LOCAL CONVEXITY ESTIMATES FOR MEAN CURVATURE FLOW

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Abstract. We develop a local version of Huisken–Stampacchia iteration, using it to obtain local versions of a host of important sharp curvature pinching estimates for mean curvature flow. The local estimates we obtain do not depend on the quality of noncollapsing of the solution and the method adopted applies in a host of other settings.

Developing a quantitative structure theory for singularities in geometric flows is a fundamental problem, since their emergence prevents the flow from reaching an equilibrium state, and hence obstructs many desired applications. One powerful tool for analyzing singularity formation in extrinsic geometric flows is Huisken–Stampacchia iteration, which is the main tool in the proof of various important curvature pinching estimates that improve at the onset of singularities. For mean curvature flow, these a priori estimates imply that a compact, strictly $(m+1)$-convex[1] hypersurface evolving by mean curvature becomes weakly convex and either becomes strictly m-convex or forms a high quality m-neck in regions of sufficiently high curvature [5–8, 11] (see [4] for a different approach for embedded flows). This provides a powerful description of singularity formation; however, there remains the major drawback that the estimates depend on *global* data from the initial (compact) hypersurface, whereas singularity formation is a *local* phenomenon. This prevents application of the estimates to noncompact solutions. It also prevents iteration of the estimates in a neighbourhood of a singularity, which seems to be a basic requirement for extending the Huisken–Sinestrari surgery algorithm for 2-convex hypersurfaces [8] to weaker intermediate convexity conditions.

We will localize these pinching estimates by introducing suitable cutoff functions and developing a local version of the Huisken–Stampacchia method. Fixing data $n \in \mathbb{N} \setminus \{1\}$, $m \in \{0, \ldots, n-1\}$, $\alpha > 0$, $\Theta < \infty$,

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[1] Recall that a hypersurface of $\mathbb{R}^{n+1}$ is *strictly m-convex* if the sum $\kappa_1 + \cdots + \kappa_m$ of its smallest m of its principal curvatures $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$ is positive.

$V<\infty$, $q>0$, $\lambda>0$ and $\delta\in(0,\lambda)$ such that $\sqrt{q}\,\Theta\geq(\lambda-\delta)^{-1}$ and $\Theta V\geq 1$, and a scale parameter $R>0$, our main result may be stated as follows.

Theorem 1 (Local convexity and cylindrical estimates)**.** *If a mean curvature flow is properly defined in $B_{\lambda R}\times(0,\frac{1}{2n}R^2)\subset\mathbb{R}^{n+1}\times\mathbb{R}$ and satisfies*

– $\displaystyle\inf_{B_{\lambda R}\times\{0\}\cup\partial B_{\lambda R}\times(0,\frac{1}{2n}R^2)}\frac{\kappa_1+\cdots+\kappa_{m+1}}{|A|}\geq\alpha>0,$

– $\displaystyle\sup_{B_{\lambda R}\times\{0\}\cup B_{\lambda R}\setminus B_{(\lambda-\delta)R}\times(0,\frac{1}{2n}R^2)}\tfrac{1}{n}H\leq\Theta R^{-1}$, *and*

– $\displaystyle\frac{R^2}{2n}\int_{B_{\lambda R}}H^q\,d\mu_0+\delta^{-2}\int_0^{\frac{R^2}{2n}}\int_{B_{\lambda R}\setminus B_{(\lambda-\delta)R}}H^q\,d\mu_t\,dt\leq(\Theta R^{-1})^q(VR)^{n+2},$

then, given any $\varepsilon>0$ and $\vartheta\in(0,1)$, it satisfies the estimates

$$\kappa_1\geq-\varepsilon H-C_\varepsilon R^{-1}\tag{1}$$

and

$$\sum_{j=m+1}^{n}(\kappa_n-\kappa_j)\leq\sum_{j=1}^{m}\kappa_j+\varepsilon H+C_\varepsilon R^{-1}\tag{2}$$

in $B_{(\lambda-\delta)\vartheta R}\times(0,\frac{1}{2n}R^2)$, where $C_\varepsilon=c(n,\alpha,q,\varepsilon)\Theta\left[\frac{\Theta V}{1-\vartheta}\right]^{\frac{2}{q}}$.

Remark 1.

