TRANSLATING SOLUTIONS TO MEAN CURVATURE FLOW

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Abstract. We describe our recent construction of a new family of translating solutions to mean curvature flow and discuss some implications for the construction of new convex ancient solutions. The full details appear in [7] and [8].

1. Introduction

A smooth one-parameter family $\{M_t^n\}_{t\in I}$ of smoothly immersed hypersurfaces M_t^n of $\mathbb{R}^{n+1}$ is a (classical) *solution to mean curvature flow* if there exists a smooth one parameter family $X : M^n \times I \to \mathbb{R}^{n+1}$ of immersions $X(\cdot, t) : M^n \to \mathbb{R}^{n+1}$ with $M_t^n = X(M^n, t)$ satisfying

$$\frac{\partial X}{\partial t}(x,t) = \vec{H}(x,t) \quad \text{for all} \quad (x,t) \in M^n \times I\,,$$

where $\vec{H}(\cdot, t)$ is the mean curvature vector field of $X(\cdot, t)$. The mean curvature flow is the L^2-gradient flow of the area functional and hence, roughly speaking, deforms a hypersurface in such a way as to decrease its area most rapidly.

An *ancient* solution to a geometric flow, such as mean curvature flow, is one which is defined on a time interval of the form $I = (-\infty, T)$, where $T \le \infty$. A special class of ancient solutions are the *translating solutions*. As the name suggests, these are solutions $\{M_t^n\}_{t\in(-\infty,\infty)}$ which evolve by translation: $M_{t+s}^n = M_t^n + se$ for some fixed vector $e \in \mathbb{R}^{n+1}$. The timeslices M_t^n of a translating solution $\{M_t^n\}_{t\in(-\infty,\infty)}$ are all congruent and satisfy the *translator equation*, which asserts that the mean curvature vector of M_t^n is equal to the projection of e onto its normal bundle. Translating solutions arise as blow-up limits of type-II singularities along an essential blow-up sequence [4, 5, 13, 24]. Type-II singularities (and, more generally, translating solutions) are still not very well understood, except in certain special cases [3, 6, 14, 17, 18, 19, 26, 28, 29]. More general blow-up sequences yield more general ancient solutions. Understanding ancient and translating solutions is therefore important to many applications of the flow which require a controlled continuation of the flow through singularities.

Further interest in ancient and translating solutions to geometric flows arise from their rigidity properties, which are analogous to those of complete minimal surfaces, harmonic maps and Einstein metrics; for example, when $n \geq 2$, under certain geometric conditions — uniform convexity, bounded eccentricity, type-I curvature decay or bounded isoperimetric ratio, for instance — the only compact[1], convex (or non-collapsing) ancient solutions to mean curvature flow are shrinking spheres [20, 15]. In fact, the convexity/noncollapsing condition can be weakened to a uniform bound for $|A|^2/H^2$, where A is the second fundamental form of the solution [21].

Compact ancient solutions to mean curvature flow are closely related to translating solutions. Indeed, if $\{M_t\}_{t\in(-\infty,0)}$ is a compact, convex ancient solution and $P(e,t) \in M_t$ satisfies $\nu(P(e,t),t) = e \in S^n$, then, by the differential Harnack inequality [2, 13], the translated flow $M_t^s := M_{t+s} - P(e,s)$ converges (in the smooth topology, uniformly on compact subsets) to a convex translating solution with velocity $-H_\infty e$, where $H_\infty \doteqdot \lim_{s\to-\infty} H(P(e,s),s)e$.

