 2.4, 2.5, and 2.6 are all valid if the second condition in* (2.2) *is replaced by* $c_1 + |c_2| \geq \alpha_0$.

Next, to prove our main results, we need to recall several definitions and analysis estimates. Let us denote the collection of balls in $\mathbb{R}^n$ by

$$\mathcal{B} = \{B_\rho(x) : \rho > 0, x \in \mathbb{R}^n\}.$$

For any locally integrable function f defined in $\mathbb{R}^n$, the Hardy-Littlewood maximal function of f is defined by

$$\mathcal{M}(f)(x) = \sup_{B\in\mathcal{B}, x\in B} \fint_B |f(y)|dy.$$

Moreover, the Fefferman-Stein sharp function of f is defined by

$$f^{\#}(x) = \sup_{B\in\mathcal{B}, x\in B} \fint_B |f(y) - (f)_B|dy,$$

where $(f)_B$ is defined as in (1.5). Note that for $p \in (1, \infty)$ and $f \in L^p(\mathbb{R}^n)$, it follows from the Fefferman-Stein theorem and Hardy-Littlewood maximal function theorem that (see [22, Chapter 3], for instance)

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C(n, p)\|f^{\#}\|_{L^p(\mathbb{R}^n)}, \quad \text{and} \quad \|\mathcal{M}(f)\|_{L^p(\mathbb{R}^n)} \leq C(n, p)\|f\|_{L^p(\mathbb{R}^n)}. \tag{2.19}$$

Also, observe that it follows directly from the definitions that

$$f^{\#}(x) \leq 2\mathcal{M}(f)(x), \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Consequently,

$$\|f^{\#}\|_{L^p(\mathbb{R}^n)} \leq 2\|\mathcal{M}(f)\|_{L^p(\mathbb{R}^n)} \leq C(n, p)\|f\|_{L^p(\mathbb{R}^n)}. \tag{2.20}$$

Proof of Theorem 2.2. By duality, we only need to consider the case $p \in [2, \infty)$. Moreover, as the case $p = 2$ is proved already by Lemma 2.3, it remains to consider the case $p > 2$.

We first prove the a-priori estimate (2.3). Let $u \in W^{2,p}(\mathbb{R}^n, \mathbb{C})$ be a solution of (2.1). By using the density of $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ in $W^{2,p}(\mathbb{R}^n, \mathbb{C})$ as in the proof of Lemma 2.3, we can assume that $u \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$. Then, by applying Lemma 2.6, we can control the Fefferman-Stein sharp function of U as

$$U^{\#}(x) \leq C(\Lambda, n)\left[\kappa\mathcal{M}(|U|^2)(x)^{1/2} + \kappa^{-\frac{n}{2}}\mathcal{M}(|f|^2)(x)^{1/2}\right], \quad \text{for a.e. } x \in \mathbb{R}^n,$$

where $U = (D^2u, \sqrt{\lambda\alpha_0}Du, \lambda\alpha_0 u)$, $C = C(\Lambda, n)$ and $\kappa \in (0, 1/4)$. By using the Fefferman-Stein theorem for sharp functions and Hardy-Littlewood maximal function theorem (see (2.19) and (2.20)), we obtain

$$\begin{aligned}\|U\|_{L^p(\mathbb{R}^n)} &\leq C(n, p)\left\|U^{\#}\right\|_{L^p(\mathbb{R}^n)} \leq C\left[\kappa\left\|\mathcal{M}(|U|^2)^{1/2}\right\|_{L^p(\mathbb{R}^n)} + \kappa^{-\frac{n}{2}}\left\|\mathcal{M}(|f|^2)^{1/2}\right\|_{L^p(\mathbb{R}^n)}\right] \\ &\leq C\left[\kappa\|U\|_{L^p(\mathbb{R}^n)} + \kappa^{-\frac{n}{2}}\|f\|_{L^p(\mathbb{R}^n)}\right].\end{aligned}$$

From this and by choosing κ sufficiently small, we obtain

$$\|U\|_{L^p(\mathbb{R}^n)} \leq C(\Lambda, p, n)\|f\|_{L^p(\mathbb{R}^n)},$$

and this proves (2.3).

Now, it remains to prove the existence of the solution as the uniqueness follows from (2.3). For given $f \in L^p(\mathbb{R}^n, \mathbb{C})$, by the density of $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ in $L^p(\mathbb{R}^n, \mathbb{C})$, we can find a sequence of smooth compactly supported functions $\{f_k\}_k \subset C_0^\infty(\mathbb{R}^n, \mathbb{C})$ such that $f_k \to f$ in $L^p(\mathbb{R}^n, \mathbb{C})$. Observe that as $p > 2$, $f_k \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. Then, by Lemma 2.3, there is a unique solution $u_k \in W^{2,2}(\mathbb{R}^n, \mathbb{C})$ of the equation

$$-\sum_{j,l=1}^{n} a_{jl}D_{jl}u_k + \lambda[c_1 + ic_2]u_k = f_k(x) \quad \text{in} \quad \mathbb{R}^n. \tag{2.21}$$

Since $f_k \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$, and the coefficients in (2.21) are constants, we can formally differentiate the equation (2.21) and then apply Lemma 2.3 iteratively to prove that $u_k \in W^{l,2}(\mathbb{R}^n, \mathbb{C})$ for all $l \in \mathbb{N}$. From this and by choosing l sufficiently large, we can apply the Sobolev imbedding theorem to infer that $u_k \in W^{2,p}(\mathbb{R}^n, \mathbb{C})$. Then, by using the a-priori estimate (2.3), we obtain

$$\left\|D^2 u_k\right\|_{L^p(\mathbb{R}^n)} + \sqrt{\lambda\alpha_0}\,\|Du_k\|_{L^p(\mathbb{R}^n)} + \lambda\alpha_0\,\|u_k\|_{L^p(\mathbb{R}^n)} \leq C(\Lambda, p, n)\,\|f_k\|_{L^p(\mathbb{R}^n)}\,.$$

Similarly to this estimate by considering the equation for $u_k - u_l$, we also see that

$$\left\|D^2(u_k - u_l)\right\|_{L^p(\mathbb{R}^n)} + \sqrt{\lambda\alpha_0}\,\|D(u_k - u_l)\|_{L^p(\mathbb{R}^n)} + \lambda\alpha_0\,\|u_k - u_l\|_{L^p(\mathbb{R}^n)} \leq C(\Lambda, p, n)\,\|f_k - f_l\|_{L^p(\mathbb{R}^n)}\,,$$

for every $k, l \in \mathbb{N}$. This estimate, and since $f_k \to f$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$, we infer that $\{u_k\}_k$ is a Cauchy sequence in $W^{2,p}(\mathbb{R}^n, \mathbb{C})$. Let $u \in W^{2,p}(\mathbb{R}^n, \mathbb{C})$ be the limit of the sequence $\{u_k\}_k$ in $W^{2,p}(\mathbb{R}^n, \mathbb{C})$. It can be proved easily that u is a solution of (2.1). The proof of the theorem is completed. □

Remark 2.8. *By observing the proof and from Remark 2.7, we see that Theorem 2.2 still holds if the second condition in* (2.2) *is replaced by* $c_1 + |c_2| \geq \alpha_0$.

2.2. **Equations with measurable variable coefficients.** We first state and prove an improved version of Lemma 2.6 for equations with constant coefficients.

Lemma 2.9. *For a given constant* $\Lambda \in (0, 1)$ *and* $q \in (1, \infty)$*, there exists* $C = C(\Lambda, q, n) > 0$ *such that the following statement holds. Suppose that the matrix* $(a_{jl})_{n\times n}$ *of complex numbers satisfies the conditions in* (1.1)*. Suppose also that* $f \in L^q(B_\rho(x_0), \mathbb{C})$*. Then, if* $u \in W^{2,q}(B_\rho(x_0), \mathbb{C})$ *is a solution of*

$$-\sum_{j,l=1}^{n} a_{jl} D_{jl} u + \lambda[c_1 + ic_2]u = f, \quad \textit{in} \quad B_\rho(x_0)$$

for some $x_0 \in \mathbb{R}^n$, $\rho > 0$, $\lambda > 0$*, and some real numbers* c_1, c_2 *satisfying* (2.2)*, the following estimate holds*

$$\fint_{B_{\kappa\rho}(x_0)} |U - (U)_{B_{\kappa\rho}(x_0)}| dx \leq C(\Lambda, q, n)\left[\kappa\left(\fint_{B_\rho(x_0)} |U(x)|^q dx\right)^{1/q} + \kappa^{-\frac{n}{q}}\left(\fint_{B_\rho(x_0)} |f(x)|^q dx\right)^{1/q}\right],$$

for every $\kappa \in (0, 1/4)$ *and for* $U = D^2 u$ *or* $\sqrt{\lambda\alpha_0}Du$ *or* $\lambda\alpha_0 u$.