– The hypotheses of the theorem are very close in nature to those of the aforementioned global estimates; however, since the estimates are local, the hypotheses must also be made at the spatial boundary. Cf. [13].
– Taking λ sufficiently large, we see that the (global) pinching estimates for mean curvature flow are an immediate consequence of Theorem 1.
– Local estimates of a similar nature have been obtained in [4] (cf. [13]) assuming a noncollapsing condition (in the sense of [1,14]); however, the mean curvature flow in Theorem 1 need not be noncollapsing, nor even embedded; it need only be *proper*, which we take to mean that the pair $(X,t):M\times(t_{\mathrm{i}},t_{\mathrm{f}})\to\mathbb{R}^{n+1}\times\mathbb{R}$ forms a proper map with respect to the subset $B_{\lambda R}\times(-\frac{1}{2n}R^2,0)\subset\mathbb{R}^{n+1}\times\mathbb{R}$, where $X:M\times(t_{\mathrm{i}},t_{\mathrm{f}})\to\mathbb{R}^{n+1}$ is a parametrization for the flow and $t:M\times(t_{\mathrm{i}},t_{\mathrm{f}})\to\mathbb{R}$ is the projection onto the second (time) factor. Moreover, the method adopted here can be applied in other contexts, such as flows by nonlinear speeds, high codimension mean curvature flow, or free-boundary mean curvature flow.

– The estimates can be localized in more general open subsets of $\mathbb{R}^{n+1}$, or parabolically open subsets of $\mathbb{R}^{n+1}\times\mathbb{R}$.
– Due to the pointwise curvature bound of the second hypothesis in Theorem 1, the integral curvature bound of the third hypothesis may be replaced by an area bound of the form

$$\frac{R^2}{2n}\int_{B_{\lambda R}} d\mu_0 + \delta^{-2}\int_0^{\frac{R^2}{2n}}\int_{B_{\lambda R}\setminus B_{(\lambda-\delta)R}} d\mu_t\,dt \le (VR)^{n+2}\,.$$

In fact, as can be easily deduced from the proof, each of these hypotheses need only be made in regions where the conclusion of the theorem does not hold.
– Even in the global setting, Theorem 1 provides a more precise accounting of the dependence of the constant C_ε on the boundary data. In particular, we see that certain rigidity results for the shrinking sphere or shrinking cylinder amongst convex ancient solutions (see [3, 9]) may be recovered from (2).
– By applying maximum principle type arguments similar to those of [8, Theorems 6.1 and 6.3], Theorem 1 yields corresponding local derivative estimates for the second fundamental form A, so long as $m < \frac{2(n-1)}{3}$, and therefore also a corresponding local neck detection lemma (cf. [8, Lemma 7.4]). This makes a local surgery algorithm possible when $m = 1$ (and $n \ge 3$).

Proof of Theorem 1. Let G be given either by

$$G \doteqdot -\kappa_1 \;\text{ or by }\; G \doteqdot \kappa_n - \tfrac{1}{n-m}H$$

and set, for any $\varepsilon > 0$ and $\sigma \in (0,1)$,

$$G_\varepsilon \doteqdot G - \varepsilon(H - \tfrac{\alpha}{2}|A|)\,,\;\; G_{\varepsilon,\sigma} \doteqdot G_\varepsilon H^{\sigma-1} \;\text{ and }\; G_{\varepsilon,\sigma,+} \doteqdot \max\{G_{\varepsilon,\sigma}, 0\}\,.$$

Well known calculations then show that

$$\frac{(\partial_t-\Delta)G_{\varepsilon,\sigma}}{G_{\varepsilon,\sigma}} \le \sigma|A|^2 - \gamma\frac{|\nabla A|^2}{H^2} + \gamma^{-1}\frac{|\nabla G_{\varepsilon,\sigma}|^2}{G_{\varepsilon,\sigma}^2} \tag{3}$$

in $B_\lambda\times(0,\frac{1}{2n})\cap \operatorname{spt} G_{\varepsilon,\sigma}$ in the distributional sense[2], where $\gamma = \gamma(n,\alpha,\varepsilon) > 0$ (see, for example, [2, Proposition 12.9] and [11, Section 3]).

[2]In what follows, all differential inequalities are intended in the distributional sense.

