Random Walk on $\mathbb{Z}$ with Drift Depending on Occupation Time at 0

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(∗) Slides reproduced from the board talk given

Model

Dynamics

$X = \{X_n : n \in \mathbb{Z}_+\}$ is a nearest neighbor process on $\mathbb{Z}$ defined as follows.
Let $\epsilon = \{\epsilon(n) : n \in \mathbb{N}\}$ be a sequence in $(0, 1)$.

$$P(|X_{n+1}| = |X_n| + 1 | X_0, \ldots, X_n) = \begin{cases} \frac{1}{2} & X_n = 0 \\ \frac{1}{2}(1 - \epsilon(\eta_n)) & \text{otherwise,} \end{cases}$$

where $\eta_n = \#\{1 \le k \le n : X_k = 0\}$.

In words: from 0 jump to ± 1 with equal probabilities, while from $x \neq 0$ jump one step away from 0 with probability $\frac{1}{2}(1 - \epsilon(\eta_n))$, a function of the number of visits to 0 increasing to $\frac{1}{2}$ as the number of visits tends to ∞, or one step towards 0 with probability $\frac{1}{2}(1 + \epsilon(\eta_n))$.

Drift of magnitude $\epsilon(\eta_n)$ directed towards the origin.

Problem

Study asymptotic behavior of X as a function of the sequence ϵ.

Motivation

Classical oscillating random walks.
Analogy with Invasion Percolation on Binary trees by Angel, Goodman, den Hollander and Slade (`http://dx.doi.org/10.1214/07-AOP346`).

Supercritical regime

If $\epsilon(n) \to 0$ sufficiently fast, then X asymptotically coincides with simple random walk.

Theorem

1. *If $n\epsilon(n) \to 0$, then the process $X([\cdot n])/\sqrt{n}$ converges weakly to Brownian motion. (Martingale CLT)*
2. *If $\sum \epsilon(n) < \infty$, then the laws of X and the simple random walk are mutually absolutely continuous (Kakutani Dichotomy)*

Corollories:

1. Running maximum: $\max_{k \leq n} X_k / \sqrt{n}$ converges to the maximum of Brownian motion on $[0, 1]$. Similarly also $\max_{k \leq n} |X_k|$.
2. LIL.
3. Arcsine law.

Subcritical regime. Definition and Limit for X_n

Suppose $\epsilon(n) = n^{-\alpha}L(n)$, where $\alpha \in [0,1]$ and L is slowly varying.
If $\alpha = 0$ assume further $\epsilon(n) \to 0$ and if $\alpha = 1$ assume $n\epsilon(n)/\ln n \to \infty$.

Define

Quantity	Value if $\epsilon(n) = n^{-\alpha}$
$a_n = E\#\{$time of n'th visit to 0$\}$	$\sim (1+\alpha)^{-1}n^{1+\alpha}$
$c_n =$ inverse of $a_n \sim$ "typical" # of visits to 0 up to n'th step	$\sim ((1+\alpha)n)^{\frac{1}{1+\alpha}}$
$b_n = \frac{1}{\epsilon(c_n)} \sim$ "typical" length of "current" excursion	$\sim ((1+\alpha)n)^{\alpha/(1+\alpha)}$

Theorem

Under above conditions X_n/b_n converges in distribution to $e^{-2|x|}dx$.

Explanation: each excursion of the process away from 0 has same distribution as random walk on $\mathbb{Z}_+$ with constant drift towards 0. This is positive recurrent and converges sufficiently fast to invariant distribution which is (almost) geometrically distributed. This property is preserved if looking at the right scale.

Subcritical regime. Running maximum.

Let $h_n = \frac{1}{2} b_n \ln(c_n/b_n)$.
When $\epsilon(n) = n^{-\alpha}$, $h_n \sim \frac{1}{2}(1-\alpha)(1+\alpha)^{(1-\alpha)/(1+\alpha)} n^{\alpha/(1+\alpha)} \ln n$. Let $M_n = \max_{k \le n} X_k$.

Theorem

Assume $\alpha < 1$.

1. $P(M_n \le x h_n) \sim (1+o(1)) c_1 e^{-(1+o(1)) c_2 n^{\frac{1+\alpha}{1-\alpha}(1-x)}}$, $x \in (0,1)$
2. $P(M_n > x h_n) \sim (1+o(1)) c_3 n^{-\frac{1+\alpha}{1-\alpha}(x-1)}$, $x > 1$

Corollary

$\limsup_{n\to\infty} X_n/h_n = \lim_{n\to\infty} M_n/h_n = 1$ *a.s.*

Remarks

1. A general theorem was proved for the entire range of α. This simplified version is brought in order to give a somewhat concrete idea.
2. For any subcritical sequence ϵ, $b_n = o(h_n)$. Therefore, the running maximum M_n and X_n scale differently, suggesting that unlike the supercritical regime, the process does not obey a (non degenerate) scaling limit.
3. Results valid for $\max_{k \le n} |X_k|$ as well.

Subcritical regime. Last Excursion.

Let $V_n = \max\{k \leq n : X_k = 0\}$, the time of the last visit to 0 before n.
Let $\mathfrak{N}$ be the Ito excursion law for Brownian motion with drift -1, and let ζ denote the (random) lifetime of an excursion.

Theorem

1. *Local limit theorem:* $\lim_{n\to\infty} b_{2n} P(X_{2n} = 0) = 2.$
2. $\lim_{n\to\infty} P((2n - V_{2n})/b_{2n}^2 \leq x) = \int_0^x \mathfrak{N}(\zeta > t)dt, \ \ x \geq 0.$
3. *Let* $\{W_k^{(n)} : k \in \mathbb{Z}_+\}$ *be the last excursion from* 0 *starting before time* n. *That is*

$$W_0^{(n)} = 0 \text{ and } W_k^{(n)} = \begin{cases} X_{V_n+k} & \text{if } X_{V_n+m} \neq 0 \text{ for } 1 \leq m \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Then the process $\frac{|W_{[\cdot b_{2n}^2]}|}{b_{2n}}$ *converges weakly to a process whose law is* $\int_0^\infty \mathfrak{N}(\cdot, \zeta > t)dt$.

1. Part (1) is proved by the method use to obtain the weak limit of X_n/b_n
2. We were able to show that the number of visits to 0 is well-localized (LLN and LD) around c_n. This allows to localize the drift and eventually leads to part (2).
3. Once drift is localized, results on convergence of drifted random walks are applied to obtain (3).

Critical regime

We were unable to treat the so-called critical regime, say when $n\epsilon(n) \to 1$.
Reasons:

1. Corrector used to apply Martingale CLT as in supercritical regime is too large.
2. Ergodicity and localization arguments used in subcritical regime fail.

Thank you.