Proof. The proof is similar to that of Lemma 2.6. The only difference is that we do not apply Lemma 2.3 as in the proof of Lemma 2.6, but instead, we apply Theorem 2.2. We provide the details of the proof for completeness. By using the translation $x \mapsto x - x_0$, we can assume that $x_0 = 0$. Let $\eta \in C_0^\infty(B_\rho)$ be a standard cut-off function which satisfies

$$\eta = 1 \quad \text{on} \quad B_{\rho/2}.$$

Then, let $w \in W^{2,q}(\mathbb{R}^n, \mathbb{C})$ be the solution of the equation

$$-\sum_{j,l=1}^{n} a_{jl} D_{jl} w + \lambda[c_1 + ic_2]w = \eta(x)f(x) \quad \text{in} \quad \mathbb{R}^n, \tag{2.22}$$

where the existence of w is obtained by using Theorem 2.2. By writing $W = (D^2 w, \sqrt{\lambda\alpha_0}Dw, \lambda\alpha_0 w)$, we can see that from Theorem 2.2 that

$$\begin{aligned} &\left(\fint_{B_{\kappa\rho}} |W|^q dx\right)^{1/q} \leq \frac{C(\Lambda, q, n)}{\kappa^{\frac{n}{q}}}\left(\fint_{B_\rho} |f(x)|^q dx\right)^{1/q} \quad \text{and} \\ &\left(\fint_{B_\rho} |W|^q dx\right)^{1/q} \leq C(\Lambda, q, n)\left(\fint_{B_\rho} |f(x)|^q dx\right)^{1/q}. \end{aligned} \tag{2.23}$$

Now, let $v = u - w$, we see that v is a solution of the equation

$$-\sum_{j,l=1}^{n} a_{jl} D_{jl} v + \lambda[c_1 + ic_2]v = 0 \quad \text{in} \quad B_{\rho/2}.$$

Again, by writing $V = (D^2 v,\ \sqrt{\lambda\alpha_0} Dv, \lambda\alpha_0 v)$, we can apply Lemma 2.5 for V to see that

$$\fint_{B_{\kappa\rho}} |V(x) - (V)_{B_{\kappa\rho}}| dx \leq \kappa C_0 \left(\fint_{B_{\rho/2}} |V(x)|^q dx \right)^{1/q}, \quad \forall\, \kappa \in (0, 1/4). \tag{2.24}$$

Recall that

$$\fint_{B_{\kappa\rho}} |U - (U)_{B_{\kappa\rho}}| dx \leq 2 \fint_{B_{\kappa\rho}} |U - c| dx, \quad \forall c \in \mathbb{R}.$$

Then, by taking $c = (V)_{B_{\kappa\rho}}$, and using the triangle inequality and Hölder's inequality, we see that

$$\begin{aligned} \fint_{B_{\kappa\rho}} |U - (U)_{B_{\kappa\rho}}| dx &\leq 2 \fint_{B_{\kappa\rho}} |U - (V)_{B_{\kappa\rho}}| dx \\ &\leq 2 \left[\fint_{B_{\kappa\rho}} |V - (V)_{B_{\kappa\rho}}| dx + \left(\fint_{B_{\kappa\rho}} |W|^q dx \right)^{1/q} \right]. \end{aligned}$$

From this, the first estimate in (2.23), and from (2.24), we see that

$$\begin{aligned} &\fint_{B_{\kappa\rho}} |U - (U)_{B_{\kappa\rho}}| dx \\ &\leq C \left[\kappa \left(\fint_{B_{\rho/2}} |V(x)|^q dx \right)^{1/q} + \kappa^{-\frac{n}{q}} \left(\fint_{B_\rho} |f(x)|^q dx \right)^{1/q} \right] \\ &\leq C \left[\kappa \left(\fint_{B_{\rho/2}} |U(x)|^q dx \right)^{1/q} + \kappa \left(\fint_{B_{\rho/2}} |W(x)|^q dx \right)^{1/q} + \kappa^{-\frac{n}{q}} \left(\fint_{B_\rho} |f(x)|^q dx \right)^{1/q} \right]. \end{aligned}$$

Now, using the second estimates in (2.23), we can control the second term on the right hand side of the last estimate and infer that

$$\fint_{B_{\kappa\rho}} |U - (U)_{B_{\kappa\rho}}| dx \leq C \left[\kappa \left(\fint_{B_\rho} |U(x)|^q dx \right)^{1/q} + \kappa^{-\frac{n}{q}} \left(\fint_{B_\rho} |f(x)|^q dx \right)^{1/q} \right],$$

where C is a constant depending only on Λ, q and n. The proof of the lemma is therefore completed. □

Lemma 2.10. *Let $\Lambda \in (0, 1)$ and assume that* (1.1) *and* (1.4) *hold. Let $q \in (1, \infty)$, $p \in (q, \infty)$ and assume that $f \in L^q(B_\rho(x_0), \mathbb{C})$ and $u \in W^{2,p}(B_\rho(x_0), \mathbb{C})$ is a strong solutions of*

$$-\sum_{j,k=1}^{n} a_{jk}(x) D_{jk} u + \lambda[c_1(x) + ic_2(x)]u(x) = f(x) \quad in \quad B_\rho(x_0),$$

with some $\lambda > 0$. Then, for every $\kappa \in (0, 1/4)$, it holds that

$$\begin{aligned}
&\fint_{B_{\kappa\rho}(x_0)} |U(x) - (U)_{B_{\kappa\rho}(x_0)}| dx \\
&\leq C \Bigg[\kappa \left(\fint_{B_\rho(x_0)} |U(x)|^q dx \right)^{1/q} + \kappa^{-\frac{n}{q}} [a^{\#}_\rho(x_0)]^{\frac{1}{q}-\frac{1}{p}} \left(\fint_{B_\rho(x_0)} |D^2 u(x)|^p dx \right)^{1/p} \\
&\qquad + \lambda\alpha_0 \kappa^{-\frac{n}{q}} [\tilde{c}^{\#}_\rho(x_0)]^{\frac{1}{p}-\frac{1}{q}} \left(\fint_{B_\rho(x_0)} |u(x)|^p dx \right)^{1/p} + \kappa^{-n/q} \left(\fint_{B_\rho(x_0)} |f(x)|^q dx \right)^{1/q} \Bigg],
\end{aligned}$$

where $C = C(\Lambda, p, q, n)$, and $U = D^2 u$ or $\sqrt{\lambda\alpha_0} Du$ or $\lambda\alpha_0 u$ and $\tilde{c}(x) = \frac{c(x)}{\alpha_0}$.

Proof. Let us denote

$$F(x) = f(x) + \sum_{j,l=1}^{n} [a_{jk}(x) - (a_{jk})_{B_\rho(x_0)}] D_{jk} u(x) + \lambda[(c)_{B_\rho(x_0)} - c(x)] u(x),$$

where $(a_{jk})_{B_\rho(x_0)}$ and $(c)_{B_\rho(x_0)}$ are defined as in (1.5). Then, we see that u is a strong solution of

$$-(a_{jk})_{B_\rho(x_0)} D_{jk} u + \lambda (c)_{B_\rho(x_0)} u = F(x).$$

From (1.4), it follows that $(c_1)_{B_\rho(x_0)} \geq 0$ and $(c_1)_{B_\rho(x_0)} + (c_2)_{B_\rho(x_0)} \geq \alpha_0$. Then, by applying Lemma 2.9, we infer that

$$\fint_{B_{\kappa\rho}(x_0)} |U - (U)_{B_{\kappa\rho}(x_0)}| dx \leq C(\Lambda, q, n) \left[\kappa \left(\fint_{B_\rho(x_0)} |U(x)|^q dx \right)^{1/q} + \kappa^{-\frac{n}{q}} \left(\fint_{B_\rho(x_0)} |F(x)|^q dx \right)^{1/q} \right],$$

for every $\kappa \in (0, 1/4)$ and for $U = D^2 u$, $\sqrt{\lambda\alpha_0} Du$, or $\lambda\alpha_0 u$ and for some fixed $q \in (1, p)$. Now, observe that by using Hölder's inequality with the power $\frac{p}{p-q}$ and $\frac{p}{q}$ and by using the boundedness of the coefficients $(a_{kl})_{n\times n}$ in (1.1), we see that

$$\begin{aligned}
&\left(\fint_{B_\rho(x_0)} |(a_{jk})_{B_\rho(x_0)} - a_{jk}(x)|^q |D_{jk} u(x)|^q dx \right)^{1/q} \\
&\leq \left(\fint_{B_\rho(x_0)} |(a_{jk})_{B_\rho(x_0)} - a_{jk}(x)|^{\frac{qp}{p-q}} dx \right)^{\frac{p-q}{pq}} \left(\fint_{B_\rho(x_0)} |D_{jk} u(x)|^p dx \right)^{1/p} \\
&\leq C(\Lambda, p, q) \left(\fint_{B_\rho(x_0)} |(a_{jk})_{B_\rho(x_0)} - a_{jk}(x)| dx \right)^{\frac{p-q}{pq}} \left(\fint_{B_\rho(x_0)} |D_{jk} u(x)|^p dx \right)^{1/p} \\
&= C(\Lambda, p, q) [a^{\#}_\rho(x_0)]^{\frac{1}{q}-\frac{1}{p}} \left(\fint_{B_\rho(x_0)} |D^2 u(x)|^p dx \right)^{1/p}.
\end{aligned}$$