Given any $\zeta \in C_0^\infty(B_\lambda)$, set $\psi \doteqdot \zeta \circ X$. Setting $v \doteqdot G_{\varepsilon,\sigma,+}^{\frac{p}{2}}$ we then obtain

$$\frac{(\partial_t - \Delta)\psi^2 v^2}{\psi^2 v^2} = 2\frac{(\partial_t - \Delta)\psi}{\psi} - 2\frac{|\nabla\psi|^2}{\psi^2} + \frac{p(\partial_t - \Delta)G_{\varepsilon,\sigma}}{G_{\varepsilon,\sigma}} - 4\frac{p-1}{p}\frac{|\nabla v|^2}{v^2} - 8\left\langle \frac{\nabla\psi}{\psi}, \frac{\nabla v}{v}\right\rangle$$

wherever $\psi v > 0$. Applying Young's inequality to the final term and recalling (3) we obtain, for $p \geq 6(1+\gamma^{-1})$,

$$\frac{(\partial_t - \Delta)\psi^2 v^2}{\psi^2 v^2} \leq 2\frac{(\partial_t - \Delta)\psi}{\psi} + 6\frac{|\nabla\psi|^2}{\psi^2} - \left(2 - \frac{4}{p}\right)\frac{|\nabla v|^2}{v^2} + p\left(\sigma|A|^2 - \gamma\frac{|\nabla A|^2}{H^2} + \frac{4}{\gamma p^2}\frac{|\nabla v|^2}{v^2}\right).$$

Fix r and R so that $0 < r < R \leq \lambda$. If we choose the function $\zeta : \mathbb{R}^{n+1} \to \mathbb{R}$ so that

(1) $\zeta(X) = 0$ when $X \notin B_R$ and $\zeta(X) = 1$ when $X \in B_r$,
(2) $|D_i\zeta|^2 \leq 10(R-r)^{-2}\zeta$ for each i, and
(3) $|D_iD_j\zeta| \leq 10(R-r)^{-2}$ for each i and j,

then

$$(\partial_t - \Delta)\psi \leq 10n(R-r)^{-2}\chi_{B_R\setminus B_r} \quad\text{and}\quad \frac{|\nabla\psi|^2}{\psi} \leq 10n(R-r)^{-2}\chi_{B_R\setminus B_r},$$

where $\chi_{B_R\setminus B_r}$ denotes the characteristic function of the set $B_R \setminus B_r$ (pulled back by the flow parametrization X).

We thus obtain

$$\begin{aligned}\frac{(\partial_t - \Delta)\psi^2 v^2}{\psi^2 v^2} &\leq \frac{100n}{(R-r)^2}\frac{\chi_{B_R\setminus B_r}}{\psi} - 2\left(1 - \frac{2+2\gamma^{-1}}{p}\right)\frac{|\nabla v|^2}{v^2} - \gamma p\frac{|\nabla A|^2}{H^2} + \sigma p|A|^2 \\ &\leq \frac{100n}{(R-r)^2}\frac{\chi_{B_R\setminus B_r}}{\psi} - \frac{4}{3}\frac{|\nabla v|^2}{v^2} - \gamma p\frac{|\nabla A|^2}{H^2} + \sigma p|A|^2 \qquad (4)\end{aligned}$$

wherever $\psi v > 0$.

The L^2-estimate. Applying (4) and the gradient flow property of mean curvature flow, we obtain

$$\begin{aligned}\frac{d}{dt}\int \psi^2 v^2 d\mu + \int \psi^2 v^2 H^2\, d\mu &= \int \partial_t(\psi^2 v^2) d\mu \\ &\le \int \psi^2 v^2 \left(\sigma p|A|^2 - \frac{4}{3}\frac{|\nabla v|^2}{v^2} - \gamma p \frac{|\nabla A|^2}{H^2}\right) d\mu \\ &\quad + \frac{100n}{(R-r)^2}\int_{B_R\setminus B_r} \psi v^2\, d\mu\,. \qquad (5)\end{aligned}$$

