On the stochastic heat equation

Davar Khoshnevisan
with M. Foondun and E. Nualart

Department of Mathematics
University of Utah
http://www.math.utah.edu/~davar

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Outline

- The heat equation with random forcing

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- The linear equation and its connections with local times of LP’s

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- The linear equation and its connections with local times of LP's
- The nonlinear equation & intermittency, and their connections with recurrence/transience of LP's

The basic problem

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- The nonlinear heat equation for L with forcing $\dot W$:

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  - What if $\dot{W}$ is replaced by spatially-colored noise?

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- Take F.T. [in x]:

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- $\therefore\ \hat{v}(t\,,\xi) = e^{-t\Psi(\xi)}$, and the solution is measure-valued:

$$v(t\,,A) := P\{X_t \in A\} := P_t(A).$$

The heat equation with forcing

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- Interpretation: Multiply by $\phi \in \mathscr{S}(\mathbf{R}_+ \times \mathbf{R}^d)$:

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- Solve by variation of parameters [Duhamel's formula].

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- Therefore (Dalang, 1999): the heat equation has function solutions iff $[1+2\mathrm{Re}\Psi]^{-1} \in L^1(\mathbf{R}^d)$.

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Theorem (Foondun-K-Nualart, 2009+)

Let $\bar{X}_t := X_t - X'_t$ by the symmetrization of X.

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Additive nonlinearities

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Suppose b is bounded and Lipschitz continuous, and the linear heat equation has a function solution u with $u(0\,,x)=0$. *Consider*

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- Using local-time theory, we can construct u with $\text{Osc}\,u\equiv\infty$.
- The blowup of u forces the blowup of U.
- Everything holds if b is Lipschitz continuous.

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- (Dalang, 1999): If the linear equation $[\sigma \equiv 0]$ has a unique solution, then the nonlinear one does too.

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Proposition (Carmona and Molchanov '94)

Intermittency holds if $\overline{\gamma}(2) > 0$ and $\overline{\gamma}(p) < \infty$ for all $p \geq 2$, if $u \geq 0$

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Then, $u(t,\cdot) \in L^2(\mathbf{R}^d)$ a.s. for all $t > 0$, and

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If σ is Lipschitz and $u(0\,,\cdot)$ is nonrandom and bounded, then

$$\frac{\partial}{\partial t} u(t\,,x) = (Lu)(t\,,x) + \sigma(u(t\,,x))\dot{F}(t\,,x)$$

has a unique solution provided that

$$\int_{\mathbf{R}^d} \frac{\hat{f}(\xi)}{1 + 2Re\Psi(\xi)}\, d\xi < \infty \qquad \text{[iff, when } \sigma := const\text{].}$$

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$$\|\exp(-(t/2)\text{Re}\Psi)\|_2^2 = (2\pi)^d \|p_{t/4} * \check{p}_{t/4}\|_2^2$$

Hawkes' Lemma (early half of '80's)

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- **Lemma.** $p_t(x) = P_t(dx)/dx$; TFAE:
  1. $p_t \exists$ and is in $L^2(\mathbf{R}^d)$ for all $t > 0$;
  2. $p_t \exists$ and is in $L^\infty(\mathbf{R}^d)$ for all $t > 0$;
  3. $\exp(-t\mathrm{Re}\Psi) \in L^1(\mathbf{R}^d)$ for all $t > 0$.
- **Proof:** Since $\|p_t\|_2^2 \le \|p_t\|_\infty$, (2)⇒(1). Since $p_t = p_{t/2} * p_{t/2}$, Young's inequality: $\|p_t\|_\infty \le \|p_{t/2}\|_2^2$; therefore, (1)⇔(2). By the inversion theorem, (3)⇒(1)+(2). Suppose (1)+(2). Let $\check{f}(x) := f(-x)$. The F.T. of $p_{t/4} * \check{p}_{t/4}$ is $\exp(-(t/2)\mathrm{Re}\Psi)$. By Plancherel,

$$\|\exp(-(t/2)\mathrm{Re}\Psi)\|_2^2 = (2\pi)^d \|p_{t/4} * \check{p}_{t/4}\|_2^2 \le (2\pi)^d \|p_{t/4} * \check{p}_{t/4}\|_\infty.$$

∴ by Young's inequality,

$$\|\exp(-t\mathrm{Re}\Psi)\|_1 = \|\exp(-(t/2)\mathrm{Re}\Psi)\|_2^2 \le (2\pi)^d \|p_{t/4}\|_2^2.$$

∴ (1)+(2)⇒(3). □

Concluding remarks

Lemma (Zabczyk, '70)

Suppose X is a radial Lévy process in $d \geq 2$. Then, $P_t(dx) \ll dx$ if and only if $\Psi(\xi) \to \infty$ as $\|\xi\| \to \infty$.

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- False if $d = 1$.
- What are good NASC conditions for $P_t(dx) \ll dx$ in terms of Ψ?