Similarly, using the fact that $\tilde{c}(x) = \frac{c(x)}{\alpha_0}$ is bounded above by a constant depending only on Λ, we also have

$$\begin{aligned}
\lambda \left(\fint_{B_\rho(x_0)} |(c)_{B_\rho(x_0)} - c(x)|^q |u(x)|^q dx \right)^{1/q} &= \lambda\alpha_0 \left(\fint_{B_\rho(x_0)} |(\tilde{c})_{B_\rho(x_0)} - \tilde{c}(x)|^q |u(x)|^q dx \right)^{1/q} \\
&\leq \lambda\alpha_0 C(\Lambda, p, q) [\tilde{c}^{\#}_\rho(x_0)]^{\frac{1}{p}-\frac{1}{q}} \left(\fint_{B_\rho(x_0)} |u(x)|^p dx \right)^{1/p}.
\end{aligned}$$

Collecting all of the above estimates, we obtain the desired result. □

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We first prove the a-priori estimate (1.7). Assume that $u \in W^{2,p}(\mathbb{R}^n, \mathbb{C})$ is a strong solution of the equation

$$\mathcal{L}_\lambda u(x) = f(x), \quad \text{for a.e.} \quad x \in \mathbb{R}^n. \tag{2.25}$$

By (1.4), the density of $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ in $W^{2,p}(\mathbb{R}^n, \mathbb{C})$, and by following the argument in the proof of Lemma 2.3, we prove (1.7) only for $u \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$. Recall that from (1.6), we have

$$a_\rho^{\#}(x) \leq \delta, \quad c_\rho^{\#}(x) \leq \delta\alpha_0, \quad \text{for a.e.} \quad x \in \mathbb{R}^n, \quad \forall\, \rho \in (0, R_0). \tag{2.26}$$

We follow the approach introduced in [21, 22] and split the proof of the estimate (1.7) into two steps.

Step I. We assume that $\text{spt}(u) \subset \mathbb{R}^{n-1} \times (\hat{x}_n - R_0\rho_0, \hat{x}_n + R_0\rho_0)$ for some $\rho_0 > 0$ sufficiently small that will be determined, and for some $\hat{x}_n \in \mathbb{R}$. For all $\rho \in (0, R_0)$, by applying Lemma 2.10 and (2.26), we infer

$$\begin{aligned} &\fint_{B_{\kappa\rho}(x)} |U(y) - (U)_{B_{\kappa\rho}(x)}| dy \\ &\leq C(\Lambda, q, n) \left[\kappa \mathcal{M}(|U|^q)(x)^{1/q} + \kappa^{-n/q} \mathcal{M}(|f|^q)(x)^{1/q}\right] \\ &\qquad + C(\Lambda, q, n)\kappa^{-n/q}\delta^{\frac{1}{q}-\frac{1}{p}} \left[\mathcal{M}(|D^2u|^p)(x)^{1/p} + \lambda\alpha_0 \mathcal{M}(|u|^p)(x)^{1/p}\right], \end{aligned} \tag{2.27}$$

for a.e. $x \in \mathbb{R}^n$, where $U = (D^2u, \sqrt{\lambda\alpha_0}Du, \lambda\alpha_0 u)$, $\kappa \in (0, 1/4)$ and some $q \in (1, p)$.

On the other hand, when $\rho \geq R_0$, we see that

$$\begin{aligned} \fint_{B_{\kappa\rho}(x)} |U(y) - (U)_{B_{\kappa\rho}(x)}| dy &\leq C(n)\kappa^{-n} \fint_{B_\rho(x)} |U(y)| dy \\ &\leq C(n)\kappa^{-n} \left(\fint_{B_\rho(x)} \mathbf{1}_{(\hat{x}_n - R_0\rho_0, \hat{x}_n + R_0\rho_0)}(y_n) dy\right)^{1-\frac{1}{q}} \left(\fint_{B_\rho(x)} |U(y)|^q dy\right)^{\frac{1}{q}} \\ &\leq C(n)\kappa^{-n} \left(\frac{R_0\rho_0}{\rho}\right)^{1-\frac{1}{q}} \left(\fint_{B_\rho(x)} |U(y)|^q dy\right)^{\frac{1}{q}} \\ &\leq C(n)\kappa^{-n} (\rho_0)^{1-\frac{1}{q}} \mathcal{M}(|U|^q)(x)^{\frac{1}{q}}. \end{aligned}$$

From this last estimate and (2.27), it follows that

$$\begin{aligned} U^{\#}(x) \leq C(\Lambda, q, n) &\left[\left(\kappa + \kappa^{-n}\rho_0^{1-\frac{1}{q}}\right)\mathcal{M}(|U|^q)(x)^{1/q} + \kappa^{-n/q}\mathcal{M}(|f|^q)(x)^{1/q}\right] \\ &+ C(\Lambda, q, n)\kappa^{-n/q}\delta^{\frac{1}{q}-\frac{1}{p}} \left[\mathcal{M}(|D^2u|^p)(x)^{1/p} + \lambda\alpha_0\mathcal{M}(|u|^p)(x)^{1/p}\right], \quad \forall\, x \in \mathbb{R}^n. \end{aligned}$$

Then, by using the Fefferman-Stein theorem for sharp functions, and Hardy-Littlewood maximal function theorem (see (2.19) and (2.20)), we obtain

$$
\begin{aligned}
\|U\|_{L^p(\mathbb{R}^n)} &\leq C(n,p)\left\|U^\#\right\|_{L^p(\mathbb{R}^n)} \\
&\leq C\left[\left(\kappa+\kappa^{-n}\rho_0^{1-\frac{1}{q}}\right)\left\|\mathcal{M}(|U|^q)^{1/q}\right\|_{L^p(\mathbb{R}^n)}+\kappa^{-\frac{n}{q}}\left\|\mathcal{M}(|f|^q)^{1/q}\right\|_{L^p(\mathbb{R}^n)}\right] \\
&\quad + C\kappa^{-\frac{n}{q}}\delta^{\frac{1}{q}-\frac{1}{p}}\left[\left\|\mathcal{M}(|D^2u|^p)^{1/p}\right\|_{L^p(\mathbb{R}^n)}+\lambda\alpha_0\left\|\mathcal{M}(|u|^q)^{1/q}\right\|_{L^p(\mathbb{R}^n)}\right] \\
&\leq C(\Lambda,q,n)\left[\left(\kappa+\kappa^{-n}\rho_0^{1-\frac{1}{q}}\right)\|U\|_{L^p(\mathbb{R}^n)}+\kappa^{-\frac{n}{2}}\|f\|_{L^p(\mathbb{R}^n)}\right] \\
&\quad + C(\Lambda,q,n)\kappa^{-\frac{n}{q}}\delta^{\frac{1}{q}-\frac{1}{p}}\left[\left\|D^2u\right\|_{L^p(\mathbb{R}^n)}+\lambda\alpha_0\|u\|_{L^p(\mathbb{R}^n)}\right] \\
&\leq C(\Lambda,q,n)\left[\left(\kappa+\kappa^{-n}\rho_0^{1-\frac{1}{q}}\right)\|U\|_{L^p(\mathbb{R}^n)}+\kappa^{-\frac{n}{2}}\|f\|_{L^p(\mathbb{R}^n)}\right] \\
&\quad + C(\Lambda,n)\kappa^{-\frac{n}{q}}\delta^{\frac{1}{q}-\frac{1}{p}}\|U\|_{L^p(\mathbb{R}^n)}.
\end{aligned} \tag{2.28}
$$

Now, we choose κ sufficiently small and then we choose ρ_0 sufficiently small so that

$$C(\Lambda,q,n)\left(\kappa+\kappa^{-n}\rho_0^{1-\frac{1}{q}}\right)<\frac{1}{2}.$$

From this and the estimate (2.28), it follows that

$$\|U\|_{L^p(\mathbb{R}^n)}\leq C(\Lambda,p,q,n)\|f\|_{L^p(\mathbb{R}^n)}+C(\Lambda,p,q,n)\delta^{\frac{1}{q}-\frac{1}{p}}\kappa^{-n/q}\|U\|_{L^p(\mathbb{R}^n)}.$$

Then, with the choice of δ so that it is sufficiently small depending only on Λ, n, p, we can deduce from the last estimate that

$$\|U\|_{L^p(\mathbb{R}^n)}\leq C(\Lambda,p,q,n)\|f\|_{L^p(\mathbb{R}^n)}.$$

This proves (1.7) when $\mathrm{spt}(u)\subset\mathbb{R}^{n-1}\times(\hat{x}_n-R_0\rho_0,\hat{x}_n+R_0\rho_0)$.