We now apply the Poincaré-type inequality [11, Proposition 2.7] to $u = \psi v$. This yields, for any $\beta > 0$,

$$\begin{aligned}\int \psi^2 v^2 |A|^2 d\mu &\le \beta \int \psi^2 v^2 \left|\frac{\nabla\psi}{\psi} + \frac{\nabla v}{v}\right|^2 d\mu + P(1+\beta^{-1}) \int \psi^2 v^2 \frac{|\nabla A|^2}{H^2} d\mu \\ &\le 2\beta \int \left(\psi^2|\nabla v|^2 + v^2 |\nabla\psi|^2\right)\, d\mu \\ &\quad + P(1+\beta^{-1}) \int \psi^2 v^2 \frac{|\nabla A|^2}{H^2}\, d\mu\,,\end{aligned}$$

where $P = P(n,\alpha,\varepsilon)$. Setting $\beta = p^{-\frac{1}{2}}$ and recalling (5), we find that

$$\begin{aligned}\frac{d}{dt}\int \psi^2 v^2\, d\mu &\le \frac{(100 + 20\sigma p^{\frac{1}{2}})n}{(R-r)^2} \int_{B_R\setminus B_r} \psi\, v^2\, d\mu \\ &\quad + \left(2\sigma p^{\frac{1}{2}} - \frac{4}{3}\right) \int \psi^2 |\nabla v|^2\, d\mu \\ &\quad + \left(\sigma P(1+p^{\frac{1}{2}}) - \gamma\right) p \int \psi^2 v^2 \frac{|\nabla A|^2}{H^2}\, d\mu\,.\end{aligned}$$

Choosing $p \ge L$ and $\sigma \le \ell p^{-\frac{1}{2}}$, where $\ell = \ell(n,\alpha,\varepsilon) \le 4$ and $L = L(n,\alpha,\varepsilon)$, yields

$$\frac{d}{dt}\int \psi^2 v^2\, d\mu \le \frac{200n}{(R-r)^2}\int_{B_R\setminus B_r} \psi v^2\, d\mu\,. \qquad (6)$$

If we choose $R = \lambda$ and $r = \lambda - \delta$, then, integrating in time, we find that

$$\sup_{t\in(0,\frac{1}{2n}R^2)} \int_{B_{\lambda-\delta}} v^2\, d\mu \le \int_{B_\lambda} v^2\, d\mu_0 + 200n\delta^{-2} \iint_{B_\lambda\setminus B_{\lambda-\delta}} v^2\, d\mu_t\, dt\,,$$

and hence

$$\iint_{B_{\lambda-\delta}} v^2\, d\mu\, dt \le \frac{1}{2n}\int_{B_\lambda} v^2\, d\mu_0 + 100\delta^{-2} \iint_{B_\lambda\setminus B_{\lambda-\delta}} v^2\, d\mu_t\, dt\,, \qquad (7)$$

so long as $p \geq L$ and $\sigma \leq \ell p^{-\frac{1}{2}}$.

From L^2 to L^∞. Stampacchia iteration will now allow us to pass from L^2 to L^∞. We assume the reader is familiar with [15, Section 5] and [10, Chapter II: Appendices B and C].

Given $k > k_0 \doteqdot \Theta^\sigma \geq \sup_{B_\lambda \setminus B_{\lambda-\delta}} G_{\varepsilon,\sigma}$ and $R < \lambda - \delta$, consider

$$v_k^2 \doteqdot (G_{\varepsilon,\sigma} - k)_+^p \quad \text{and} \quad A_{k,R} \doteqdot \{(x,t) \in X^{-1}(B_R) : v_k(x,t) > 0\}$$

and set

$$u(k,R) \doteqdot \iint_{A_{k,R}} v_k^2 \, d\mu_t \, dt \quad \text{and} \quad a(k,R) \doteqdot \iint_{A_{k,R}} d\mu_t \, dt \, .$$

Note that, for any $h \geq k > 0$ and any $0 < r \leq R \leq \lambda - \delta$,

$$(h-k)^p a(h,r) \leq u(k,r) \, . \tag{8}$$

We need an estimate for $u(k,r)$. First observe that, computing as in (5), we can estimate

$$\frac{d}{dt}\int \psi^2 v_k^2 \, d\mu + \int_{A_{k,r}} |\nabla v_k|^2 \, d\mu + \int_{A_{k,r}} v_k^2 H^2 \, d\mu \leq \frac{100n}{(R-r)^2}\int_{A_{k,R}} v_k^2 \, d\mu + \sigma p \int_{A_{k,R}} G_{\varepsilon,\sigma}^p |A|^2 \, d\mu$$

for any $k > 0$ and $r < R \leq \lambda - \delta$, where ψ is a cut-off function satisfying $\psi \equiv 1$ on B_r and $\psi \equiv 0$ outside of B_R. On the other hand, the Sobolev inequality of Michael and Simon [12, Theorem 2.1] yields