2. Translators

As we mentioned in the introduction, a translating solution of mean curvature flow is one which evolves purely by translation and, in that case, the time slices are all congruent and satisfy

$$H(x) = -\langle \nu(x), e\rangle \tag{1}$$

for some $e \in \mathbb{R}^{n+1}$, where ν is a choice of local unit normal field near x and $H = \operatorname{div}\nu$ is the corresponding mean curvature. Since we are interested in the classification problem, it is useful to eliminate the scaling invariance and isotropy of (1) by restricting attention to translating solutions which move with unit speed in the 'upwards' direction. That is, we henceforth assume that $e = e_{n+1}$. We will refer to a hypersurface $M^n \subset \mathbb{R}^{n+1}$ satisfying (1) with $e = e_{n+1}$ as a *translator*.

The most prominent example of a translator is the Grim Reaper curve, $\Gamma^1 \subset \mathbb{R}^2$, defined by

$$\Gamma^1 := \left\{(x, -\log\cos x) : |x| < \tfrac{\pi}{2}\right\} .$$

Taking products with lines then yields the Grim hyperplanes

$$\Gamma^n := \left\{(x_1, \ldots, x_n, -\log\cos x_1) : |x_1| < \tfrac{\pi}{2}\right\} .$$

[1]We refer to a solution $\{M_t^n\}_{t\in I}$ to mean curvature flow as *compact, convex, embedded, etc* if this is the case for each time slice M_t^n.

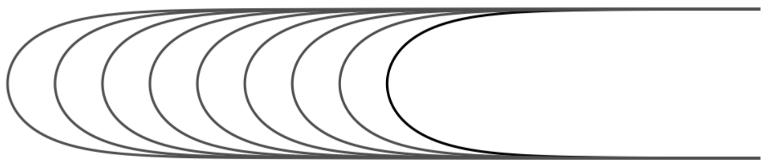


FIGURE 1. The Grim Reaper translating to the right under curve shortening flow, killing every compact solution in its way (by the avoidance principle).

The Grim hyperplane Γ^n lies in the slab $\{(x_1, \ldots, x_n) : |x_1| < \frac{\pi}{2}\}$ (and in no smaller slab). More generally, if M^{n-k} is a translator in $\mathbb{R}^{n-k+1}$ then $M^{n-k} \times \mathbb{R}^k$ is a translator in $\mathbb{R}^{n-k+1} \times \mathbb{R}^k \cong \mathbb{R}^{n+1}$.

There is also a family of 'oblique' Grim planes $\Gamma^n_{\theta,\phi}$ parametrized by $(\theta, \phi) \in [0, \frac{\pi}{2}) \times S^{n-2}$. These are obtained by rotating the 'standard' Grim plane Γ^n through the angle $\theta \in [0, \frac{\pi}{2})$ in the plane $\operatorname{span}\{\phi, e_{n+1}\}$ for some unit vector $\phi \in \operatorname{span}\{e_2, \ldots e_n\}$ and then scaling by the factor $\cos\theta$. To see that the result is indeed a translator, we need only check that

$$-H_\theta = -\cos\theta H = \cos\theta \left\langle \nu, e_{n+1} \right\rangle = \left\langle \cos\theta\nu + \sin\theta\phi, e_{n+1} \right\rangle = \left\langle \nu_\theta, e_{n+1} \right\rangle ,$$

where H_θ and ν_θ are the mean curvature and unit normal to $\Gamma^n_{\theta,\phi}$ respectively. We also set $\Gamma^n_\theta := \Gamma^n_{\theta,e_2}$.

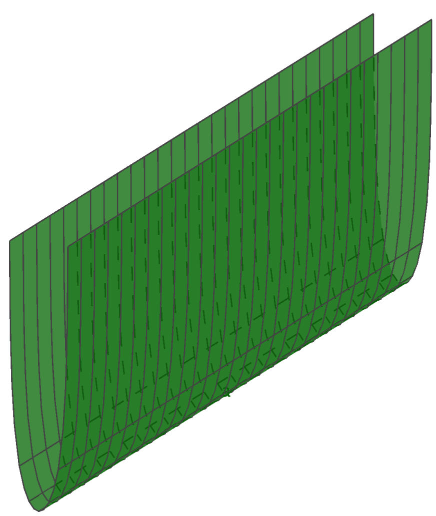


FIGURE 2. The oblique Grim plane Γ^2_θ with $\theta = \pi/6$. The translation direction is vertical.