Step II. We remove the condition on the smallness of the support of the solution u and proving (1.7) for $\lambda>\frac{N_0}{\alpha_0R_0^2}$ with some sufficiently large constant $N_0>0$ that depends only on n,Λ,p. The essential idea is to use the partition of unity. Let $\rho_0>0$ be the number defined in **Step I** which depends only on n,q,Λ. Let $\xi\in C_0^\infty(-R_0\rho_0,R_0\rho_0)$ be the standard non-negative cut-off function satisfying

$$\int_{\mathbb{R}}\xi^p(s)ds=1,\quad |\xi'|\leq\frac{2}{R_0\rho_0},\quad \text{and}\quad |\xi''|\leq\frac{4}{(R_0\rho_0)^2}. \tag{2.29}$$

For any $s\in\mathbb{R}$, let $w_s(x)=u(x)\xi(x_n-s)$ for every $x=(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$. We observe that for each s, w_s is a solution of

$$\mathcal{L}_\lambda w_s(x)=F_s(x)\quad x\in\mathbb{R}^n,$$

where

$$
\begin{aligned}
F_s(x)&=f(x)\xi(x_n-s)+2\sum_{l=1}^{n-1}a_{nl}(x)D_lu\xi'(x_n-s) \\
&\quad + a_{nn}(x)\Big[2D_nu(x)\xi'(x_n-s)+u(x)\xi''(x_n-s)\Big],\quad x=(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}.
\end{aligned} \tag{2.30}
$$

As $\mathrm{spt}(w_s)\in\mathbb{R}^{n-1}\times(s-R_0\rho_0,s+R_0\rho_0)$, we can apply the result in **Step I** to conclude that

$$\left\|D^2w_s\right\|_{L^p(\mathbb{R}^n)}+\sqrt{\lambda\alpha_0}\|Dw_s\|_{L^p(\mathbb{R}^n)}+\lambda\alpha_0\|w_s\|_{L^p(\mathbb{R}^n)}\leq C(\Lambda,p,n)\|F_s\|_{L^p(\mathbb{R}^n)}. \tag{2.31}$$

Now, we observe that for each multi-index $\sigma \in (\mathbb{N} \cup \{0\})^n$, it follows from the first identity in (2.29) that

$$|D^\sigma u(x)|^p = \int_{\mathbb{R}} |D^\sigma u(x)|^p \xi^p(x_n - s) ds, \quad \text{for a.e. } x \in \mathbb{R}^n.$$

From this and the Fubini's theorem, we find that

$$\left\|D^\sigma u\right\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}} \left[\int_{\mathbb{R}^n} |D^\sigma u(x)|^p \xi^p(x_n - s) dx\right] ds.$$

On the other hand, observe that

$$|Du|\xi \leq C\Big[|Dw_s| + |u||\xi'|\Big], \quad |D^2 u|\xi \leq C\Big(|D^2 w_s| + 2|Du||\xi'| + |u||\xi''|\Big).$$

Then, we deduce that

$$\|Du\|_{L^p(\mathbb{R}^n)}^p \leq C(p)\left[\int_{\mathbb{R}} \|Dw_s\|_{L^p(\mathbb{R}^n)}^p ds + (R_0\rho_0)^{-p}\|u\|_{L^p(\mathbb{R}^n)}^p\right]$$

and

$$\left\|D^2 u\right\|_{L^p(\mathbb{R}^n)}^p \leq C(p)\left[\int_{\mathbb{R}} \|D^2 w_s\|_{L^p(\mathbb{R}^n)}^p ds + (R_0\rho_0)^{-p}\|Du\|_{L^p(\mathbb{R}^n)}^p + (R_0\rho_0)^{-2p}\|u\|_{L^p(\mathbb{R}^n)}^p\right].$$

As a consequence, we obtain

$$\begin{aligned}
&\left\|D^2 u\right\|_{L^p(\mathbb{R}^n)}^p + \Big(\sqrt{\lambda\alpha_0}\,\|Du\|_{L^p(\mathbb{R}^n)}\Big)^p + \Big(\lambda\alpha_0\|u\|_{L^p(\mathbb{R}^n)}\Big)^p \\
&\leq C(p)\left[\int_{\mathbb{R}} \|D^2 w_s\|_{L^p(\mathbb{R}^n)}^p ds + (\lambda\alpha_0)^{\frac{p}{2}} \int_{\mathbb{R}} \|Dw_s\|_{L^p(\mathbb{R}^n)}^p ds + (\lambda\alpha_0)^p \int_{\mathbb{R}} \|w_s\|_{L^p(\mathbb{R}^n)}^p ds\right] \\
&\quad + C(p,\rho_0) R_0^{-2p}\left[R_0^p\|Du\|_{L^p(\mathbb{R}^n)}^p + \{R_0^p(\lambda\alpha_0)^{\frac{p}{2}} + 1\}\|u\|_{L^p(\mathbb{R}^n)}^p\right].
\end{aligned}$$

From this last estimate and (2.31), we obtain

$$\begin{aligned}
&\left\|D^2 u\right\|_{L^p(\mathbb{R}^n)} + \sqrt{\lambda\alpha_0}\,\|Du\|_{L^p(\mathbb{R}^n)} + \lambda\alpha_0\|u\|_{L^p(\mathbb{R}^n)} \\
&\leq C(\Lambda, p, n)\left(\int_{\mathbb{R}} \|F_s\|_{L^p(\mathbb{R}^n)}^p ds\right)^{\frac{1}{p}} \\
&\qquad + C(\Lambda, p, \rho_0, n) R_0^{-2}\left[R_0\|Du\|_{L^p(\mathbb{R}^n)} + \Big(R_0\sqrt{\lambda\alpha_0} + 1\Big)\|u\|_{L^p(\mathbb{R}^n)}\right],
\end{aligned}$$

From this estimate, the definition of F_s in (2.30), and the fact that $R_0 \in (0, 1)$, we infer that

$$\begin{aligned}
&\left\|D^2 u\right\|_{L^p(\mathbb{R}^n)} + \sqrt{\lambda\alpha_0}\,\|Du\|_{L^p(\mathbb{R}^n} + \lambda\alpha_0\|u\|_{L^p(\mathbb{R}^n)} \\
&\leq C(\Lambda, p, n)\,\|f\|_{L^p(\mathbb{R}^n)} + C_*(\Lambda, n, p) R_0^{-2}\left[R_0\|Du\|_{L^p(\mathbb{R}^n)} + \Big(R_0\sqrt{\lambda\alpha_0} + 1\Big)\|u\|_{L^p(\mathbb{R}^n)}\right],
\end{aligned} \tag{2.32}$$

where in the last step, we have used the fact that ρ_0 is a constant depending only on Λ, n, p. Now, let $N_0 = 16C_*^2$, where C_* is the constant defined in the right hand side of (2.32) which can be assumed to be greater than one. Then, with $\gamma_0 := N_0 R_0^{-2}$, we easily deduce that

$$C_*(\Lambda, n, p) R_0^{-1} \leq \frac{\sqrt{\gamma_0}}{2} \quad \text{and} \quad C_*(\Lambda, n, p)\Big(R_0^{-1}\sqrt{\gamma} + R_0^{-2}\Big) \leq \frac{\gamma}{2}, \quad \gamma \geq \gamma_0,$$

From this and (2.32), we conclude that

$$\left\|D^2 u\right\|_{L^p(\mathbb{R}^n)} + \sqrt{\lambda\alpha_0}\,\|Du\|_{L^p(\mathbb{R}^n)} + \lambda\alpha_0\|u\|_{L^p(\mathbb{R}^n)} \leq C(\Lambda, n, p)\,\|f\|_{L^p(\mathbb{R}^n)}$$

for all $\lambda > \frac{\gamma_0}{\alpha_0} = \frac{N_0}{\alpha_0 R_0^2}$. This completes the proof of (1.7).