$$\left(\int_{A_{k,r}} v_k^{2^*} \, d\mu\right)^{\frac{2}{2^*}} \leq c_S \int_{A_{k,r}} \left(|\nabla v_k|^2 + H^2 v_k^2\right) \, d\mu \, ,$$

where[3], for $n \geq 3$, $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}$ and c_S depends only on n, so that

$$\frac{d}{dt}\int \psi^2 v_k^2 \, d\mu + \frac{1}{c_S}\left(\int_{A_{k,r}} v_k^{2^*} \, d\mu\right)^{\frac{2}{2^*}} \leq \frac{100n}{(R-r)^2}\int_{A_{k,R}} v_k^2 \, d\mu + \sigma p \int_{A_{k,R}} G_{\varepsilon,\sigma}^p |A|^2 \, d\mu \, .$$

[3]We can interpret the left hand side as the square of the L^∞-norm when $n = 2$ with 2^* any fixed number bigger than one and the constant c_S depending additionally on 2^* and the measure of the support of v_k. Cf. [5].

Integrating with respect to t then yields

$$\sup_{t\in(0,\frac{1}{2n}R^2)}\int_{A_{k,r}} v_k^2\,d\mu+\int\left(\int_{A_{k,r}} v_k^{2^*}\,d\mu\right)^{\frac{2}{2^*}}dt \leq \frac{100nc_S}{(R-r)^2}\iint_{A_{k,R}} v_k^2\,d\mu\,dt + c_S\sigma p\iint_{A_{k,R}} G^p_{\varepsilon,\sigma}|A|^2\,d\mu\,dt\,.$$

By the interpolation inequality,

$$\int_{A_{k,r}} v_k^{\frac{2(n+2)}{n}}\,d\mu \leq \left(\int_{A_{k,r}} v_k^2\,d\mu\right)^{\frac{2}{n}}\left(\int_{A_{k,r}} v_k^{2^*}\,d\mu\right)^{\frac{2}{2^*}}$$

and hence, by Young's inequality,

$$\left(\iint_{A_{k,r}} v_k^{\frac{2(n+2)}{n}}\,d\mu\,dt\right)^{\frac{n}{n+2}} \leq \left(\sup_{t\in(0,\frac{1}{2n}R^2)}\int_{A_{k,r}} v_k^2\,d\mu\right)^{\frac{2}{n+2}}\left(\int\left(\int_{A_{k,r}} v_k^{2^*}\,d\mu\right)^{\frac{2}{2^*}}dt\right)^{\frac{n}{n+2}}$$
$$\leq \tfrac{2}{n+2}\sup_{t\in(0,\frac{1}{2n}R^2)}\int_{A_{k,r}} v_k^2\,d\mu + \tfrac{n}{n+2}\int\left(\int_{A_{k,r}} v_k^{2^*}\,d\mu\right)^{\frac{2}{2^*}}dt\,.$$

Thus,

$$\left(\iint_{A_{k,r}} v_k^{\frac{2(n+2)}{n}}\,d\mu\,dt\right)^{\frac{n}{n+2}} \leq \frac{100nc_S}{(R-r)^2}\iint_{A_{k,R}} v_k^2\,d\mu\,dt + c_S\sigma p\iint_{A_{k,R}} G^p_{\varepsilon,\sigma}|A|^2\,d\mu\,dt\,. \tag{9}$$

Applying Hölder's inequality and (choosing ℓ slightly smaller[4]), we estimate, for $\sigma' \doteqdot \sigma + \frac{2}{p}$ and some soon-to-be-determined $\rho \geq 1$,

$$\iint_{A_{k,R}} H^2 G^p_{\varepsilon,\sigma}\,d\mu\,dt = \iint_{A_{k,R}} G^p_{\varepsilon,\sigma',+}\,d\mu\,dt \leq a(k,R)^{1-\frac{1}{\rho}}\left(\iint_{A_{k,R}} G^{p\rho}_{\varepsilon,\sigma',+}\,d\mu\,dt\right)^{\frac{1}{\rho}} \leq a(k,R)^{1-\frac{1}{\rho}}\left(\iint_{B_{\lambda-\delta}} G^{p\rho}_{\varepsilon,\sigma',+}\,d\mu\,dt\right)^{\frac{1}{\rho}}\,. \tag{10}$$

[4]Depending now also on ρ, which will be fixed momentarily.