The oblique Grim hyperplane $\Gamma^n_{\theta,\phi}$ lies in the slab

$$\Sigma^{n+1}_\theta := \{(x_1, \dots, x_{n+1}) : |x_1| < \tfrac{\pi}{2}\sec\theta\}$$

(and in no smaller slab). More generally, if M^{n-k} is a translator in $\mathbb{R}^{n-k+1}$ then the hypersurface $M^n_{\theta,\phi}$ obtained by rotating $M^{n-k}\times\mathbb{R}^k$ counterclockwise through angle θ in the plane $\phi\wedge e_{n+1}$ and then scaling by $\sec\theta$ is a translator in $\mathbb{R}^{n+1}$, so long as ϕ is a non-zero vector in $\mathrm{span}\{e_{n-k+1},\dots e_n\}$.

For each $n\geq 2$, Altschuler and Wu constructed an $O(n)$-invariant, convex, entire translating graph in $\mathbb{R}^{n+1}$ asymptotic to a paraboloid [1] (see also [11]). X.-J. Wang proved that this solution is the only convex entire translator in $\mathbb{R}^3$ and constructed further convex entire examples in higher dimensions [28]. He also proved the existence of strictly convex translating solutions which lie in slab regions in $\mathbb{R}^{n+1}$ for all $n\geq 2$ and showed that these are the only possibilities:

Theorem 2.1 (X.-J. Wang's dichotomy for translators [28])**.** *Every proper, convex translator is either entire or lies in a slab region.*

A major difficulty in the construction of solutions to the translator equation is to obtain curvature estimates for the Dirichlet problem for the graphical translator equation (note that convexity is not guaranteed). Wang sidesteps this problem by exploiting the Legendre transform and the existence of *convex* solutions of certain fully nonlinear equations. Unfortunately, this method loses track of the precise geometry of the domain on which the solution is defined and so it remained unclear exactly which slabs admit translators. On the other hand, there can exist no strictly convex translator in a slab of width less than or equal to π (the Grim hyperplane is a barrier).

Shariyari [25] and Spruck–Xiao [27] obtained curvature estimates for graphical translators in $\mathbb{R}^3$ by exploiting their stability properties (cf. [10]). Using their curvature estimates, Spruck and Xiao were then able to deduce that every mean convex translator in $\mathbb{R}^3$ is actually weakly convex [27, Theorem 1.1]. Inspired by their work, we were able to obtain a complete resolution to the existence question in all dimensions.

Theorem 2.2 (Existence of convex translators in all admissible slabs [8])**.** *For every $n\geq 2$ and every $\theta\in(0,\frac{\pi}{2})$ there exists a strictly convex translator W^n_θ which lies in*

$$\Sigma^{n+1}_\theta := \{(x,y,z)\in\mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R} : |x| < \tfrac{\pi}{2}\sec\theta\}\subset\mathbb{R}^{n+1}$$

and in no smaller slab.

Around the same time our work was completed, Hoffman, Ilmanen, Martìn and White provided an existence theorem for all slabs of width greater than π in the case $n = 2$ [16, Theorem 1.1]. They were also able to prove uniqueness in this case, thereby completing the classification of translating graphs in $\mathbb{R}^3$. Finally, they gave a different construction of examples of translating graphs in slabs in $\mathbb{R}^{n+1}$, extending an earlier construction of Ilmanen for the case $n = 2$ [16]. These solutions are parametrized by the vector of principal curvatures at the 'tip' (the unique point at which the downward unit normal is $-e_n$).

