The inverse Laplace Transform.

If $\mathcal{L}\{f(t)\} = F(s)$ exists then the inverse of this is $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

ex.) $\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$

The inverse LT is also additive and scalar mult. is preserved.

$$\mathcal{L}^{-1}\{c_1 F_1(s) + c_2 F_2(s)\} = c_1 \mathcal{L}^{-1}\{F_1(s)\} + c_2 \mathcal{L}^{-1}\{F_2(s)\}$$

$$= c_1 f_1(t) + c_2 f_2(t).$$

ex) $\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+4}\right\}$ is not in our table, so

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+4}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= 2\cos(2t) + \frac{1}{2}\sin(2t).$$

$$\mathcal{L}^{-1}\{F(s)\}$$

ex.) Evaluate $\mathcal{L}^{-1}\left\{\frac{s+5}{s^2-2s-3}\right\}$

$$\frac{s+5}{(s-3)(s+1)} = \frac{\overset{2}{A}}{s-3} + \frac{\overset{-1}{B}}{s+1} \Longrightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{s+5}{s^2-2s-3}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 2e^{3t} - e^{-t}.$$

ex.) Find $\mathcal{L}^{-1}\left\{\frac{s^2}{(s+1)^3}\right\}$

$$\frac{s^2}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{\overset{1}{C}}{(s+1)^3}$$

let $s=0 \Rightarrow 0 = A + B + 1 \Longrightarrow A+B=-1$

let $s=1 \Longrightarrow \frac{1}{8} = \frac{A}{2} + \frac{B}{4} + \frac{1}{8} \Longrightarrow \underline{2A+B=0} \Longrightarrow$ $A=1$, $B=-2$

So then

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s+1)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$= e^{-t} - 2te^{-t} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2!}{(s+1)^{2+1}}\right\}$$

$$= e^{-t} - 2te^{-t} + \frac{1}{2}t^2e^{-t}.$$

ex.) Find $\mathcal{L}^{-1}\left\{\frac{9s+14}{(s-2)(s^2+4)}\right\}$

See middle pg. 399

$$\frac{9s+14}{(s-2)(s^2+4)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+4} = \frac{A}{s-2} + \frac{Bs+2D}{s^2+2^2}$$

$$\frac{9s+14}{(s-2)(s^2+4)} = \frac{\overset{4}{A}}{s-2} + \frac{\overset{-4}{B}s+2\overset{1/2}{D}}{s^2+2^2}$$

Let $s=0 \Rightarrow -\frac{14}{8} = -2 + \frac{2D}{4} \Rightarrow D = \frac{1}{2}$

Let $s=1 \Rightarrow -\frac{23}{5} = -4 + \frac{B+1}{5} \Rightarrow B=4$

Then

$$\mathcal{L}^{-1}\left\{\frac{9s+14}{(s-2)(s^2+4)}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + (-4)\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + (-4)\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= 4e^{2t} - 4\cos(2t) + \frac{1}{2}\sin(2t)$$

Recall $\mathcal{L}\left\{e^{at} f(t)\right\} = F(s-a)$

So $\mathcal{L}^{-1}\left\{F(s-a)\right\} = e^{at} f(t)$

Sometimes called the 1st shifting THM.

ex.) Find $\mathcal{L}^{-1}\left\{\dfrac{3}{(s-4)^2+9}\right\}$

we know $\mathcal{L}^{-1}\left\{\dfrac{3}{s^2+3^2}\right\} = \sin(3t)$

so $\mathcal{L}^{-1}\left\{\dfrac{3}{(s-4)^2+3^2}\right\} = e^{4t}\sin(3t)$

ex) Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4s+10}\right\}$

$s^2+4s+10 = s^2+4s+4+10-4 = (s+2)^2+6 = (s+2)^2+(\sqrt{6})^2$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+(\sqrt{6})^2}\right\}$$

$$= \frac{1}{\sqrt{6}}\mathcal{L}^{-1}\left\{\frac{\sqrt{6}}{(s+2)^2+(\sqrt{6})^2}\right\} = \frac{1}{\sqrt{6}} e^{-2t}\sin(\sqrt{6}\,t)$$