It now remains to prove the existence and uniqueness of solutions of (1.2). Observe that the uniqueness of solution $u \in W^{2,p}(\mathbb{R}^n, \mathbb{C})$ follows from the a-priori estimate (1.7) that we just proved. Therefore, we only prove the solvability of (1.2) in $W^{2,p}(\mathbb{R}^n, \mathbb{C})$. We use the method of continuity (see [22, Theorem 1.4.4, p. 15 and Theorem 6.4.1 p. 139] for instance). As this is standard, we only provide some important steps in the proof. For fixed $\lambda > \frac{N_0}{\alpha_0 R_0^2}$ and for each $\mu \in [0, 1]$, we define the operator

$$\mathcal{T}_\mu u = \mu \mathcal{L}_\lambda u + (1-\mu)[-\Delta u + \lambda \alpha_0 u].$$

By simple calculations, we see that

$$\mathcal{T}_\mu u = -\sum_{k,l=1}^n \tilde{a}_{kl}(x) D_{kl} u(x) + \lambda \tilde{c}(x) u(x),$$

where

$$\tilde{a}_{kl}(x) = \mu a_{kl}(x) + (1-\mu)\delta_{kl}, \quad \tilde{c}(x) = \mu c(x) + (1-\mu)\alpha_0$$

with $\delta_{kl} = 0$ for $k \neq l$ and $\delta_{kk} = 1$, for $k, l = 1, 2, \cdots, n$. Observe that the new coefficients $(\tilde{a}_{kl})_{k,l=1}^n$ and $\tilde{c}$ satisfy the conditions (1.1), (1.4), and (1.6). Therefore, by the a-priori estimate (1.7) that we just proved, there is a constant $C = C(\Lambda, n, R_0, p)$ independent on μ such that

$$\left\|D^2 u\right\|_{L^p(\mathbb{R}^n)} + \sqrt{\lambda \alpha_0}\, \|Du\|_{L^p(\mathbb{R}^n)} + \lambda \alpha_0 \|u\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)},$$

where $u \in W^{2,p}(\mathbb{R}^n, \mathbb{C})$ is a solution of

$$\mathcal{T}_\mu u = f \quad \text{in} \quad \mathbb{R}^n$$

and for $\mu \in [0, 1]$. On the other hand, by Theorem 2.2, we see that for every $f \in L^p(\mathbb{R}^n, \mathbb{C})$, there exists unique solution $u \in W^{2,p}(\mathbb{R}^n, \mathbb{C})$ of the equation

$$\mathcal{T}_0 u = f \quad \text{in} \quad \mathbb{R}^n.$$

Hence, by the method of continuity (see [22, Theorem 1.4.4, p. 15 and Theorem 6.4.1 p. 139] for details), for every $f \in L^p(\mathbb{R}^n, \mathbb{C})$, there is a solution $u \in W^{2,p}(\mathbb{R}^n, \mathbb{C})$ of the equation

$$\mathcal{T}_1 u = f \quad \text{in} \quad \mathbb{R}^n.$$

As $\mathcal{T}_1 = \mathcal{L}_\lambda$, the proof of the theorem is completed. □

3. Schrödinger equations in divergence form

In this section, we prove Theorem 1.2. The proof is similar to that of Theorem 1.1 using equations with frozen coefficients and Fefferman-Stein sharp functions, see also [9, 10, 21–23]. To take advantage of the imaginary part of the potentials, we freeze the spatial variables of the potentials.

3.1. **Equations with constant coefficients.** This section derives basic estimates for solutions of second order divergence form elliptic equations with constant complex coefficients. We consider the following equation

$$-\sum_{j,l=1}^n D_l[a_{jl} D_j u] + \lambda[c_1 + ic_2]u = \lambda f + \operatorname{div}[g(x)], \quad \text{in} \quad \mathbb{R}^n, \tag{3.1}$$

where $\lambda > 0$ is a constant, $f : \mathbb{R}^n \to \mathbb{C}$ is a given measurable function, $g = (g_1, g_2, \cdots, g_n) : \mathbb{R}^n \to \mathbb{C}^n$ is a given measurable vector field, and $u : \mathbb{R}^n \to \mathbb{C}$ is an unknown solution. Moreover, $(a_{jl})_{n\times n}$ is a given $n\times n$ matrix of complex numbers, and c_1, c_2, λ are given constants. We say that $u \in W^{1,p}(\mathbb{R}^n, \mathbb{C})$ is a weak solution of (3.1) if

$$\begin{aligned}
&\sum_{j,l=1}^n \int_{\mathbb{R}^n} a_{jl} D_j u(x) \overline{D_l \varphi(x)} dx + \lambda \int_{\mathbb{R}^b} [c_1 + i c_2] u(x) \overline{\varphi(x)} dx \\
&= \lambda \int_{\mathbb{R}^n} f(x) \overline{\varphi(x)} dx - \sum_{k=1}^n \int_{\mathbb{R}^n} g_k(x) \cdot \overline{D_k \varphi(x)} dx,
\end{aligned}$$

for every $\varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$.

The main result of this subsection is the following theorem which is a special case of Theorem 1.2 when coefficients are constants.

Theorem 3.1. *Let $\Lambda \in (0,1), \alpha_0 > 0$ and $(a_{kl})_{n\times n}$ be a matrix of complex numbers satisfying the conditions in* (1.1)*. Then, for every real numbers c_1, c_2 satisfying* (2.2) *and for $\lambda > 0$, $f \in L^p(\mathbb{R}^n, \mathbb{C})$, $g \in L^p(\mathbb{R}^n, \mathbb{C})^n$ with some $p \in (1, \infty)$, there exists a unique weak solution $u \in W^{1,p}(\mathbb{R}^n, \mathbb{C})$ of* (3.1)*. Moreover,*

$$\|Du\|_{L^p(\mathbb{R}^n)} + \sqrt{\lambda \alpha_0} \|u\|_{L^p(\mathbb{R}^n)} \leq C(\Lambda, p, n) \Big[\sqrt{\frac{\lambda}{\alpha_0}} \|f\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)} \Big]. \tag{3.2}$$

As Theorem 3.1 is new and important in our approach, we prove it in the remaining part of the section. We start with the proof of Theorem 3.1 when $p = 2$ in the following lemma.

Lemma 3.2. *Let $\Lambda \in (0,1)$, $\alpha_0 > 0$ and assume that the matrix $(a_{kl})_{n\times n}$ of complex numbers satisfies* (1.1)*. Moreover, let c_1, c_2 be real numbers satisfying* (2.2) *and $\lambda > 0$ be a given number. Then, for every $f \in L^2(\mathbb{R}^n, \mathbb{C})$ and $g \in L^2(\mathbb{R}^n, \mathbb{C})^n$, there exists unique weak solution $u \in W^{1,2}(\mathbb{R}^n, \mathbb{C})$ of* (3.1)*. Moreover,*

$$\|Du\|_{L^2(\mathbb{R}^n)} + \sqrt{\lambda \alpha_0} \|u\|_{L^2(\mathbb{R}^n)} \leq C(\Lambda) \Big[\sqrt{\frac{\lambda}{\alpha_0}} \|f\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} \Big].$$

Proof. The proof is similar to that of Lemma 2.3 and we only provide main steps. The existence and uniqueness of weak solution can be done exactly as that in the proof of Lemma 2.3. Therefore, it remains to prove the estimate in the lemma. By using density, we can assume that u, f are smooth and compactly supported. By using $\overline{u}$ as a test function for (3.1), we obtain

$$\begin{aligned}
&\sum_{j,l=1}^n \int_{\mathbb{R}^n} a_{jl} D_l u D_j \bar{u} dx + \lambda [c_1 + i c_2] \int_{\mathbb{R}^n} |u|^2 dx \\
&= \lambda \int_{\mathbb{R}^n} f(x) \bar{u}(x) dx - \sum_{k=1}^n \int_{\mathbb{R}^n} g_k(x) \overline{D_k u(x)} dx.
\end{aligned} \tag{3.3}$$

