Cauchy – Euler Eqns. (As in textbook)

Def: Linear 2^{nd} order eqn.

$$\star \quad at^2y'' + bty' + cy = g(t)$$

guess soln is of form $y = t^r$

$$y = t^r$$

$$y' = rt^{r-1}$$

$$y'' = r(r-1)t^{r-2}$$

Then $\quad ty' = t\left(rt^{r-1}\right) = rt^r$

$$t^2y'' = t^2\left(r(r-1)t^{r-2}\right) = r(r-1)t^r$$

So sub into $\star$ letting $g(t) \equiv 0$

$$ar(r-1)t^r + brt^r + ct^r = 0$$

$$t^r\left(ar(r-1) + br + c\right) = 0$$

$$ar^2 - ar + br + c = 0$$

$$ar^2 + (b-a)r + c = 0 \longleftrightarrow$$

$ar^2 + (b-a)r + c = 0$ char eqn: Aux eqn for CE DE's of order 2.

example 2: Find two LI solutions to

$$3t^2 y'' + 11t y' - 3y = 0, \quad t > 0$$

Let $y = t^r \Rightarrow 3r^2 + (11-3)r - 3 = 0$

$$* \quad 3r^2 + 8r - 3 = 0$$

$$(3r-1)(r+3) = 0$$

$$r = \tfrac{1}{3} \;\&\; r = -3$$

$\therefore \; y_1 = t^{1/3} \;\&\; y_2 = t^{-3}$ are two LI solns.

Note: This matches what we did earlier letting $y = e^{rt}$ in homogeneous eqns with constant coefficients.

This means we will have the same three possibilities.

$ar^2 + (b-a)r + c = 0$

A.) roots real, distinct (Example 2)

B.) roots real, equal

C.) roots $\mathbb{C}$.

for B.) if roots real, equal

$y_1 = t^r \qquad y_2 = \underline{t^r \ln t}$ (why?)

C.) if roots $\mathbb{C}$, then $r = \alpha \pm \beta i$ $\quad b = e^{\ln b}$

$\not{=}$ $t^{\alpha + \beta i}$ and let $\underline{t = e^{\ln t}}$

$t^{\alpha+\beta i} = t^\alpha \cdot t^{\beta i} = t^\alpha (e^{\ln t})^{\beta i} = t^\alpha e^{i\beta \ln t}$

$= t^\alpha \left[\cos(\beta \ln t) + i \sin(\beta \ln t) \right]$

$\{ t^\alpha \cos(\beta \ln t), \quad t^\alpha \sin(\beta \ln t) \}$

ex.) $1\,t^2 \dfrac{d^2y}{dt^2} + 1\,t\dfrac{dy}{dt} + 4y = 0$

let $y = t^r$

$r^2 + \cancel{(1-1)r} + 4 = 0 \implies r^2 + 4 = 0$

$r = \pm 2i \qquad \{\sin(2\ln t), \cos(2\ln t)\}$

$y = C_1 \sin(2\ln t) + C_2 \cos(2\ln t)$