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Pb. 3

8.3 $(\underline{\underline{\theta^2-2}})y'' + 2y' + (\sin\theta)\, y = 0$

$$y'' + \underbrace{\frac{2}{\theta^2-2}}_{P_1(\theta)} y' + \underbrace{\frac{(\sin\theta)}{\theta^2-2}}_{P_2(\theta)} y = 0$$

$\theta = \pm\sqrt{2}$ singular Points

$$\underline{(\theta^2-2)y''} + \underline{2y'} + \boxed{(\sin\theta)}\, y = 0$$

Series Solution

$$y = \sum_{n=0}^{\infty} C_n \theta^n = C_0 + C_1\theta + C_2\theta^2 + \cdots$$

Maclaurren

$$\sin\theta = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} = \boxed{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}} - \frac{\theta^7}{7!} + \cdots$$

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$$y'' - xy = 0 \qquad x_0 = -2$$

Our answer $\quad y = \sum_{n=0}^{\infty} C_n (x+2)^n$ *

$$y = C_0 + C_1(x+2) + C_2(x+2)^2 + C_3(x+2)^3 + \cdots$$

$$y' = \sum_{n=1}^{\infty} n C_n (x+2)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) C_n (x+2)^{n-2}$$

$$* \quad \sum_{n=2}^{\infty} n(n-1) C_n (x+2)^{n-2} - x \sum_{n=0}^{\infty} C_n (x+2)^n = 0 \quad *$$

slick

$$\sum_{n=2}^{\infty} n(n-1) C_n (x+2)^{n-2} - (x+2-2) \sum_{n=0}^{\infty} C_n (x+2)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) C_n (x+2)^{n-2} - (x+2) \sum_{n=0}^{\infty} C_n (x+2)^n + 2 \sum_{n=0}^{\infty} C_n (x+2)^n = 0$$

$$* \quad \sum_{n=2}^{\infty} n(n-1) C_n (x+2)^{n-2} - \sum_{n=0}^{\infty} C_n (x+2)^{n+1} + 2 \sum_{n=0}^{\infty} C_n (x+2)^n = 0$$

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$$\sum_{n=2}^{\infty} n(n-1) C_n (x+2)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} (x+2)^n$$

replace n with $n+2$

$$\sum_{n=0}^{\infty} C_n (x+2)^{n+1} = \sum_{n=1}^{\infty} C_{n-1} (x+2)^n$$

replace n with $n-1$

$\therefore$

$$\sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} (x+2)^n - \sum_{n=1}^{\infty} C_{n-1} (x+2)^n + 2 \sum_{n=0}^{\infty} C_n (x+2)^n = 0$$

Now wish indices begin @ $n=1$

$$2C_2 + \sum_{n=1}^{\infty} (n+2)(n+1) C_{n+2} (x+2)^n - \sum_{n=1}^{\infty} C_{n-1} (x+2)^n$$

$$+ 2\left(C_0 + \sum_{n=1}^{\infty} C_n (x+2)^n \right) = 0$$

$\therefore$ $2C_2 + 2C_0 = 0$

$$\boxed{C_2 = -C_0}$$

$$+ \sum_{n=1}^{\infty} \left((n+2)(n+1) C_{n+2} - C_{n-1} + 2C_n \right) (x+2)^n = 0$$

$$C_{n+2} = \frac{C_{n-1} - 2C_n}{(n+2)(n+1)}, \quad n \geq 1$$

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$C_2 = -C_0$

$$C_{n+2} = \frac{C_{n-1} - 2C_n}{(n+2)(n+1)}, \quad n \geq 1$$

Let $n=1$ $\quad * \ C_3 = \dfrac{C_0 - 2C_1}{6} = \frac{1}{6}C_0 - \frac{1}{3}C_1$

Let $n=2$ $\quad C_4 = \dfrac{C_1 - 2C_2}{12} = \frac{1}{12}C_1 - \frac{1}{6}(-C_0)$

$* \ C_4 = \frac{1}{12}C_1 + \frac{1}{6}C_0$

Let $n=3$ $\quad C_5 = \dfrac{C_2 - 2C_3}{20} = \frac{1}{20}C_2 - \frac{1}{10}C_3$

$$C_5 = -\frac{1}{20}C_0 - \frac{1}{10}\left(\frac{1}{6}C_0 - \frac{1}{3}C_1\right)$$

$$= -\frac{1}{20}C_0 - \frac{1}{60}C_0 + \frac{1}{30}C_1$$

$$= -\frac{4}{60}C_0 + \frac{1}{30}C_1$$

$* \ C_5 = -\frac{1}{15}C_0 + \frac{1}{30}C_1$

$$y = C_0 + C_1(x+2) + C_2(x+2)^2 + \cdots$$

$$y = C_0 + \underline{C_1(x+2)} - C_0(x+2)^2 + \left(\tfrac{1}{6}C_0 - \tfrac{1}{3}C_1\right)(x+2)^3$$
$$+ \left(\tfrac{1}{12}C_1 + \tfrac{1}{6}C_0\right)(x+2)^4$$

$$y = C_0\left(1 - (x+2)^2 + \tfrac{1}{6}(x+2)^3 + \tfrac{1}{6}(x+2)^4 + \cdots\right)$$
$$+ C_1\left((x+2)^1 - \tfrac{1}{3}(x+2)^3 + \tfrac{1}{12}(x+2)^4 + \cdots\right)$$

