MATH 241- EXAM 2-Feb. 22, 2005

1. [4,4] The position vector of a moving particle is given by:

$$\mathbf{r}(t) = (2\cos t + 2t\sin t, 2\sin t - 2t\cos t, \frac{\sqrt{5}}{2}t^2), t \in \mathbb{R}.$$

(i) Find the speed v and the unit tangent vector $\mathbf{T}$ when $t = \pi/4$.

$$r'(t) = (2t\cos t, 2t\sin t, \sqrt{5}t) = \frac{\pi}{4}(\sqrt{2}, \sqrt{2}, \sqrt{5})$$

$$v = \frac{\pi}{4}(2+2+5)^{1/2} = \frac{3\pi}{4} \quad T = \frac{1}{3}(\sqrt{2}, \sqrt{2}, \sqrt{5})$$

(ii) The time derivative of the speed is constant: $v' = 3$. Compute the remaining terms in the decomposition of the acceleration vector at $t = \pi/4$:

$$\mathbf{r}''(\pi/4) = v'\mathbf{T} + v^2\kappa\mathbf{N}$$

(that is, find $\mathbf{r}''$, the curvature κ and the unit normal vector $\mathbf{N}$ when $t = \pi/4$). *Hint:* Note $v^2\kappa\mathbf{N} = \mathbf{r}'' - v'\mathbf{T}$ (where the right-hand side is easy to compute) and $\mathbf{N}$ has length one.

$$r''(t) = (2\cos t - 2t\sin t, 2\sin t + 2t\cos t, \sqrt{5}) = (\sqrt{2} - \sqrt{2}\frac{\pi}{4}, \sqrt{2} + \sqrt{2}\frac{\pi}{4}, \sqrt{5})$$

$$r'' - v'T = (-\sqrt{2}\frac{\pi}{4}, \sqrt{2}\frac{\pi}{4}, 0) \quad N = \frac{1}{\sqrt{2}}(-1, 1, 0)$$

$$v^2\kappa = \frac{\pi}{2}, \quad \kappa = \frac{\pi}{2}\frac{16}{9\pi^2} = \frac{8}{9\pi}$$

2. [2,4,2,4] Consider the function of three variables:

$$F(x, y, z) = xyz - \cos(x + y + z).$$

The equation $F(x, y, z) = 0$ implicitly defines a function $z = z(x, y)$, for (x, y, z) close to $(\pi/4, 0, \pi/4)$.

(i) Write down the *definition* of the partial derivative $\frac{\partial z}{\partial x}(\pi/4, 0)$ (using limits).

$$z_x(\pi/4, 0) = \lim_{h\to 0} \frac{z(\pi/4 + h, 0) - z(\pi/4, 0)}{h}$$

(ii) Find the **linear function** $L(x,y)$ that best approximates $z(x,y)$, for (x,y) close to $(\pi/4,0)$.

$$yz + xyz_x + \sin(x+y+z)[1+z_x] = 1 + z_x = 0, \quad z_x = -1$$

$$xz + xyz_y + \sin(x+y+z)[1+z_y] = \frac{\pi^2}{16} + 1 + z_y = 0, z_y = -(1+\frac{\pi^2}{16}).$$

(*The computation of these partial derivatives implicitly was HW problem 11.3(42).*)

$$L(x,y) = \frac{\pi}{4} - (x - \frac{\pi}{4}) - (1 + \frac{\pi^2}{16})y.$$

(iii) Find an equation for the **tangent plane** to the graph of $z(x,y)$ at the point $(\pi/4, 0, \pi/4)$. (Near this point, the graph coincides with the surface $\{F(x,y,z) = 0\}$).

$$z = \frac{\pi}{4} - (x - \frac{\pi}{4}) - (1 + \frac{\pi^2}{16})y$$

(iv) What statement can be made about the **remainder**:

$$R(\Delta x, \Delta y) = z(x,y) - L(x,y),$$

as $(\Delta x, \Delta y) \to 0$? (Here $\Delta x = x - \pi/4, \Delta y = y$).

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{R(\Delta x, \Delta y)}{||(\Delta x, \Delta y)||} = 0.$$

3. [4,4,4,4] The diagram shows level sets (contour lines) for the function $f(x,y) = (4x^2 + 2xy + y^2)e^{(-x^2-y^2)}$. Think of f as describing elevation on a landscape. (*Scale:* one unit in x, y or f =100m.) Eight level sets are plotted (at equal intervals of 20m). The outermost level set consists of one piece, the innermost of three, the others of two pieces.

(i) At the point A(2,0), what is the direction of **steepest ascent** on this landscape? (Directions are given by unit vectors.)

$$\nabla f = (8x+2y+(4x^2+2xy+y^2)(-2x), 2x+2y+(4x^2+2xy+y^2)(-2y))e^{-x^2-y^2}.$$

$$\nabla f(2,0) = (-48, 4)e^{-4}, \quad u = \frac{1}{\sqrt{145}}(-12, 1).$$

(ii) Without computation, would you expect the **length of the gradient** of f to be greater at the point (0.5, 0.3) or at (1, 1.3)? Why?

Expect greater length at (0.5, 0.3)- the level sets are more densely spaced there, indicating faster change in the value of f for the same (x,y) displacement (steeper slope).

(iii) Suppose a vehicle is moving on a road on this landscape with **speed** 10m/s (bearing roughly NW), and at a certain moment the road is at a *constant elevation* and goes through point B(1,1). What is the vehicle's **velocity vector** at this moment? (Include the units in your answer.)

$$\nabla f(1,1) = (-4,-10)e^{-2} = 2e^{-2}(-2,-5), \quad N = (-2,-5)/\sqrt{29}$$

This is the unit normal to the level set through (1,1). Therefore the unit tangent to the level set (pointing roughly NW) is $T = (-5,2)/\sqrt{29}$*, and multiplying by the speed we obtain the velocity vector:*

$$\mathbf{v} = \frac{10}{\sqrt{29}}(-5,2) \quad m/s.$$

(iv) Suppose the vehicle at that moment leaves the road and turns due **South** without changing its speed. Will it start climbing or descending, and how fast will its elevation start changing at that moment? (Include the units.)

Since the rate of change (R.O.C)in f *along a curve is the speed times the directional derivative:*

$$R.O.C. = v(0,-1)\cdot\nabla f(1,1) = 100e^{-2} \quad m/s \approx 13.5 m/s.$$

(Positive, hence climbing.)