A limit theorem for the trajectory of particle moving in random medium

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Motivation: the Lorentz model (1905)

A spherical particle is driven through a random medium by a constant external field a.

The medium is a set of immobile identical spherical obstacles whose centers form a Poisson point process in $\mathbb{R}^d$, $d \geq 2$.

At collisions with obstacles, the particle inelastically reflects with the restitution coefficient $\alpha \in [0, 1]$:

$$v \mapsto v - (1 + \alpha)(v, \nu)\nu,$$

where v is the velocity of the particle and ν is the unit normal to the obstacle.

New model: Markov approximation of the LM

Motion in $\mathbb{R}^d, d \geq 1$, under a constant external field a. Initial position $X(0) = 0$, nonrandom initial velocity v_0.

Assumptions:
By η_n denote the length of the trajectory between nth and $(n+1)$st collisions.
A1: $\{\eta_n\}_{n\geq 0}$ are i.i.d. exponential r.v.'s with mean λ.

By v_n denote the velocity of the particle at nth collision.
A2: At collisions, $v_n \mapsto v_n - \frac{1+\alpha}{2}(v_n + |v_n|\sigma_n)$, where $\{\sigma_n\}_{n\geq 1}$ are i.i.d. random vectors unif. distrd. on the unit sphere $S^{d-1} \subset \mathbb{R}^d$.

A3: $\{\eta_n\}_{n\geq 0}, \{\sigma_n\}_{n\geq 1}$ are independent.

V. ('06), Ravishankar and Triolo ('99) for $\alpha = 1$.

Velocities at collisions v_n form a *Markov chain*:

$$v_{n+1} = v_n - \frac{1+\alpha}{2}\left(v_n + |v_n|\sigma_n\right) + aF\left(v_n - \frac{1+\alpha}{2}\left(v_n + |v_n|\sigma_n\right), \eta_n\right),$$

where the nonrandom function $F : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$\int_0^{F(v,z)} |v + at| dt = z.$$

The main result

By $X(T)$ denote the position of the particle at time T.
Choose a basis of $\mathbb{R}^d$ such that $a = (0, \ldots, 0, |a|)$.

Theorem (V., '06)

Suppose that $0 < \alpha < 1$ and $a \neq 0$. Then there exist constants $c_v > 0$ and $c_1, c_2 \geq 0$ (depending on $|a|, \alpha, \lambda$ and d) such that for any initial velocity $v_0 \in \mathbb{R}^d$,

$$\frac{X(sT) - c_v a s T}{\sqrt{T}} \xrightarrow{\mathcal{D}} (c_1 W_1(\cdot), \ldots, c_1 W_{d-1}(\cdot), c_2 W_d(\cdot))$$

in $C([0,1] \to \mathbb{R}^d)$ as $T \to \infty$, where $W_i(\cdot)$ are independent Wiener processes.

A brief sketch of the proof

Let $N(T)$ be the number of collisions before T, then $\frac{N(T)}{T} \to c > 0$ a.s.
By τ_n denote the moment of the nth collision.
Consider a new Markov chain $\Phi_n := \binom{v_n}{\sigma_n}$. Now

$$
\begin{aligned}
X(sT) &= \sum_{i=1}^{N(sT)} X(\tau_k) - X(\tau_{k+1}) + o(\sqrt{T}) \\
&= \sum_{i=1}^{N(sT)} g(\Phi_k, \Phi_{k-1}) + o(\sqrt{T}) \\
&= \sum_{i=1}^{N(sT)} \tilde{g}(\Phi_k) + o(\sqrt{T}) = \sum_{i=1}^{csT} \tilde{g}(\Phi_k) + o(\sqrt{T})
\end{aligned}
$$

Relations between the models

Collisions in the Lorentz model:

$$v \mapsto v - (1+\alpha)(v,\nu)\nu = v - \frac{1+\alpha}{2}\Big(v + |v|\sigma\Big),$$

where σ is a unit vector such that ν is directed along the bisectrix of the angle between v and σ.
It is easier to describe a collision by σ rather then by ν.

Lemma

We have[1]

1. $\eta_0^{LM} \stackrel{\mathcal{D}}{=} \eta_0$ *for* $\mathbb{R}^d$ *with any* $d \geq 2$*;*

2. $(\eta_0^{LM}, \sigma_1^{LM}) \stackrel{\mathcal{D}}{=} (\eta_0, \sigma_1)$ *for* $\mathbb{R}^3$*.*

[1] under condition that $|v_0|^{\perp} > (r+R)|a|$, where r and R are the radii of the particle and an obstacle.

The Markov approximation model is the limiting case of the LM:

The following approach is by Gallavotti and Lanford ('70s).

Recall that r and R are the radii of the particle and an obstacle. By ρ denote the intensity of the Poisson point process which defines positions of obstacles.

Consider the Grad limit: $r+R \to 0, \rho \to \infty, (r+R)^{d-1}\rho = const$. The last condition implies that the mean free path $\lambda(r+R, \rho)$ also stays constant, denote $\lambda(r+R,\rho) =: \lambda$.

We believe that

$$X^{LM}_{\rho,r,R}(\cdot) \xrightarrow{\mathcal{D}} X^{MALM}_{\lambda}(\cdot).$$

Some references

- K. Ravishankar and L. Triolo (1999) Diffusive limit of the Lorentz model with a uniform field starting from the Markov approximation. *Markov Proc. Rel. Fields*, **5**, 385–421.
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