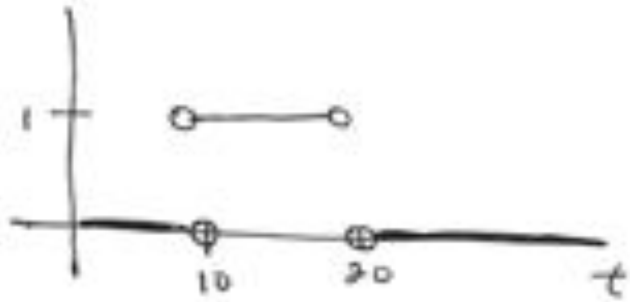
Ex.) Solve $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} = \begin{cases} 0, & 0 < t < 10 \\ 1, & 10 < t < 20 \\ 0, & t > 20 \end{cases}$, $y(0) = 0$, $y'(0) = 0$

The right side :



in terms of unit step functions can rewrite as

$$u(t-10) - u(t-20) \quad \text{so then}$$

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = u(t-10) - u(t-20)$$

Take $\mathcal{L}T$ of both sides,

$$\left(s^2Y(s) - sy(0) - y'(0)\right) + 2\left(sY(s) - y(0)\right) = \frac{e^{-10s}}{s} - \frac{e^{-20s}}{s}$$

$$s^2Y(s) + 2sY(s) = \frac{e^{-10s} - e^{-20s}}{s}$$

$$Y(s) = \frac{e^{-10s} - e^{-20s}}{s^2(s+2)}$$

For the moment consider just $\dfrac{1}{s^2(s+2)}$

$$\frac{1}{s^2(s+2)} = \overset{-\frac{1}{4}}{\frac{A}{s}} + \overset{\frac{1}{2}}{\frac{B}{s^2}} + \overset{\frac{1}{4}}{\frac{C}{s+2}}$$

Let $s=1 \Rightarrow \frac{1}{3} = A + \frac{1}{2} + \frac{1}{12} \Rightarrow A = -\frac{1}{4}$

So then $$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2)}\right\} = -\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$
$$= -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t}$$

Now we have

$$Y(s) = \frac{1}{s^2(s+2)}e^{-10s} - \frac{1}{s^2(s+2)}e^{-20s}$$

$$y(t) = -\frac{1}{4}u(t-10) + \frac{1}{2}(t-10)u(t-10) + \frac{1}{4}e^{-2(t-10)}u(t-10)$$
$$-\left(-\frac{1}{4}u(t-20) + \frac{1}{2}(t-20)u(t-20) + \frac{1}{4}e^{-2(t-20)}u(t-20)\right)$$

$$y(t) = \frac{1}{4}\left(2t - 21 + e^{-2(t-10)}\right)u(t-10) -$$
$$\frac{1}{4}\left(-41 + 2t + e^{-2(t-20)}\right)u(t-20)$$