Let us briefly sketch the proof of Theorem 2.2. The idea is to take a limit of solutions to an appropriate sequence of Dirichlet problems. Since translators automatically satisfy $H \leq 1$, general methods of geometric measure theory can be used to obtain curvature estimates when $n \leq 6$. In order to obtain curvature estimates in higher dimensions, one needs to rule out singular (minimal) tangent cones. This can be achieved using the rotational symmetry hypothesis [9, 22, 23].

Proposition 2.3 (Curvature estimates up to the boundary for translating graphs [8])**.** *Given any $K > 0$ and $\ell \in \mathbb{N}$, there exists a constant $C_\ell < \infty$ with the following property: Let u be a solution to*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{1}{\sqrt{1+|Du|^2}} \quad \text{in } \ \Omega$$
$$u = \psi \quad \text{on } \ \partial\Omega\,,$$

with $\partial\Omega$ and ψ bounded in $C^{\ell_0,\alpha}$ by K for some $\ell_0 \geq 2$ and $\alpha \in (0,1]$ (and rotationally symmetric with respect to the subspace $\mathbb{E}^{n-1} := \operatorname{span}\{e_2,\dots,e_n\}$ if $n \geq 7$). Then

$$\sup_{p\in\operatorname{graph} u} |\nabla^\ell A(p)| \leq C_\ell \quad \text{for all} \quad \ell \in \{0,\dots,\ell_0 - 2\}\,,$$

where A is the second fundamental form of $\operatorname{graph} u$ and $\nabla^0 A := A$.

Remark 2.4. *In case $\ell_0 = 1$ we obtain uniform estimates in $C^{1,\alpha}$.*

We emphasize that the estimates of Proposition 2.3 hold all the way to the boundary of Ω. This will be needed later.

We are then able to extend the convexity estimate of Spruck and Xiao to higher dimensions under the rotational symmetry hypothesis.

Proposition 2.5. *Let $M \subset \mathbb{R}^{n+1}$ be a mean convex translator with at most two distinct principal curvatures at each point and bounded norm of the second fundamental form. Then M is convex.*

In order to obtain the solution as a limit of solutions to Dirichlet problems, it then remains to obtain height estimates (to ensure that the limit is complete) and to rule out a 'width-drop' in the limit. We achieve this by constructing appropriate barriers: The function $\underline{u} : \{(x,y) \in \mathbb{R} \times \mathbb{R}^{n-1} : |x| < \frac{\pi}{2}\sec\theta\} \to \mathbb{R}$ defined by

$$\underline{u}(x,y) := -\sec^2\theta \log\cos\left(\tfrac{x}{\sec\theta}\right) + \tan^2\theta \log\cosh\left(\tfrac{|y|}{\tan\theta}\right)$$

is a subsolution to the graphical translator equation. It was discovered by modifying the arrival time of the Angenent oval so that it lies in the correct slab and is asymptotic to the correct oblique Grim hyperplanes.

A suitable supersolution is obtained by rotating the Angenent oval of width $\pi\sec\theta$ and cutting off at an appropriate height (see Figure 3).

