$$y'' + y = e^{-2t}\sin t \quad , \quad y(0) = 0 = y'(0) \qquad 1$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{(s+2)^2+1}$$

$$s^2Y(s) + Y(s) = \frac{1}{(s+2)^2+1}$$

$$Y(s)(s^2+1) = \frac{1}{(s+2)^2+1} \Rightarrow Y(s) = \frac{1}{((s+2)^2+1)(s^2+1)}$$

Need $\mathcal{L}^{-1}$ of both sides after PFD.

$$\frac{1}{((s+2)^2+1)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{C(s+2)+D}{(s+2)^2+1}$$

let $S=0$ $\quad \frac{1}{5} = B + \frac{2C+D}{5}$

$1 = 5B + 2C + D$

$1 = 5B + 2C + D$

let $S=1$ $\quad \frac{1}{20} = \frac{A+B}{2} + \frac{3C+D}{10}$

$1 = 10A + 10B + 6C + 2D$

$10A + 10B + 6C + 2D = 1$

$S=-1$ $\quad \frac{1}{4} = \frac{-A+B}{2} + \frac{C+D}{2}$

$1 = -2A + 2B + 2C + 2D$

$-2A + 2B + 2C + 2D = 1$

$S=-2$ $\quad \frac{1}{5} = \frac{-2A+B}{5} + D$

$1 = -2A + B + 5D$

$-2A + B + 5D = 1$

So we have
$$\begin{cases} -5B + 2C + D = 1 & (i) \\ 10A + 10B + 6C + 2D = 1 & (ii) \\ -2A + 2B + 2C + 2D = 1 & (iii) \\ -2A + B + 5D = 1 & (iv) \end{cases}$$

Let's eliminate "A"

$-(iv)+(iii) \Rightarrow$ $B + 2C - 3D = 0$

$5(iv)+(ii) \Rightarrow$ $15B + 6C + 27D = 6$

(i) $5B + 2C + D = 1$

use top eqn. to eliminate "C" in eqs 2 & 3

$$\begin{cases} 12B + 36D = 6 \\ 4B + 4D = 1 \end{cases}$$

Now for $\begin{cases} 12B + 36D = 6 \\ (-3)\{ 4B + 4D = 1 \end{cases} \Rightarrow \begin{cases} 12B + 36D = 6 \\ -12B - 12D = -3 \end{cases} \Rightarrow 24D = 3 \Rightarrow D = \frac{1}{8}$

So $D = \frac{1}{8}$, begin back substituting. Then

$A = -\frac{1}{8}$

$B = \frac{1}{8}$

$C = \frac{1}{8}$

$D = \frac{1}{8}$

So $$Y(s) = \frac{1}{((s+2)^2+1)(s^2+1)} = \frac{-\frac{1}{8}s + \frac{1}{8}}{s^2+1} + \frac{\frac{1}{8}(s+2) + \frac{1}{8}}{(s+2)^2+1}$$

$$Y(s) = -\frac{1}{8}\frac{s}{s^2+1} + \frac{1}{8}\frac{1}{s^2+1} + \frac{1}{8}\frac{s+2}{(s+2)^2+1} + \frac{1}{8}\frac{1}{(s+2)^2+1}$$

Now take INV. LT of both sides.

$$y(t) = -\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} + \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$

$$y(t) = -\frac{1}{8}\cos(t) + \frac{1}{8}\sin(t) + \frac{1}{8}e^{-2t}\cos(t) + \frac{1}{8}e^{-2t}\sin(t)$$