$$y'' - (x+1)y' + x^2 y = x \quad , \quad y(0) = y'(0) = 1$$

$$y'' + (-x-1)y' + x^2 y = x \quad , \quad \text{first let } y = \sum_{n=0}^{\infty} C_n x^n$$

$$y = C_0 + C_1 x + C_2 x^2 + \cdots$$

clearly using $y(0) = 1 \Rightarrow \underline{\underline{C_0 = 1}}$ and

$$y' = C_1 + 2C_2 x + 3C_3 x^2 + \cdots \Rightarrow \underline{\underline{C_1 = 1}}$$

So $y = \sum_{n=0}^{\infty} C_n x^n$, $y' = \sum_{n=1}^{\infty} n C_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$

Then

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - x\sum_{n=1}^{\infty} n C_n x^{n-1} - \sum_{n=1}^{\infty} n C_n x^{n-1} + x^2 \sum_{n=0}^{\infty} C_n x^n = x$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - \sum_{n=1}^{\infty} n C_n x^n - \sum_{n=1}^{\infty} n C_n x^{n-1} + \sum_{n=0}^{\infty} C_n x^{n+2} = x$$

4th term

rewrite 1st, 3rd & 5th terms to power of "n".

$$\sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n - \sum_{n=1}^{\infty} n C_n x^n - \sum_{n=0}^{\infty} (n+1) C_{n+1} x^n + \sum_{n=2}^{\infty} C_{n-2} x^n = x$$

expand as needed so that all sums begin @ $n=2$.

$$(2C_2 - C_1) + (6C_3 - 2C_2 - C_1)x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)C_{n+2} - nC_n - (n+1)C_{n+1} + C_{n-2}\right]x^n = x$$

remember $C_0 = 1$ & $C_1 = 1$. Comparing coefficients on each side of the equation gives

$$2C_2 - C_1 = 0 \Rightarrow C_2 = \tfrac{1}{2}C_1 = \tfrac{1}{2} \quad \text{and}$$

$$6C_3 - 2C_2 - C_1 = 1$$

$$6C_3 - 2(\tfrac{1}{2}) - 1 = 1 \qquad \text{\&} \qquad \underline{\underline{C_2 = \tfrac{1}{2}}}$$

$$6C_3 = 3$$

$$\underline{\underline{C_3 = \tfrac{1}{2}}}$$

then $(n+2)(n+1)C_{n+2} - nC_n - (n+1)C_{n+1} + C_{n-2} = 0$, $n \geq 2$

OR

$$C_{n+2} = \frac{nC_n + (n+1)C_{n+1} - C_{n-2}}{(n+2)(n+1)}, \quad n \geq 2$$

$n=2$

$$C_4 = \frac{2C_2 + 3C_3 - C_0}{12} = \frac{1 + \frac{3}{2} - 1}{12} = \frac{1}{8}$$

$n=3$

$$C_5 = \frac{3C_3 + 4C_4 - C_1}{20} = \frac{3(\frac{1}{2}) + 4(\frac{1}{8}) - 1}{20} = \frac{1}{20}$$

So for y we have

$$y = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 + \tfrac{1}{8}x^4 + \tfrac{1}{20}x^5 + \ldots$$

Now for successive differentiation

$$y'' - (x+1)y' + x^2 y = x \quad , \quad y(0) = 1$$

rewrite

$$y'(0) = 1 \qquad y''(0) = 1$$

$$y'' + (-x-1)y' + x^2 y = x \quad , \text{ find } y''(0)$$

$$y''(0) + (\cancel{x} - 1)(1) + \cancel{x}^2(1) = \cancel{x}$$

$$y''(0) - 1 = 0 \implies y''(0) = 1$$

Differentiate

$$y''' + (-x-1)y'' - y' + x^2 y' + 2xy = 1 \quad , \text{ Then find } y'''(0)$$

$$y'''(0) + (0-1)(1) - 1 + \underline{0^2(1)} + \underline{2(0)(1)} = 1$$

$$y'''(0) - 2 = 1 \implies \boxed{y'''(0) = 3}$$

rewrite $\quad y''' + (-x-1)y'' + (x^2-1)y' + 2xy = 1 \quad$ Then

$$y^4 + (-x-1)y''' - y'' + (x^2-1)y'' + 2xy' + 2xy' + 2y = 0$$

so $\quad y^4(0) + (0-1)(3) - 1 + (0-1)(1) + \underline{2(0)(1)} + \underline{2(0)(1)} + 2(1) = 0$

$$y^4(0) - 3 - 1 - 1 + 2 = 0 \implies \boxed{y^4(0) = 3}$$

rewrite

$$y^4 + (-x-1)y''' + (x^2-2)y'' + 4xy' + 2y = 0$$

differentiate once more

$$y^{5} + (-x-1)y^{4} - y''' + (x^2-2)\,y''' + 2xy'' + 4xy'' + 4y' + 2y' = 0$$

Then

$$y^{5}(0) - 3 - 3 - 6 + 4 + 2 = 0$$

$$y^{5}(0) = 6$$

OK, remember $y = \sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} \frac{y^{n}(0)}{n!} x^n$

$$y = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{4}(0)}{4!}x^4 + \frac{y^{5}(0)}{5!}x^5 + \cdots$$

$$y = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 + \tfrac{1}{8}x^4 + \tfrac{1}{20}x^5 + \cdots$$

as before.