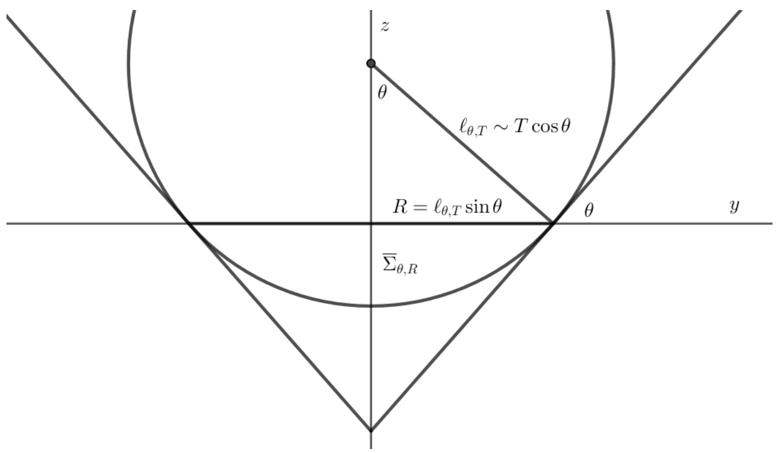


FIGURE 3. Given any $\varepsilon \in (0, \varepsilon_0(n,\theta))$, the portion of the rotated time $T = \sec^2\theta\cosh\left(\frac{R}{\tan\theta}\right)$ slice of the Angenent oval of width $\pi\sec\theta$ lying below height $z = -R\frac{\cos(\theta-\varepsilon)}{\sin\theta}$ is a supersolution of the translator equation when $R > R_\varepsilon := \frac{2(n-1)}{\varepsilon}$.

Given $R > 0$, set

$$\underline{u}_R := \underline{u} - \tan^2\theta \log\cosh\left(\tfrac{R}{\tan\theta}\right)$$

and let u_R be the solution to

$$\begin{cases} \operatorname{div}\left(\dfrac{Du_R}{\sqrt{1+|Du_R|^2}}\right) = \dfrac{1}{\sqrt{1+|Du_R|^2}} & \text{in } \Omega_R \\ u_R = 0 \quad \text{on } \partial\Omega_R\,, \end{cases}$$

where Ω_R is the set of points where $\underline{u}_R < 0$. Since the equation admits upper and lower barriers (0 and $\underline{u}_R$, respectively), existence and uniqueness of a smooth solution follows from well-known methods (see, for example, [12, Chapter 15]). Uniqueness implies that u_R is rotationally symmetric with respect to the subspace $\mathbb{E}^{n-1} = \operatorname{span}\{e_2, \ldots, e_n\}$. Since $\underline{u}_R$ is a subsolution, its graph lies below graph u_R. Since the two surfaces coincide on the boundary $\partial\Omega_R$, the mean curvature H_R of $\operatorname{graph} u_R$ satisfies

$$
\begin{aligned}
H_R = -\langle \nu_R, e_{n+1}\rangle \geq{} & -\langle \underline{\nu}_R, e_{n+1}\rangle \\
\geq{} & \cos\theta\cos\left(x\cos\theta\right) \\
\geq{} & \cos\theta\left(1-\frac{x}{\frac{\pi}{2}\sec\theta}\right)
\end{aligned} \tag{2}
$$

on $\partial\Omega_R$, where $\underline{\nu}_R$ is the downward pointing unit normal to $\operatorname{graph}\underline{u}_R$. On the other hand, using the ancient pancake as an upper barrier yields

$$
-u_R(0) \gtrsim \frac{1-\cos\theta}{\sin\theta} R \to \infty \quad \text{as} \quad R\to\infty\,. \tag{3}
$$

Let $R_i \to \infty$ be a diverging sequence and consider the translators-with-boundary

$$
M_i := \operatorname{graph} u_{R_i} - u_{R_i}(0)e_{n+1}\,.
$$

By Proposition 2.3 and the height estimate (3), some subsequence converges locally uniformly in the smooth topology to some limiting translator, M, with bounded second fundamental form. By Proposition 2.5, M is convex.

Certainly M lies in the slab Σ^{n+1}_θ, so it remains only to prove that it lies in no smaller slab (strict convexity will then follow from the splitting theorem and uniqueness of the Grim Reaper). Set

$$
v := 1-\frac{x}{\frac{\pi}{2}\sec\theta}\,,
$$

where $x(X) := \langle X, e_1\rangle$. We claim that

$$
\inf_{M\cap\{x>0\}} \frac{H}{v} > 0\,. \tag{4}
$$

Since $\inf_M H = 0$, we conclude that $\sup_M x = \frac{\pi}{2}\sec\theta$ as desired. To prove (4), first observe that