Similarly, we may estimate, for any $k > k_0$ and $R < \lambda - \delta$,

$$u(k,R) \le a(k,R)^{1-\frac{1}{\rho}} \left(\iint_{B_{\lambda-\delta}} G_{\varepsilon,\sigma,+}^{p\rho}\, d\mu\, dt \right)^{\frac{1}{\rho}} . \tag{11}$$

Finally, we estimate

$$u(k,r) \le a(k,r)^{\frac{2}{n+2}} \left(\iint_{A_{k,r}} v_k^{\frac{2(n+2)}{n}}\, d\mu\, dt \right)^{\frac{n}{n+2}} . \tag{12}$$

Since $|A| \le \alpha^{-1} H$ and $G \le H$, combining (8)–(12) and the L^2-estimate (7) yields

$$\begin{aligned}
(h-k)^p (R-r)^2 a(h,r) &\le c a(k,R)^\gamma \big(1 + \sigma p \Theta^2 (R-r)^2\big) \cdot \\
&\quad \left(\frac{1}{2n} \int_{B_\lambda} G_{\varepsilon,\sigma,+}^{p\rho} d\mu_0 + \frac{100}{\delta^2} \iint_{B_\lambda \setminus B_{\lambda-\delta}} G_{\varepsilon,\sigma,+}^{p\rho} d\mu\, dt \right)^{\frac{1}{\rho}} \\
&\le c a(k,R)^\gamma \big(1 + \sigma p \Theta^2 (R-r)^2\big) \cdot \\
&\quad \left(\frac{1}{2n} \int_{B_\lambda} H^{\sigma p\rho} d\mu_0 + \frac{100}{\delta^2} \iint_{B_\lambda \setminus B_{\lambda-\delta}} H^{\sigma p \rho} d\mu\, dt \right)^{\frac{1}{\rho}} \\
&\le c a(k,R)^\gamma \big(1 + \sigma p \Theta^2 (\lambda-\delta)^2\big) \Theta^{\sigma p (1-\frac{1}{\rho})} \Lambda^{\frac{\sigma p}{\rho}}
\end{aligned}$$

so long as $p \ge L(n,\alpha,\varepsilon,\rho)$ and $\sigma \le \ell(n,\alpha,\varepsilon,\rho) p^{-\frac{1}{2}}$, where $c = c(n,\alpha,\rho)$, $\gamma \doteqdot 1 + \frac{2}{n+2} - \frac{1}{\rho}$, and

$$\Lambda \doteqdot \left(\frac{1}{2n} \int_{B_\lambda} H^{\sigma p}\, d\mu_0 + \frac{100}{\delta^2} \iint_{B_\lambda \setminus B_{\lambda-\delta}} H^{\sigma p}\, d\mu\, dt \right)^{\frac{1}{\sigma p}} .$$

At this point, we fix some $\rho = \rho(n) > 1 + \frac{n}{2}$ (so that $\gamma = \gamma(n) > 1$) and choose $p = p(n,\alpha,\varepsilon,q) \ge L$ and $\sigma = \sigma(n,\alpha,\varepsilon,q) \le \ell p^{-\frac{1}{2}}$ such that $\sigma p = q$. Stampacchia's Lemma [15, Lemma 5.1] then yields

$$a\left(k_0 + d, \vartheta R_0\right) = 0\,,$$

where $R_0 \doteqdot \lambda - \delta$ and

$$d^p \doteqdot \frac{2^{\frac{(p+2)\gamma}{\gamma-1}} c \big(1 + \sigma p \Theta^2 (\lambda-\delta)^2\big) \Theta^{\sigma p (1-\frac{1}{\rho})} \Lambda^{\frac{\sigma p}{\rho}} a(k_0,R_0)^{\gamma-1}}{(1-\vartheta)^2 (\lambda-\delta)^2} .$$

