2.5 Special Int. Factors (~~Non~~-exact –exact)

Consider $$\overset{N}{\frac{x^2}{y}}\,dy + \overset{M}{2x}\,dx = 0 \qquad *$$

$$\frac{\partial M}{\partial y} = 0 \neq \frac{\partial N}{\partial x} = \frac{2x}{y}$$ so not exact!

i.) Mult. by $\frac{1}{x^2}\left(\frac{x^2}{y}\,dy + 2x\,dx\right) = 0 \cdot \frac{1}{x^2}$

$$\overset{M}{\frac{2}{x}}\,dx + \overset{N}{\frac{1}{y}}\,dy = 0$$

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$$ so now exact.

ii.) Now mult. $*$ by y

$$y\left(\frac{x^2}{y}\,dy + 2x\,dx\right) = 0 \cdot y$$

$$\overset{N}{x^2}\,dy + \overset{M}{2xy}\,dx = 0$$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$ so exact.

$\frac{1}{x^2}$ & y integrating factor(s).

So what gives?

$M\,dx + N\,dy = 0$ & $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so not exact.

Spse there is a $\mu = \mu(x,y)$ st

$\mu M\,dx + \mu N\,dy = 0$ is now exact.

Then $\mu = \mu(x,y)$ defined as integ. factor.

If this is the case then:

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu = 0$$ generally tough p.de.

Instead what if $\mu(x,y) \equiv \mu(x)$ function of x alone

$$* \quad \overset{M}{\mu M}\,dx + \overset{N}{\mu N}\,dy = 0 \qquad \$ \qquad \mu = \mu(x)$$

we then have, if exact $\quad \frac{\partial}{\partial y}(\underbrace{\mu M}_{M}) = \frac{\partial}{\partial x}(\underbrace{\mu N}_{N})$

and if $\boxed{\mu = \mu(x)}$ then

$$\mu \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\mu + N\frac{d\mu}{dx} \qquad \text{(Product Rule)}$$

$$\mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{d\mu}{dx} \Longrightarrow$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx = \frac{d\mu}{\mu} \qquad **$$

So $\boxed{\text{IF}}$ $\quad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is anything <u>but</u> some function of x then trouble.

but suppose just a function of x then

$$\frac{d\mu}{\mu} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}dx \Rightarrow \ln\mu = \int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}dx$$

then $\mu(x) = e^{\int \frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N}dx}$ is an integrating factor.

A similar discussion will give

$$\mu(y) = e^{\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} dy} \quad \text{if } \mu = \mu(y).$$

So back to original problem

$$2x\,dx + \frac{x^2}{y}\,dy = 0$$

i.) $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{0 - \frac{2x}{y}}{\frac{x^2}{y}} = -\frac{2x}{\cancel{y}} \cdot \frac{\cancel{y}}{x^2} = -\frac{2}{x}$$

Then $\mu(x) = e^{-2\int \frac{1}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$

so $\mu(x) = \frac{1}{x^2}$ is an integrating factor.

ii) $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{\frac{2x}{y} - 0}{2x} = \frac{\frac{2x}{y}}{2x} = \frac{1}{y}$$

So

$\mu(y) = e^{\int \frac{1}{y} dy} = e^{\ln y} = y$ is an integrating factor.

ex 2.) $\overset{M}{(2x^2+y)}\,dx + \overset{N}{(x^2y-x)}\,dy = 0$

$$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = 2xy - 1$$

Try $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{1-(2xy-1)}{x^2y-x} = \dfrac{2(1-xy)}{-x(1-xy)} = -\dfrac{2}{x}$

so $\mu(x) = e^{-2\int \frac{1}{x}dx} = e^{\ln x^{-2}} = x^{-2} = \dfrac{1}{x^2}$

is an integrating factor:

$$\frac{1}{x^2}(2x^2+y)\,dx + \frac{1}{x^2}(x^2y-x)\,dy = 0$$

$$(2+\frac{y}{x^2})\,dx + (y-\frac{1}{x})\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2} = \frac{\partial N}{\partial x}$$ so now exact!

Now solve as before $\Rightarrow$

$$F(x,y) = 2x - \frac{y}{x} + \frac{1}{2}y^2 = C.$$