When is a Moving Average a Semimartingale?

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Semimartingales

- Let $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ denote a filtration. Then $X = (X_t)_{t\geq 0}$ is said to be an $\mathcal{F}$-semimartingale if it can be written as

$$X_t = X_0 + M_t + A_t, \qquad t \geq 0,$$

  where M is a càdlàg $\mathcal{F}$-local martingale, A is an $\mathcal{F}$-adapted càdlàg process of bounded variation and X_0 is $\mathcal{F}_0$-measurable.
- A càdlàg process X induces a “reasonable” stochastic integral $\int_0^t Y_s \, dX_s$ if and only if it is a semimartingale.

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Moving averages

The focus of this talk is on the semimartingale property of moving averages $X = (X_t)_{t\geq 0}$, given by

$$X_t = \int_0^t \varphi(t-s)\,dZ_s, \qquad t \geq 0,$$

where $Z = (Z_t)_{t\geq 0}$ is a Lévy process and φ is a deterministic function for which the integral exists.

Some examples:

- If φ is constant then X is a semimartingale.
- If Z Brownian motion, $\varphi = 1_{[0,1]}$ then $X_t = Z_t - Z_{(t-1)\vee 0}$ (is not a semimartingale).
- The Ornstein-Uhlenbeck type process; here $\varphi(t) = e^{-\beta t}$ (is a semimartingale).
- The Riemann-Liouville fractional integral; here $\varphi(t) = t^{\beta-1/2}$ (in most cases not a semimartingale).

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F. Knight

Let $X = (X_t)_{t \geq 0}$ be given by

$$X_t = \int_0^t \varphi(t-s)\,dZ_s, \qquad t \geq 0,$$

where Z is a Brownian motion.

Theorem (F. Knight (1992))

X is an $\mathcal{F}^Z$-semimartingale if and only if $\varphi \in W^{1,2}_{loc}(\mathbb{R}_+)$, i.e.

$$\varphi(t) = \alpha + \int_0^t h(s)\,ds, \qquad t \geq 0,$$

where $\alpha \in \mathbb{R}$ and $h \in L^2_{loc}(\lambda)$.

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Lévy driving moving averages

Let $X = (X_t)_{t\geq 0}$ be given by

$$X_t = \int_0^t \varphi(t-s)\, dZ_s, \qquad t \geq 0,$$

where $Z = (Z_t)_{t\geq 0}$ is a Lévy process with characteristic triplet (γ, σ^2, ν).

Theorem

X is an $\mathcal{F}^Z$-semimartingale if and only if

1. *Z is of bounded variation: φ is of bounded variation,*
2. *$\sigma^2 > 0$: $\varphi \in W^{1,2}_{loc}(\mathbb{R}_+)$,*
3. *Z is of unbounded variation and $\sigma^2 = 0$: φ is absolutely continuous on $\mathbb{R}_+$ with a density φ' satisfying*

$$\int_0^t \int_{-1}^1 \big(|x\varphi'(s)|^2 \wedge |x\varphi'(s)|\big)\nu(dx)\, ds < \infty, \qquad \textit{for all } t > 0.$$

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A corollary

Let X be given by

$$X_t = \int_0^t (t-s)^{\beta - \frac{1}{2}} \, dZ_s, \qquad \left(\beta \neq \frac{1}{2}\right).$$

Corollary (The Riemann-Liouville fractional integral)

- $\sigma^2 > 0$: *X is an $\mathcal{F}^Z$-semimartingale if and only if $\beta > 1$.*
- $\sigma^2 = 0$: *X is an $\mathcal{F}^Z$-semimartingale if and only if*
    - $\beta > 1$,
    - $\beta = 1$ *and* $\int_{-1}^{1} x^2 |\log(|x|)| \, \nu(dx) < \infty$,
    - $\frac{1}{2} < \beta < 1$ *and* $\int_{-1}^{1} |x|^{2/(3-2\beta)} \, \nu(dx) < \infty$.

Thus if Z is α-stable then X is an $\mathcal{F}^Z$-semimartingale if and only if

- $\beta > \frac{3}{2} - \frac{1}{\alpha}$ when $\alpha \in [1,2]$,
- $\beta > \frac{1}{2}$ when $\alpha \in (0,1)$.

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Gaussian chaos processes

- Let $(Z_t)_{t\geq 0}$ denote a Brownian motion.
- $(Y_t)_{t\geq 0}$ is said to be a Gaussian chaos process if there exists $n \geq 1$ such that $Y_t \in \bigoplus_{k=0}^{n} \mathcal{H}_k$ for all $t \geq 0$.
- Let us now consider X of the form

$$X_t = \int_0^t \varphi(t-s)\sigma_s \, dZ_s, \qquad t \geq 0,$$

where σ is an $\mathcal{F}^Z$-adapted Gaussian chaos process which is continuous in probability and $\varphi \in L^2_{loc}$ is a deterministic function.

Theorem

X is an $\mathcal{F}^Z$-semimartingale if and only if $\varphi \in W^{1,2}_{loc}(\mathbb{R}_+)$.

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Chaos semimartingales

- Let $\mathcal{S}^p$ denote the space of special semimartingales $X = (X_t)_{t\in[0,T]}$ with canonical decomposition $X = X_0 + M + A$ satisfying $[M]_T^{p/2}, \mathrm{V}_{[0,T]}(A) \in L^1$.
- To show the previous theorem we use and prove the following result:

Theorem

Let X denote an $\mathcal{F}$-semimartingale satisfying

$$E[X_t|\mathcal{F}_s] \in \oplus_{k=0}^n \mathcal{H}_k, \qquad \forall\, 0 \le s \le t \le T.$$

Then $X \in \mathcal{S}^p$ for all $p \ge 1$, and in particular a quasimartingale. Futhermore, if $X = X_0 + A + M$ denotes the $\mathcal{F}$-canonical decomposition of X then $A_t, M_t \in \oplus_{k=0}^n \mathcal{H}_k$ for all $t \in [0,T]$.

When X is a Gaussian processes this result was shown by Stricker (1983).

Key references

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