Let $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2+2)y = 0$ and

find a PS solution centered @ $x_0 = 0$.

Remember this DE is analytic $\forall x$ and we will have <u>two</u> linearly indep. solutions.

Assume $\divideontimes$ $y = \sum_{n=0}^{\infty} C_n x^n$ is a soln.

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots + C_n x^n + \cdots$$

then

$$y' = C_1 + 2C_2 x + 3C_3 x^2 + \cdots + nC_n x^{n-1} + \cdots$$

$\divideontimes$ $\frac{dy}{dx} = y' = \sum_{n=1}^{\infty} nC_n x^{n-1}$ and also

$$\frac{d^2y}{dx^2} = y'' = \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} \quad \divideontimes$$

Now just plug into the DE and we have

$$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + x\sum_{n=1}^{\infty} nC_n x^{n-1} + x^2\sum_{n=0}^{\infty} C_n x^n + 2\sum_{n=0}^{\infty} C_n x^n = 0$$

$$\underset{*}{\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2}} + \sum_{n=1}^{\infty} nC_n x^n + \underset{*}{\sum_{n=0}^{\infty} C_n x^{n+2}} + 2\sum_{n=0}^{\infty} C_n x^n = 0.$$

we want to rewrite the two * terms so that each exponent becomes $\underline{n}$.

Look at first $\displaystyle\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2}$

$$\text{Let} \quad K = n-2 \implies n = K+2$$

Look at the circled Lower index for the sum and note for n to equal 2, $K = 0$. So we have

$$\sum_{K=0}^{\infty} (K+2)(K+1)C_{K+2}\, x^K$$

and K is just a temporary dummy variable just sub n for K and we have

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, C_{n+2}\, x^n$$

for the second * term $\sum_{n=0}^{\infty} C_n x^{n+2}$

Let $K = n+2 \Rightarrow n = K-2$

and for $n=0 \Rightarrow K=2$ so then

$\sum_{K=2}^{\infty} C_{K-2} x^K$ OR $\sum_{n=2}^{\infty} C_{n-2} x^n$ so we

now have in the orig. DE

$$\sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2}x^n + \sum_{n=1}^{\infty} n c_n x^n + \underset{\text{ok}}{\sum_{n=2}^{\infty} C_{n-2}x^n} + 2\sum_{n=0}^{\infty} C_n x^n = 0$$

we now want all the Σ to start at $n=2$

$$\sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2}x^n = 2C_2 + 6C_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)C_{n+2}x^n$$

$$\sum_{n=1}^{\infty} n c_n x^n = c_1 x + \sum_{n=2}^{\infty} n c_n x^n$$

and

$$2\sum_{n=0}^{\infty} C_n x^n = 2C_0 + 2C_1 x + 2\sum_{n=2}^{\infty} C_n x^n$$

Almost there! We now have

$$\underline{2C_2} + \underline{\underline{6C_3 x}} + \sum_{n=2}^{\infty}(n+2)(n+1)C_{n+2}x^n + \underline{\underline{C_1 x}} + \sum_{n=2}^{\infty} nC_n x^n$$

$$+ \sum_{n=2}^{\infty} C_{n-2}x^n + 2\underline{C_0} + 2\underline{\underline{C_1 x}} + 2\sum_{n=2}^{\infty} C_n x^n = 0$$

$$(2C_2 + 2C_0) + (3C_1 x + 6C_3 x) + \sum_{n=2}^{\infty}\left\{(n+2)(n+1)C_{n+2} + (n+2)C_n + C_{n-2}\right\}x^n$$

$$= 0$$

on left we have a <u>Big</u> polynomial and on the right 0 is a polynomial so corresponding coefficients must be zero. So it must be that

$$2C_0 + 2C_2 = 0$$
$$3C_1 + 6C_3 = 0 \quad \text{and}$$

$$(n+2)(n+1)C_{n+2} + (n+2)C_n + C_{n-2} = 0, \quad \text{for } n \geq 2.$$

from the first eqn. $\quad C_2 = -C_0$

from the 2nd eqn $\quad C_3 = -\frac{1}{2}C_1$

and

$$C_{n+2} = -\frac{C_{n-2} + (n+2)C_n}{(n+1)(n+2)}, \quad n \geq 2$$

and we have a recurrence formula.

so if we let $n=2$

$$C_4 = -\frac{C_0 + 4C_2}{12} = \underline{\frac{1}{4}C_0}$$

for $n=3$ $$C_5 = -\frac{C_1 + 5C_3}{20} = \frac{3}{40}C_1$$

we can write the "even" coefficients in terms of C_0 and "odd" coefficients in term of C_1.

using what we have currently we have

$$y = C_0 + C_1 x - C_0 x^2 - \frac{1}{2}C_1 x^3 + \frac{1}{4}C_0 x^4 + \frac{3}{40}C_1 x^5 + \dots$$

OR

$$y = C_0\left(1 - x^2 + \frac{1}{4}x^4 + \dots\right) + C_1\left(x - \frac{1}{2}x^3 + \frac{3}{40}x^5 + \dots\right)$$

So we have y in terms of two linearly indep. solns.

$$y = C_0\left(1 - x^2 + \frac{x^4}{4} + \dots\right) + C_1\left(x - \frac{1}{2}x^3 + \frac{3}{40}x^5 + \dots\right)$$

∎

where C_0 & C_1 are arbitrary constants.