Now, let $\epsilon > 0$ be sufficiently small which will be determined. By taking the real part of (3.3) and using Lemma 2.1, we see that

$$\begin{aligned}
&\Lambda \int_{\mathbb{R}^n} |Du(x)|^2 dx + \lambda c_1 \int_{\mathbb{R}^n} |u(x)|^2 dx \\
&\leq \lambda \int_{\mathbb{R}^n} |f(x)||u(x)| dx + \int_{\mathbb{R}^n} |g(x)||Du(x)| dx \\
&\leq C(\Lambda) \left[\lambda \alpha_0 \epsilon \int_{\mathbb{R}^n} |u(x)|^2 dx + \frac{\lambda}{\alpha_0 \epsilon} \int_{\mathbb{R}^n} |f(x)|^2 dx + \int_{\mathbb{R}^n} |g(x)|^2 dx \right] + \frac{\Lambda}{2} \int_{\mathbb{R}^n} |Du(x)|^2 dx.
\end{aligned}$$

Therefore, by cancelling similar terms, we obtain

$$\begin{aligned}
&\frac{\Lambda}{2}\int_{\mathbb{R}^n}|Du(x)|^2dx + \lambda c_1\int_{\mathbb{R}^n}|u(x)|^2dx\\
&\leq C(\Lambda)\left[\lambda\alpha_0\epsilon\int_{\mathbb{R}^n}|u(x)|^2dx + \frac{\lambda}{\alpha_0\epsilon}\int_{\mathbb{R}^n}|f(x)|^2dx + \int_{\mathbb{R}^n}|g(x)|^2dx\right].
\end{aligned}$$

From this and as $c_1 \geq 0$, we infer that

$$\begin{aligned}
\int_{\mathbb{R}^n}|Du(x)|^2dx &\leq C(\Lambda)\left[\lambda\alpha_0\epsilon\int_{\mathbb{R}^n}|u(x)|^2dx + \frac{\lambda}{\alpha_0\epsilon}\int_{\mathbb{R}^n}|f(x)|^2dx + \int_{\mathbb{R}^n}|g(x)|^2dx\right],\\
\lambda c_1\int_{\mathbb{R}^n}|u(x)|^2dx &\leq C(\Lambda)\left[\lambda\alpha_0\epsilon\int_{\mathbb{R}^n}|u(x)|^2dx + \frac{\lambda}{\alpha_0\epsilon}\int_{\mathbb{R}^n}|f(x)|^2dx + \int_{\mathbb{R}^n}|g(x)|^2dx\right]
\end{aligned}\tag{3.4}$$

Also, by taking a imaginary part of (3.3) and by using the boundedness condition of $(a_{kl})_{n\times n}$ in (1.1) and Young's inequality, we obtain

$$\begin{aligned}
\lambda c_2\int_{\mathbb{R}^n}|u|^2dx &\leq \Lambda^{-1}\int_{\mathbb{R}^n}|\nabla u|^2dx + \lambda\int_{\mathbb{R}^n}|f(x)||u(x)|dx + \int_{\mathbb{R}^n}|g(x)||Du(x)|dx\\
&\leq [\Lambda^{-1}+1]\int_{\mathbb{R}^n}|Du|^2dx + \frac{\lambda\alpha_0\epsilon}{2}\int_{\mathbb{R}^n}|u(x)|^2dx + \frac{\lambda}{2\alpha_0\epsilon}\int_{\mathbb{R}^n}|f(x)|^2dx + \int_{\mathbb{R}^n}|g(x)|^2dx.
\end{aligned}$$

Then, combining the last estimate with (3.4), we can derive the following estimate

$$\lambda(c_1+c_2)\int_{\mathbb{R}^n}|u(x)|^2dx \leq C(\Lambda)\left[\lambda\alpha_0\epsilon\int_{\mathbb{R}^n}|u(x)|^2dx + \frac{\lambda}{\alpha_0\epsilon}\int_{\mathbb{R}^n}|f(x)|^2dx + \int_{\mathbb{R}^n}|g(x)|^2dx\right].$$

Since $c_1 + c_2 \geq \alpha_0$, we can choose ϵ such that $C(\Lambda)\epsilon = 1/2$ to obtain

$$\lambda\alpha_0\int_{\mathbb{R}^n}|u|^2dx \leq C(\Lambda)\left[\frac{\lambda}{\alpha_0}\int_{\mathbb{R}^n}|f(x)|^2dx + \int_{\mathbb{R}^n}|g(x)|^2dx\right].\tag{3.5}$$

From (3.4) and (3.5), it follows that

$$\int_{\mathbb{R}^n}|Du(x)|^2dx \leq C(\Lambda)\left[\frac{\lambda}{\alpha_0}\int_{\mathbb{R}^n}|f(x)|^2dx + \int_{\mathbb{R}^n}|g(x)|^2dx\right].$$

This last estimate and (3.5) imply our desired estimate. □

We next state and prove an important result similar to that of Lemma 2.5.

Lemma 3.3. *Let $\Lambda \in (0,1)$ and $q \in (1,\infty)$ be fixed. Then, there is $C_0 = C(\Lambda, q, n) > 0$ such that the following statement holds true. Suppose that $\rho > 0$ and assume that the conditions in* (1.1) *hod for the matrix of complex numbers $(a_{kl})_{n\times n}$. Assume also that* (2.2) *holds for two real numbers c_1 and c_2. Suppose also that $u \in W^{1,2}(B_\rho,\mathbb{C})$ is a weak solution of*

$$-\sum_{j,l=1}^{n}D_j[a_{jl}D_lu] + \lambda[c_1+ic_2]u = 0 \quad in \quad B_\rho,$$

with some $\lambda > 0$. Then, for every $\kappa \in (0,1/2)$, the following estimate hold

$$\begin{aligned}
&\fint_{B_{\kappa\rho}}|Du-(Du)_{B_{\kappa\rho}}|dx \leq \kappa C_0\left(\fint_{B_\rho}|Du(x)|^qdx\right)^{1/q},\\
&\fint_{B_{\kappa\rho}}|u-(u)_{B_{\kappa\rho}}|dx \leq \kappa C_0\left(\fint_{B_\rho}|u(x)|^qdx\right)^{1/q}.
\end{aligned}$$

Proof. The proof is similar to that of Lemma 2.5. Note that by standard regularity theory, $u \in C^\infty(B_\rho, \mathbb{C})$. Therefore, we can apply Lemma 2.4. Because of this, the proof is now the same as that of Lemma 2.5. □

Our next lemma gives the mean oscillation estimates for solutions u of the equation (3.1), which is the same fashion as that of Lemma 2.6.

Lemma 3.4. *For a given constant $\Lambda \in (0,1)$, there exists $C = C(\Lambda, n)$ such that the following statement holds. Suppose that the matrix $(a_{jl})_{n\times n}$ of complex numbers satisfies* (1.1). *Suppose also that two given numbers c_1, c_2 satisfy* (2.2), *$f \in L^2(B_\rho(x_0), \mathbb{C}), g \in L^2(B_\rho(x_0), \mathbb{C})^n$. Then, if $u \in W^{1,2}(B_\rho(x_0), \mathbb{C})$ is a weak solution of*

$$-\sum_{j,l=1}^{n} D_l[a_j D_j u] + \lambda[c_1 + ic_2]u = \lambda f + \text{div}[g], \quad in \quad B_\rho(x_0)$$

for some $x_0 \in \mathbb{R}^n$, some $\rho > 0$ and some $\lambda > 0$, the following estimates hold

$$\begin{aligned}
&\fint_{B_{\kappa\rho}(x_0)} |U - (U)_{B_{\kappa\rho}(x_0)}| dx \\
&\leq C\left[\kappa\left(\fint_{B_\rho(x_0)} |U(x)|^2 dx\right)^{1/2} + \kappa^{-\frac{n}{2}}\sqrt{\lambda\alpha_0^{-1}}\left(\fint_{B_\rho(x_0)} |f(x)|^2 dx\right)^{1/2} + \kappa^{-\frac{n}{2}}\left(\fint_{B_\rho(x_0)} |g(x)|^2 dx\right)^{1/2}\right],
\end{aligned}$$

for every $\kappa \in (0, 1/4)$ and for $U = Du$ or for $U = \sqrt{\lambda\alpha_0}u$.