We may estimate, using the L^2-estimate (7),

$$\begin{aligned} a(k_0,R_0) &\le k_0^{-p}\iint_{U_{R_0}} G^p_{\varepsilon,\sigma,+}\,d\mu\,dt\\ &\le k_0^{-p}\left(\frac{1}{2n}\int_{B_\lambda} G^p_{\varepsilon,\sigma,+}\,d\mu_0+\frac{100}{\delta^2}\iint_{B_\lambda\setminus B_{\lambda-\delta}} G^p_{\varepsilon,\sigma,+}\,d\mu\,dt\right)\\ &\le k_0^{-p}\Lambda^{\sigma p}\,. \end{aligned}$$

Since we chose $k_0=\Theta^\sigma$ and $\sigma p=q$, we thus obtain

$$\begin{aligned} G_{\varepsilon,\sigma} &\le k_0+d\\ &= k_0\left(1+2^{\frac{(p+2)\gamma}{p(\gamma-1)}}\left[\frac{c\big(1+\sigma pR_0^2\Theta^2\big)}{(1-\vartheta)^2R_0^2}\right]^{\frac{1}{p}}\frac{\Theta^{\sigma(1-\frac{1}{\rho})}}{k_0^{1-\frac{1}{\rho}}}\frac{\Lambda^{\frac{2}{n+2}\sigma}}{k_0^{\frac{2}{n+2}}}\right)\\ &\le \Theta^\sigma\left(1+c(n,\alpha,q,\varepsilon)\left[\frac{(\lambda-\delta)^{-2}+q\,\Theta^2}{(1-\vartheta)^2}V^2\right]^{\frac{\sigma}{q}}\right) \end{aligned}$$

in $B_{\vartheta(\lambda-\delta)}$. Young's inequality then yields

$$G\le 2\varepsilon H+c(n,\alpha,q,\varepsilon)\Theta\left(\frac{\Theta V}{1-\vartheta}\right)^{\frac{2}{q}}$$

in $B_{\vartheta(\lambda-\delta)}$. This completes the proof of Theorem 1. □

References

[1] Ben Andrews. Noncollapsing in mean-convex mean curvature flow. *Geom. Topol.*, 16(3):1413–1418, 2012.

[2] Ben Andrews, Bennett Chow, Christine Guenther, and Mat Langford. *Extrinsic Geometric Flows*, volume 206 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, first edition, 2020.

[3] Robert Haslhofer and Or Hershkovits. Ancient solutions of the mean curvature flow. *Commun. Anal. Geom.*, 24(3):593–604, 2016.

[4] Robert Haslhofer and Bruce Kleiner. Mean curvature flow of mean convex hypersurfaces. *Comm. Pure Appl. Math.*, 70(3):511–546, 2017.

[5] Gerhard Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.*, 20(1):237–266, 1984.

[6] Gerhard Huisken and Carlo Sinestrari. Convexity estimates for mean curvature flow and singularities of mean convex surfaces. *Acta Math.*, 183(1):45–70, 1999.

[7] Gerhard Huisken and Carlo Sinestrari. Mean curvature flow singularities for mean convex surfaces. *Calc. Var. Partial Differential Equations*, 8(1):1–14, 1999.

[8] Gerhard Huisken and Carlo Sinestrari. Mean curvature flow with surgeries of two-convex hypersurfaces. *Invent. Math.*, 175(1):137–221, 2009.

[9] Gerhard Huisken and Carlo Sinestrari. Convex ancient solutions of the mean curvature flow. *J. Differential Geom.*, 101(2):267–287, 2015.

[10] David Kinderlehrer and Guido Stampacchia. *An introduction to variational inequalities and their applications*, volume 31 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. Reprint of the 1980 original.

[11] Mat Langford. A general pinching principle for mean curvature flow and applications. *Calculus of Variations and Partial Differential Equations*, 56(4):107, Jul 2017.

[12] J. H. Michael and L. M. Simon. Sobolev and mean-value inequalities on generalized submanifolds of R^n. *Comm. Pure Appl. Math.*, 26:361–379, 1973.

[13] Alexander Mramor. Regularity and stability results for the level set flow via the mean curvature flow with surgery. Preprint, arxiv.org/abs/1710.09989.

[14] Weimin Sheng and Xu-Jia Wang. Singularity profile in the mean curvature flow. *Methods Appl. Anal.*, 16(2):139–155, 2009.

[15] Guido Stampacchia. *Èquations elliptiques du second ordre à coefficients discontinus*. Séminaire de Mathématiques Supérieures, No. 16 (Été, 1965). Les Presses de l'Université de Montréal, Montreal, Que., 1966.

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