Final

This is a take-home exam: You cannot talk to *anyone* (except me) about *anything* on this exam and you can only look at *our* book (Velleman), our class notes [on Sage Math Cloud] and our solutions. *No other reference is allowed*, including the Internet. Failing to follow these instructions will result in a zero for the exam. Moreover, I will report the incident to the university and do all in my power to get the maximal penalty for the infraction.

As you will have a lot of time, I expect the solutions to be well written and clearly explained.

If you need *clarifications on any statement*, please use Piazza (non-private). *Questions related to content/math of the exam or that should not be seen by all should be submitted as **private messages** in Piazza!* Please use your best judgment on what is appropriate to ask in the forum.

Due date: Your solutions must be uploaded on Blackboard by Wednesday 07/06 by 11:59pm. Please send as a PDF and make sure your scanned/typed exam is clear and legible.

1) Suppose that R is a partial order on A, $B_1 \subseteq A$, $B_2 \subseteq A$ and

$$\forall x \in B_1[\exists y \in B_2(xRy)] \qquad \text{and} \qquad \forall x \in B_2[\exists y \in B_1(xRy)].$$

(a) [10 points] Prove that if $x \in A$ is an upper bound of B_1, then x is also an upper bound of B_2.

(b) [10 points] Prove that if B_1 and B_2 are disjoint, then B_1 has no maximal element.

2) [10 points] Let $\mathcal{F}$ and $\mathcal{G}$ be partitions of A and let

$$\mathcal{H} = \{Z \in \mathscr{P}(A) \mid Z \neq \varnothing \text{ and } \exists X \in \mathcal{F}[\exists Y \in \mathcal{G}(Z = X \cap Y)]\}.$$

Prove that $\mathcal{H}$ is also a partition of A.

3) [10 points] Let A be a non-empty set and $f : A \to A$. Prove that if f is either a partial order or an equivalence relation, then f is the identity function i_A.

4) [10 points] Let $f : A \to B$ be an *invertible* function [i.e., $f^{-1} : B \to A$] and R be an equivalence relation on B. Prove that $S = f^{-1} \circ R \circ f$ is an equivalence relation on A. [**Hint:** You can use, without proof, the following: $(a, a') \in S$ iff there are $b, b' \in B$ such that $(a, b) \in f$, $(b, b') \in R$ and $(b', a') \in f^{-1}$.]

5) [10 points] Prove that for all integers $n \geq 1$ we have

$$\sum_{i=1}^{n} (2i + 1)3^i = n3^{n+1}.$$

6) [10 points] Prove that for all $n \geq 0$ we have

$$\frac{2}{n!} \leq 3^{2-n}.$$

[**Hint:** If you are having trouble, try to prove it for $n \geq 2$ instead and then see if you can finish it from there.]

7) [10 points] Consider the sequence $a_0, a_1, a_2, \ldots$ given by the recursive formula:

$$a_0 = 1$$
$$a_1 = 1$$
$$a_n = a_{n-1} + 2a_{n-2}, \text{ for } n \geq 2.$$

Prove that for all $n \in \mathbb{N}$, we have that $a_n = (2^{n+1} + (-1)^n)/3$.