Proof. The proof is similar to that of Lemma 2.6, but instead we useLemma 3.2 and Lemma 3.3. By using the translation $x \mapsto x - x_0$, we can assume that $x_0 = 0$. Let $\eta \in C_0^\infty(B_\rho)$ be a standard cut-off function which satisfies

$$\eta = 1, \quad \text{on} \quad B_{\rho/2}.$$

Then, let $w \in W^{1,2}(\mathbb{R}^n, \mathbb{C})$ be the solution of the equation

$$-\sum_{j,l=1}^{n} D_l[a_{jl} D_j w] + \lambda[c_1 + ic_2]w = \eta(x)\lambda f(x) + \text{div}[\eta(x) g(x)] \quad \text{in} \quad \mathbb{R}^n, \tag{3.6}$$

whose existence is assured by Lemma 3.2. Then, by writing $W = (Dw, \sqrt{\lambda\alpha_0}w)$, we can infer from Lemma 3.2 that

$$\begin{aligned}
&\left(\fint_{B_{\kappa\rho}} |W|^2 dx\right)^{1/2} \leq \frac{C(\Lambda, n)}{\kappa^{\frac{n}{2}}}\left[\sqrt{\lambda\alpha_0^{-1}}\left(\fint_{B_\rho} |f(x)|^2 dx\right)^{1/2} + \left(\fint_{B_\rho} |g(x)|^2 dx\right)^{1/2}\right], \quad \text{and} \\
&\left(\fint_{B_\rho} |W|^2 dx\right)^{1/2} \leq C(\Lambda, n)\left[\sqrt{\lambda\alpha_0^{-1}}\left(\fint_{B_\rho} |f(x)|^2 dx\right)^{1/2} + \left(\fint_{B_\rho} |g(x)|^2 dx\right)^{1/2}\right].
\end{aligned} \tag{3.7}$$

Now, let $v = u - w$ and we see that v is a weak solution of the equation

$$-\sum_{j,l=1}^{n} D_l[a_{jl} D_j v] + \lambda[c_1 + ic_2]v = 0 \quad \text{in} \quad B_{\rho/2}.$$

Again, by writing $V = (Dv, \sqrt{\lambda\alpha_0}v)$, we can apply Lemma 3.3 for V to see that

$$\fint_{B_{\kappa\rho}} |V(x) - (V)_{B_{\kappa\rho}}| dx \leq \kappa C_0 \left(\fint_{B_{\rho/2}} |V(x)|^2 dx\right)^{1/2}, \quad \forall\, \kappa \in (0, 1/4). \tag{3.8}$$

Recall that

$$\fint_{B_{\kappa\rho}} |U - (U)_{B_{\kappa\rho}}| dx \leq 2 \fint_{B_{\kappa\rho}} |U - c| dx, \quad \forall c \in \mathbb{R}.$$

Then, by taking $c = (V)_{B_{\kappa\rho}}$, and using the triangle inequality and Hölder's inequality, we see that

$$\begin{aligned}\fint_{B_{\kappa\rho}} |U - (U)_{B_{\kappa\rho}}| dx &\leq 2 \fint_{B_{\kappa\rho}} |U - (V)_{B_{\kappa\rho}}| dx \\ &\leq 2 \left[\fint_{B_{\kappa\rho}} |V - (V)_{B_{\kappa\rho}}| dx + \left(\fint_{B_{\kappa\rho}} |W|^2 dx \right)^{1/2} \right].\end{aligned}$$

From this, the first estimate in (3.7), and from (3.8), we see that

$$\begin{aligned}&\fint_{B_{\kappa\rho}} |U - (U)_{B_{\kappa\rho}}| dx \\ &\leq C \left[\kappa \left(\fint_{B_{\rho/2}} |V(x)|^2 dx \right)^{1/2} + \kappa^{-\frac{n}{2}} \sqrt{\lambda \alpha_0^{-1}} \left(\fint_{B_\rho} |f(x)|^2 dx \right)^{1/2} + \kappa^{-\frac{n}{2}} \left(\fint_{B_\rho} |g(x)|^2 dx \right)^{1/2} \right] \\ &\leq C \left[\kappa \left(\fint_{B_{\rho/2}} |U(x)|^2 dx \right)^{1/2} + \kappa \left(\fint_{B_{\rho/2}} |W(x)|^2 dx \right)^{1/2} \right. \\ &\quad \left. + \kappa^{-\frac{n}{2}} \sqrt{\lambda \alpha_0^{-1}} \left(\fint_{B_\rho} |f(x)|^2 dx \right)^{1/2} + \kappa^{-\frac{n}{2}} \left(\fint_{B_\rho} |g(x)|^2 dx \right)^{1/2} \right].\end{aligned}$$

Now, using the second estimates in (2.23), we can control the second term on the right hand side of the last estimate and infer that

$$\begin{aligned}&\fint_{B_{\kappa\rho}} |U - (U)_{B_{\kappa\rho}}| dx \\ &\leq C \left[\kappa \left(\fint_{B_\rho} |U(x)|^2 dx \right)^{1/2} + \kappa^{-\frac{n}{2}} \sqrt{\lambda \alpha_0^{-1}} \left(\fint_{B_\rho} |f(x)|^2 dx \right)^{1/2} + \kappa^{-\frac{n}{2}} \left(\fint_{B_\rho} |g(x)|^2 dx \right)^{1/2} \right],\end{aligned}$$

where C is a constant depending only on Λ and n. The proof of the lemma is therefore completed. □

Proof of Theorem 3.1. From Lemma 3.4, the proof of Theorem 3.1 follows exactly the same as that of Theorem 2.2. We therefore skip it. □

Remark 3.5. *As in Remark 2.8, Theorem 3.1 is still valid if the second condition in* (2.2) *is replaced by* $c_1 + |c_2| \geq \alpha_0$.

3.2. **Equations with measurable coefficients.** This section provides the proof of Theorem 1.2. To this end, we first apply Theorem 3.1 to establish an improved version of Lemma 3.4.

Lemma 3.6. *For a given constant* $\Lambda \in (0,1)$ *and* $q \in (1,\infty)$*, there exists* $C = C(\Lambda, q, n)$ *such that the following statement holds. Suppose that the matrix* $(a_{jl})_{n\times n}$ *of complex numbers satisfies the conditions in* (1.1)*. Suppose also that* c_1, c_2 *are fixed numbers satisfying* (2.2)*, and* $f \in L^q(B_\rho(x_0), \mathbb{C}), g \in L^q(B_\rho(x_0), \mathbb{C})^n$*. Then, if* $u \in W^{1,q}(B_\rho(x_0))$ *is a weak solution of*

$$-\sum_{j,l=1}^n D_l[a_{jl} D_j u] + \lambda[c_1 + ic_2] u = \lambda f + \mathrm{div}[g] \quad in \quad B_\rho(x_0),$$

for some $x_0 \in \mathbb{R}^n$, some $\rho > 0$ and some $\lambda > 0$, the following estimates hold

$$
\begin{aligned}
&\fint_{B_{\kappa\rho}(x_0)} |U - (U)_{B_{\kappa\rho}(x_0)}| dx \\
&\leq C \left[\kappa \left(\fint_{B_\rho(x_0)} |U(x)|^q dx \right)^{1/q} + \kappa^{-\frac{n}{q}} \sqrt{\lambda \alpha_0^{-1}} \left(\fint_{B_\rho(x_0)} |f(x)|^q dx \right)^{1/q} + \left(\fint_{B_\rho(x_0)} |g(x)|^q dx \right)^{1/q} \right],
\end{aligned}
$$

for every $\kappa \in (0, 1/4)$ and for $U = Du$ or for $U = \sqrt{\lambda\alpha_0} u$.

Proof. Similar to that of Lemma 2.9, but we use Theorem 3.1 and Lemma 3.3 instead of Theorem 2.2 and Lemma 2.5. We therefore skip the proof. □

From Lemma 3.6 we can establish the following result on the mean oscillation of solutions that is similar to Lemma 2.10.

Lemma 3.7. *Let $\Lambda \in (0, 1)$ and assume that (1.1) and (1.4) hold. Let $q \in (1, \infty)$, $p \in (q, \infty)$ and assume that $f \in L^q(B_\rho(x_0), \mathbb{C})$, $g \in L^q((B_\rho(x_0), \mathbb{C}))^n$ and assume also that $u \in W^{1,p}(B_\rho(x_0), \mathbb{C})$ is a weak solution of*

$$
-\sum_{j,k=1}^{n} D_k[a_{jk}(x) D_j u] + \lambda[c_1(x) + i c_2(x)] u(x) = \lambda f(x) + \operatorname{div}[g], \quad \text{in} \quad B_\rho(x_0),
$$

with some $\lambda > 0$. Then, for every $\kappa \in (0, 1/4)$, it holds that

$$
\begin{aligned}
&\fint_{B_{\kappa\rho}(x_0)} |U(x) - (U)_{B_{\kappa\rho}(x_0)}| dx \\
&\leq C \left[\kappa \left(\fint_{B_\rho(x_0)} |U(x)|^q dx \right)^{1/q} + \kappa^{-\frac{n}{q}} [a_\rho^\#(x_0)]^{\frac{1}{q} - \frac{1}{p}} \left(\fint_{B_\rho(x_0)} |Du(x)|^p dx \right)^{1/p} \right. \\
&\quad + \sqrt{\lambda\alpha_0} \kappa^{-\frac{n}{q}} [\tilde{c}_\rho^\#(x_0)]^{\frac{1}{p} - \frac{1}{q}} \left(\fint_{B_\rho(x_0)} |u(x)|^p dx \right)^{1/p} + \kappa^{-n/q} \sqrt{\lambda \alpha_0^{-1}} \left(\fint_{B_\rho(x_0)} |f(x)|^q dx \right)^{1/q} \\
&\quad \left. + \kappa^{-n/q} \left(\fint_{B_\rho(x_0)} |g(x)|^q dx \right)^{1/q} \right],
\end{aligned}
$$

where $C = C(\Lambda, p, q, n)$, $U = Du$ or $U = \sqrt{\lambda\alpha_0} u$, and $\tilde{c}(x) = \frac{c(x)}{\alpha_0}$.