$$
-(\Delta+\nabla_V)v = 0
$$

and hence

$$
-(\Delta+\nabla_V)\frac{H}{v} = |A|^2\frac{H}{v} + 2\left\langle \nabla\frac{H}{v}, \frac{\nabla v}{v}\right\rangle\,,
$$

where V is the tangential projection of e_{n+1}. The maximum principle then yields

$$\min_{M_i\cap\{x>0\}}\frac{H}{v}\geq\min\left\{\min_{\partial M_i\cap\{x>0\}}\frac{H}{v},\min_{M_i\cap\{x=0\}}\frac{H}{v}\right\}$$
$$=\min\left\{\cos\theta,\min_{M_i\cap\{x=0\}}H\right\}.$$

If $\liminf_{i\to\infty}\min_{M_i\cap\{x=0\}}H>0$ then we are done. So suppose that $\liminf_{i\to\infty}H(X_i)=0$ along some sequence of points $X_i\in M_i\cap\{x=0\}$. Then, by Proposition 2.3, after passing to a subsequence, the translators-with-boundary

$$\hat{M}_i:=M_i-X_i$$

converge locally uniformly in C^∞ to a translator (possibly with boundary) $\hat{M}$ which lies in the slab Σ^{n+1}_θ and satisfies $H\geq 0$ with equality at the origin. By Proposition 2.3 the origin must be an interior point since, recalling (2), $H>\cos\theta$ on $\partial M_i\cap\{x=0\}$ for all i. The strong maximum principle then implies that $H\equiv 0$ on $\hat{M}$ and we conclude that $\hat{M}$ is either a hyperplane or half-hyperplane. Since, by the reflection symmetry, the limit cannot be parallel to $\{0\}\times\mathbb{R}^{n-1}\times\mathbb{R}$, neither option can be reconciled with the fact that $\hat{M}$ lies in Σ^{n+1}_θ. This proves that the width cannot drop in the limit, and with it Theorem 2.2.

As was the case for ancient pancakes, these solutions necessarily converge to oblique Grim hyperplanes (of width potentially smaller than that of the original slab) after translation parallel to the slab. The following theorem shows that the asymptotic Grim hyperplanes are of full width.

Theorem 2.6 (Unique asymptotics and reflection symmetry [27, 8]). *Given $n\geq 2$ and $\theta\in(0,\frac{\pi}{2})$ let M^n_θ be a convex translator which lies in the slab Σ^{n+1}_θ and in no smaller slab. If $n\geq 3$, assume in addition that M^n_θ is rotationally symmetric with respect to the subspace $\mathbb{E}^{n-1}:=\operatorname{span}\{e_2,\dots,e_n\}$. Given any unit vector $e\in\mathbb{E}^{n-1}$, the curve $\{\sin\omega e-\cos\omega e_{n+1}:\omega\in[0,\theta)\}$ lies in the normal image of M^n_θ and the translators*

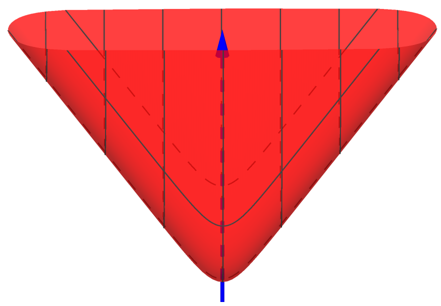
$$M^n_{\theta,\omega}:=M^n_\theta-P(\sin\omega e-\cos\omega e_{n+1})$$

converge locally uniformly in the smooth topology to the oblique Grim hyperplane $\Gamma^n_{\theta,e}$ as $\omega\to\theta$, where $P:S^n\to M^n_e$ is the inverse of the Gauss map.

Moreover, M^n_θ is reflection symmetric across the hyperplane $\{0\}\times\mathbb{R}^n$.

This result was already obtained by Spruck and Xiao when $n = 2$ using different methods [27]. Note that the translators we construct in Theorem 2.2 satisfy the hypotheses of Theorem 2.6.

