7.2

Def: Let $f(t)$ be a function defined on $[0,\infty)$. The LT of f is

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

The domain of $F(s)$ are those values of s so that the integral converges.

ex.) for $t \geq 0$, $\mathcal{L}\{1\}$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{A\to\infty} \int_0^A e^{-st}\,dt$$

(i) Consider $\displaystyle\int e^{-st}\,dt = -\frac{1}{s} e^{-st}$

(2) $\displaystyle -\frac{1}{s}\lim_{A\to\infty} e^{-st}\Big|_0^A = -\frac{1}{s}\lim_{A\to 0}\left[\frac{1}{e^{sA}} - 1\right]$

$$= \frac{1}{s}$$

$$\therefore \quad \mathcal{L}\{1\} = \frac{1}{s}$$

Ex 2 $\mathcal{L}\{e^{at}\}$, a constant.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\, dt =$$

$$\lim_{A\to\infty} \int_0^A e^{-(s-a)t}\, dt$$

i.) $\displaystyle\int e^{-(s-a)t}\, dt = \frac{-1}{s-a} e^{-(s-a)t}$ Then

$$\frac{-1}{s-a} \lim_{A\to\infty} e^{-(s-a)t} \Big|_0^A = \frac{-1}{s-a} \lim_{A\to\infty} \frac{1}{e^{(s-a)t}} \Big|_0^A$$

$$= \frac{-1}{s-a} \lim_{A\to\infty} \left[\cancel{\frac{1}{e^{(s-a)A}}}_{0} - 1 \right] = \frac{1}{s-a}$$

only if $s > a$

$$\therefore \mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a.$$

Ex 4

Let $f(t) = \begin{cases} 2, & 0 < t < 5 \\ 0, & 5 < t < 10 \\ e^{4t}, & t > 10 \end{cases}$

$$\mathcal{L}\{f(t)\} = \int_0^5 e^{-st} \cdot 2\,dt \quad (1) + \int_5^{10} e^{-st} \cdot 0\,dt \quad (2)$$

$$+ \int_{10}^{\infty} e^{-st} \cdot e^{4t}\,dt \quad (3)$$

for (1) $\quad 2\int_0^5 e^{-st}\,dt = -\frac{2}{s} e^{-st}\Big|_0^5$

$$= -\frac{2}{s}\left[e^{-5s} - 1\right] = \frac{2}{s} - \frac{2e^{-5s}}{s}$$

(2) $\quad \int_5^{10} e^{-st} \cdot 0\,dt = 0.$

(3) $\quad \int_{10}^{\infty} e^{-(s-4)t}\,dt = \lim_{A\to\infty} \int_{10}^{A} e^{-(s-4)t}\,dt$

Ex 4 continued

$$\int_{10}^{\infty} e^{-(s-4)t}\,dt = \lim_{A\to\infty} \int_{10}^{A} e^{-(s-4)t}\,dt$$

$$= \frac{-1}{s-4} \lim_{A\to\infty} \left[\cancelto{0}{\frac{1}{e^{(s-4)A}}} - e^{-10(s-4)} \right]$$

if $s > 4$

$$= \frac{e^{-10(s-4)}}{s-4}$$

$\therefore$

$$\mathcal{L}\{f(t)\} = \frac{2}{s} - \frac{2e^{-5s}}{s} + \frac{e^{-10(s-4)}}{s-4}, \quad s > 4.$$

The Laplace Transform is linear (a linear transformation)

A function (transformation) is linear if

a.) $\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$

b.) $\mathcal{L}\{\alpha f(t)\} = \alpha \mathcal{L}\{f(t)\}$

Pf: Consider $\mathcal{L}\{f(t)+g(t)\} =$

a) $$\int_0^\infty e^{-st}\big(f(t)+g(t)\big)\,dt = \int_0^\infty \left(e^{-st}f(t) + e^{-st}g(t)\right)dt$$

$$= \int_0^\infty e^{-st} f(t)\,dt + \int_0^\infty e^{-st} g(t)\,dt$$

$$= \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \quad \blacksquare$$

b) $$\mathcal{L}\{\alpha f(t)\} = \int_0^\infty e^{-st}\big(\alpha f(t)\big)\,dt$$

$$= \alpha \int_0^\infty e^{-st} f(t)\,dt$$

$$= \alpha \mathcal{L}\{f(t)\} \quad \blacksquare$$

Ex 5

Find $\mathcal{L}\left\{11 + 5e^{4t} - 6\sin(2t)\right\}$

$$= \mathcal{L}\{11\} + \mathcal{L}\{5e^{4t}\} + \mathcal{L}\{-6\sin(2t)\}$$

$$= 11\,\mathcal{L}\{1\} + 5\,\mathcal{L}\{e^{4t}\} - 6\,\mathcal{L}\{\sin(2t)\}$$

$$= \frac{11}{s} + \frac{5}{s-4} - 6\left(\frac{2}{s^2+2^2}\right)$$

$$= \frac{11}{s} + \frac{5}{s-4} - \frac{12}{s^2+4}$$

<u>Exponential Order</u>

$f(t)$ is said to be of <u>exponential</u> <u>order</u> α if there are pos. constants M & T st

$$|f(t)| \le Me^{\alpha t}, \quad t \ge T$$

$f(t)$ grows no faster than $Me^{\alpha t}$ (bounded)

Implication is integral in $\mathcal{L}T$ definition will converge if $f(t)$ is bdd by $Me^{\alpha t}$