Proof. Let us denote

$$
\tilde{f}(x) = f(x) + [(c)_{B_\rho(x_0)} - c(x)] u(x), \quad \tilde{g}_k = g_k(x) + \sum_{j=1}^{n} [a_{jk}(x) - (a_{jk})_{B_\rho(x_0)}] D_j u(x).
$$

Then, we see that u is a weak solution of

$$
-\sum_{k=1}^{n} D_k[(a_{jk})_{B_\rho(x_0)} D_j u] + \lambda (c)_{B_\rho(x_0)} u = \lambda \tilde{f}(x) + \operatorname{div}[\tilde{g}(x)].
$$

From (1.4), it follows that $(c_1)_{B_\rho(x_0)} \geq 0$ and $(c_1)_{B_\rho(x_0)} + (c_2)_{B_\rho(x_0)} \geq \alpha_0$. Then, we can apply Lemma 3.6 to infer that

$$
\begin{aligned}
&\fint_{B_{\kappa\rho}(x_0)} |U - (U)_{B_{\kappa\rho}(x_0)}| dx \\
&\leq C(\Lambda, n) \left[\kappa \left(\fint_{B_\rho} |U(x)|^q dx \right)^{1/q} + \kappa^{-\frac{n}{q}} \sqrt{\lambda \alpha_0^{-1}} \left(\fint_{B_\rho} |\tilde{f}(x)|^q dx \right)^{1/q} + \kappa^{-\frac{n}{q}} \left(\fint_{B_\rho} |\tilde{g}(x)|^q dx \right)^{1/q} \right],
\end{aligned}
$$

for every $\kappa \in (0, 1/4)$ and for $U = Du$ or $U = \sqrt{\lambda\alpha_0}u$. From this, the proof can be done exactly the same as that of Lemma 2.9 and we then skip it. □

Proof of Theorem 1.2. The proof is similar to that of Theorem 1.1 and we only outline some main steps. As in the proof of Theorem 1.1, it is sufficient to prove the estimate (1.9) for $u \in C^\infty(\mathbb{R}^n, \mathbb{C})$.

Step I. We prove that there is $\rho_0 = \rho_0(\Lambda, n, p) > 0$ and sufficiently small such that if $\mathrm{spt}(u) \subset \mathbb{R}^{n-1} \times (\hat{x}_n - R_0\rho_0, \hat{x}_n + R_0\rho_0)$ for some $\hat{x}_n \in \mathbb{R}$, then (1.9) holds. The proof of this claim can be done exactly the same as that of **Step I** in the proof of Theorem 1.1 in which we use Lemma 3.7 instead of Lemma 2.10 .

Step II. We use partition of unity to remove the condition on the smallness of the support of the solution u. Let $\xi \in C_0^\infty(-R_0\rho_0, R_0\rho_0)$ be the standard non-negative cut-off function satisfying

$$\int_{\mathbb{R}} \xi^p(s)ds = 1 \quad \text{and} \quad |\xi'| \le \frac{2}{R_0\rho_0}. \tag{3.9}$$

For any $s \in \mathbb{R}$, let $w_s(x) = u(x)\xi(x_n - s)$ where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. We observe that for each fixed s, w_s is a weak solution of

$$\mathcal{Q}_\lambda w_s(x) = \lambda \tilde{f}_s(x) + \mathrm{div}\tilde{g}_s(x) \quad x \in \mathbb{R}^n, \tag{3.10}$$

where

$$\begin{aligned} \tilde{f}_s(x) &= f(x)\xi(x_n - s) - \frac{\xi'(x_n - s)}{\lambda}\Big[g_n(x) + \sum_{l=1}^n a_{nl}(x)D_l u(x))\Big], \\ \tilde{g}_s(x) &= g(x)\xi(x_n - s) - u(x)\xi'(x_n - s)a_n(x), \quad x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}, \end{aligned} \tag{3.11}$$

where $a_n(x) = (a_{1n}(x), a_{2n}(x), \ldots, a_{nn}(x))$. As $\mathrm{spt}(w_s) \in \mathbb{R}^{n-1} \times (s - R_0\rho_0, s + R_0\rho_0)$, by applying **Step I** to the equation (3.10), we obtain

$$\|Dw_s\|_{L^p(\mathbb{R}^n)} + \sqrt{\lambda\alpha_0}\, \|w_s\|_{L^p(\mathbb{R}^n)} \le C(\Lambda, n, p)\Big[\sqrt{\lambda\alpha_0^{-1}}\, \big\|\tilde{f}_s\big\|_{L^p(\mathbb{R}^n)} + \|\tilde{g}_s\|_{L^p(\mathbb{R}^n)}\Big]. \tag{3.12}$$

Since ρ_0 depends only on Λ, n, p, it follows from (3.9), (3.11), and the boundedness of $(a_{kl})_{n\times n}$ in (1.1) that

$$\sqrt{\lambda\alpha_0^{-1}}\left(\int_{\mathbb{R}} \|\tilde{f}_s\|^p_{L^p(\mathbb{R}^n)}\, ds\right)^{\frac{1}{p}} \le C(\Lambda, n, p)\left[\sqrt{\lambda\alpha_0^{-1}}\|f\|_{L^p(\mathbb{R}^n)} + \frac{1}{R_0\sqrt{\lambda\alpha_0}}\Big(\|g\|_{L^p(\mathbb{R}^n)} + \|Du\|_{L^p(\mathbb{R}^n)}\Big)\right]$$

and

$$\left(\int_{\mathbb{R}} \|\tilde{g}_s\|^p_{L^p(\mathbb{R}^n)}\, ds\right)^{\frac{1}{p}} \le C(\Lambda, n, p)\left[\|g\|_{L^p(\mathbb{R}^n)} + \frac{1}{R_0}\|u\|_{L^p(\mathbb{R}^n)}\right].$$

From (3.12) and the last two estimates, we can follow the calculation as in **Step II** of the proof of Theorem 1.1 to conclude that

$$
\begin{aligned}
&\|Du\|_{L^p(\mathbb{R}^n)} + \sqrt{\lambda\alpha_0}\,\|u\|_{L^p(\mathbb{R}^n)} \\
&\leq C(\Lambda,n,p)\left[\sqrt{\lambda\alpha_0^{-1}}\left(\int_{\mathbb{R}} \|\tilde{f}_s\|^p_{L^p(\mathbb{R}^n)}\,ds\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}} \|\tilde{g}_s\|^p_{L^p(\mathbb{R}^n)}\,ds\right)^{\frac{1}{p}} + \frac{1}{R_0}\,\|u\|_{L^p(\mathbb{R}^n)}\right] \\
&\leq C_0(\Lambda,n,p)\left[\sqrt{\lambda\alpha_0^{-1}}\|f\|_{L^p(\mathbb{R}^n)} + \Big(1+\frac{1}{R_0\sqrt{\lambda\alpha_0}}\Big)\|g\|_{L^p(\mathbb{R}^n)}\right. \\
&\qquad\left. + \frac{1}{R_0\sqrt{\lambda\alpha_0}}\,\|Du\|_{L^p(\mathbb{R}^n)} + \frac{1}{R_0}\,\|u\|_{L^p(\mathbb{R}^n)}\right].
\end{aligned}
\tag{3.13}
$$

Now, we choose $N_0 = 4C_0^2$ where $C_0 = C_0(\Lambda,p,n)$ is the number defined in (3.13). Then, for all $\lambda > \frac{N_0}{\alpha_0 R_0^2}$, it can be deduced from (3.13) that

$$
\|Du\|_{L^p(\mathbb{R}^n)} + \sqrt{\lambda\alpha_0}\,\|u\|_{L^p(\mathbb{R}^n)} \leq C(\Lambda,n,p)\left[\sqrt{\lambda\alpha_0^{-1}}\|f\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)}\right].
$$

The proof is then completed. □

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