It would be useful to have a better understanding of the location of the point $P(\sin\omega e - \cos\omega e_{n+1})$. For example, it remains unclear whether or not the 'flying wing' solution constructed in [8] lies above a (translated) oblique Grim hyperplane $\Gamma^n_\theta - Ce_{n+1}$. This information will be of use in constructing new examples of ancient and translating solutions.



The 'flying wing' W^2_θ of [8] with $\theta = \frac{\pi}{4}$ (right). The translation direction is vertical.

The rotational symmetry hypothesis — which is not required when $n = 2$ — may be necessary in higher dimensions. We note that, in higher dimensions, 'oblique' products of lower dimensional wing families with flat directions provide additional possible asymptotics for higher dimensional translators, so the description of higher dimensional translators is therefore to be far more complex. It is conceivable that there exist convex translators in the slab $\Sigma^4_\theta \subset \mathbb{R}^4$, for example, which are asymptotic to an 'oblique' $M^2_\theta \times \mathbb{R}$, where $M^2_\theta \subset \mathbb{R}^3$ is the translator from Theorem 3.1.

3. Ancient solutions with discrete symmetry groups

The Angenent oval provides an example of a compact, convex ancient solution to mean curvature flow that lies on a slab.

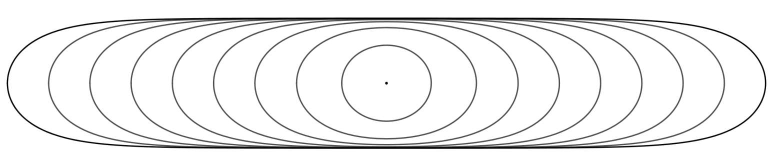


FIGURE 4. The Angenent oval solution to curve shortening flow. It shrinks to a round point as $t \to 0$ and sweeps out a strip of width π as $t \to -\infty$.

We shall refer to an ancient solution that satisfies these hypotheses as an *ancient pancake*. In higher dimensions, Xu-Jia Wang has constructed ancient pancakes in $\mathbb{R}^{n+1}$ by taking a limit of solutions to the Dirichlet problem for the level set flow [28].

Recently, we have provided a different construction of an $O(1) \times O(n)$-invariant ancient pancake, including a precise description of its asymptotics using methods that are rather different from Wang's.

Theorem 3.1 (Existence of ancient pancakes [7] (cf. [28]))**.** *There exists a compact, convex, $O(1)\times O(n)$-invariant ancient solution $\{M_t^n\}_{t\in(-\infty,0)}$ to mean curvature flow in $\mathbb{R}^{n+1}$ which lies in the stationary slab $\Sigma := \{x \in \mathbb{R}^{n+1} : |x_1| < \frac{\pi}{2}\}$ and has the following properties.*

- *(1 a) $\{\lambda M_{\lambda^{-2}t}\}_{t\in(-\infty,0)}$ converges uniformly in the smooth topology to the shrinking sphere $S^n_{\sqrt{-2nt}}$ as $\lambda \to 0$,*
- *(1 b) $\{M_{t+s}\}_{t\in(-\infty,-s)}$ converges locally uniformly in the smooth topology to the stationary solution $\partial\Sigma$ as $s \to -\infty$, and*
- *(1 c) for any unit vector $e \in \{e_1\}^\perp$, $\{M_{t+s} - P(e, s)\}_{t\in(-\infty,-s)}$ converges locally uniformly in the smooth topology as $s \to -\infty$ to the Grim hyperplane (see §2) which translates with unit speed in the direction e, where, given any $v \in S^n$, $P(v, t)$ denotes the unique point of M_t^n with outward pointing unit normal v.*