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$$w'' + 3x w' - w = 0 \qquad w(0) = 2 \quad w'(0) = 0$$

$$w = \sum_{n=0}^{\infty} C_n x^n \qquad w' = \sum_{n=1}^{\infty} n C_n x^{n-1}$$

$$w'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + 3x \sum_{n=1}^{\infty} n C_n x^{n-1} - \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n + \sum_{n=1}^{\infty} 3n C_n x^n - \sum_{n=0}^{\infty} C_n x^n = 0$$

$$2C_2 + \sum_{n=1}^{\infty} (n+1)(n+2) C_{n+2} x^n + \sum_{n=1}^{\infty} 3n C_n x^n - C_0 - \sum_{n=1}^{\infty} C_n x^n = 0$$

$$\underbrace{2C_2 - C_0 = 0}_{} + \sum_{n=1}^{\infty} \left((n+1)(n+2) C_{n+2} + 3n C_n - C_n \right) x^n = 0$$

$$\boxed{C_2 = \tfrac{1}{2} C_0}$$

Then $$C_{n+2} = \frac{C_n - 3_n C_n}{(n+1)(n+2)}, \quad n \geq 1$$

$w(0) = 2 \qquad w'(0) = 0$

$$w = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \cdots$$

$\boxed{2 = C_0}$ using initial conditions

$w' = C_1 + 2C_2 x + \cdots$

$\boxed{0 = C_1}$

$\therefore \; C_2 = \frac{1}{2}(2) = 1$

Let $n=1$ $\quad C_3 = \frac{C_1 - 3C_1}{6} = -\frac{2C_1}{6} = -\frac{1}{3}C_1$

$\therefore \; C_3 = 0$

Let $n=2$ $\quad C_4 = \frac{C_2 - 6C_2}{12} = -\frac{5}{12}C_2 = -\frac{5}{12}$

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$$W = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \cdots$$

$$W = 2 + x^2 - \frac{5}{12} x^4 + \cdots$$

8.3

Problem 3 $\quad (\theta^2-2)y'' + 2y' + (\sin\theta)y = 0$

rewrite as $\quad y'' + \frac{2}{\theta^2-2}y' + \frac{\sin\theta}{\theta^2-2}y = 0$

Singular points are $\theta^2-2=0$ OR $\theta = \pm\sqrt{2}$
all other values of θ are ordinary.

#17 $\quad w'' - x^2 w' + w = 0$

assume solution is of form $w = \sum_{n=0}^{\infty} C_n x^n$ Then

$w' = \sum_{n=1}^{\infty} n C_n x^{n-1}$ & $w'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$ giving

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - x^2 \sum_{n=1}^{\infty} n C_n x^{n-1} + \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - \sum_{\substack{n=1 \\ n-1=1}}^{\infty} n C_n x^{n+1} + \sum_{n=0}^{\infty} C_n x^n = 0$$

now write the sums so that the exponent for x is n.

$$\sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n - \sum_{n=2}^{\infty} (n-1) C_{n-1} x^n + \sum_{n=0}^{\infty} C_n x^n = 0$$

Now, we want the index of the sums to begin with $n=2$

$$\sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2}X^n = 2C_2 + 6C_3X + \sum_{n=2}^{\infty}(n+2)(n+1)C_{n+2}X^n$$

and

$$\sum_{n=0}^{\infty}C_nx^n = C_0 + C_1x + \sum_{n=2}^{\infty}C_nx^n$$

This gives

$$2C_2 + 6C_3X + C_0 + C_1X + \sum_{n=2}^{\infty}\left((n+2)(n+1)C_{n+2} - (n-1)C_{n-1} + C_n\right)X^n = 0$$

$$\therefore \quad 2C_2 + C_0 = 0 \implies \boxed{C_2 = -\tfrac{1}{2}C_0}$$

$$6C_3 + C_1 = 0 \implies \boxed{C_3 = -\tfrac{1}{6}C_1}$$

and

$$C_{n+2} = \frac{(n-1)C_{n-1} - C_n}{(n+2)(n+1)}, \quad n \geq 2$$

let $n=2$ $\qquad C_4 = \dfrac{C_1 - C_2}{12} = \frac{1}{12}C_1 - \frac{1}{12}\left(-\frac{1}{2}C_0\right)$

$$* \; C_4 = \frac{1}{12}C_1 + \frac{1}{24}C_0$$

let $n=3$ $\qquad C_5 = \dfrac{2C_2 - C_3}{20} = \frac{2}{20}\left(-\frac{1}{2}C_0\right) - \frac{1}{20}\left(-\frac{1}{6}C_1\right)$

$$C_5 = -\frac{1}{20}C_0 + \frac{1}{120}C_1$$

then recall $\quad w = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + \ldots$

$$w = C_0 + C_1 x - \frac{1}{2}C_0 x^2 - \frac{1}{6}C_1 x^3 + \left(\frac{1}{12}C_1 + \frac{1}{24}C_0\right)x^4$$
$$+ \left(-\frac{1}{20}C_0 + \frac{1}{120}C_1\right)x^5 + \ldots$$

$$w = C_0\left(1 - \frac{1}{2}x^2 + \frac{1}{24}\underline{x^4} - \frac{1}{20}\underline{x^5} + \ldots\right) +$$

$$C_1\left(x - \frac{1}{6}x^3 + \frac{1}{12}\underline{x^4} + \frac{1}{120}\underline{x^5} + \ldots\right)$$