Cauchy-Euler (EXAMPLE NON Homogenous)

Rewrite $t^2 \frac{d^2y}{dx^2} - 2t\frac{dy}{dt} + 2y = t^3$ using $t = e^x$

as a second order DE with constant coefficients

and solve.

From the derivation handout

$t = e^x \Rightarrow x = \ln t$

$$\frac{dy}{dt} = \frac{1}{t}\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{1}{t^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = t^2\frac{d^2y}{dt^2}$$

$$\frac{dy}{dx} = t\frac{dy}{dt} \Rightarrow \left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) - 2\frac{dy}{dx} + 2y = e^{3x}$$

$$\frac{d^2y}{dx^2} + (-2-1)\frac{dy}{dx} + 2y = e^{3x}$$

$$\boxed{\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}}$$

Now solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}$

(A) Solve homogenous equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

Aux: $r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0$ $\quad \{e^x, e^{2x}\}$ fund. Solu set

$r = 1, 2$

(B) Nature of $y_p = Ae^{3x}$ (Since e^{3x} & e^x, e^{2x} are LI)

$y_p' = 3Ae^{3x}$

$y_p'' = 9Ae^{3x}$

$(9A - 9A + 2A)e^{3x} = e^{3x}$

$2A = 1 \Rightarrow A = \frac{1}{2}$

$y_p = \frac{1}{2}e^{3x} \quad \therefore \quad y = C_1e^x + C_2e^{2x} + \frac{1}{2}e^{3x}$

Now translate back to "t"

$t = e^x$ so the origional equation has as a solution

$$y = C_1 t + C_2 t^2 + \frac{1}{2}t^3$$