Moreover, as $t \to -\infty$,

- *(2 a) $\min_{M_t} H = H(P(e_1, t)) \leq o\left(\frac{1}{(-t)^k}\right)$ for any $k > 0$,*
- *(2 b) $\min_{p\in M_t} |p| = |P(e_1, t)| \geq \frac{\pi}{2} - o\left(\frac{1}{(-t)^k}\right)$ for any $k > 0$ and*
- *(3 a) $\max_{M_t} H = H(P(\varphi, t)) \geq \left(1 + \frac{n-1}{-t} + o\left(\frac{1}{(-t)^{2-\varepsilon}}\right)\right)$ for any unit vector $\varphi \in \{e_1\}^\perp$ and any $\varepsilon > 0$, and*
- *(3 b) $\max_{p\in M_t} |p| = |P(\varphi, t)| = -t + (n-1)\log(-t) + C + o(1)$ for any unit vector $\varphi \in \{e_1\}^\perp$, where $C \in \mathbb{R}$ is some constant.*

One of our motivations for studying translators lying in slab regions is to study ancient pancake solutions which are not necessarily rotationally symmetric. Based on the description of the translators with $O(1) \times O(n-1)$-symmetry contained in the previous section, the following conjecture appears natural.

Conjecture 3.2 (Dihedral pancakes)**.** *Given any $k \geq 3$ there exists an ancient pancake lying in the slab $(-\frac{\pi}{2}\sec\frac{\pi}{k}, \frac{\pi}{2}\sec\frac{\pi}{k}) \times \mathbb{R}^2 \subset \mathbb{R}^3$ (and in no smaller slab) with symmetry group $O(1) \times D_k$, where D_k is the symmetry group of the regular k-sided polygon. Modulo translations and*

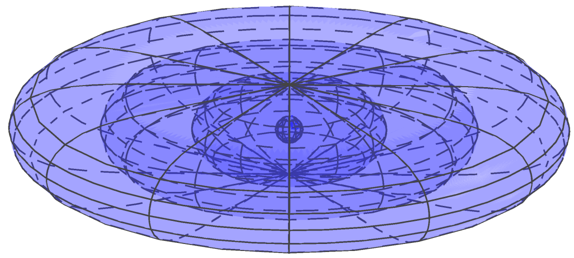


FIGURE 5. The rotationally symmetric ancient pancake. It shrinks to a round point as $t \to 0$ and sweeps out a slab of width π as $t \to -\infty$.

rotations, this is the unique such solution. Let $\{\phi_i\}_{i=1}^k \subset \mathbb{C} \cong \{0\} \times \mathbb{R}^2$ *be the* k*-th roots of unity in the* $\{x_1 = 0\}$ *plane. Up to a rotation, the solution has the following asymptotics: Given a unit vector* $\phi \in \{0\} \times \mathbb{R}^2$*, the asymptotic translator in the* ϕ*-direction is the oblique Grim plane* $\Gamma^2_{\frac{\pi}{k}}$*, except when* $\phi \in \{\phi_i\}_{i=1}^k$*, in which case the asymptotic translator is the flying wing translator* $W^2_{\frac{\pi}{k}}$*.*

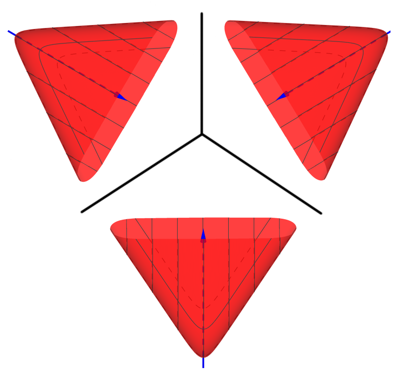


FIGURE 6. Gluing three flying wing translators at infinity to form a compact ancient solution.

Generalizing these principles leads to the following natural question.

Question 3.3. *Do there exist translators contained in slab regions of* $\mathbb{R}^{n+1}$ *with symmetry groups* $O(1) \times G^{n-1}$*, where* G^{n-1} *is the symmetry group of a regular* $(n-1)$*-polytope?*

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