1) Compute the following limits. If they do not exist or are infinite, check if the side limits exist.

(a) [5 points] $\lim_{x \to 1} \frac{x^3 - x}{x^2 - x + 1}$

Solution. The function is continuous at $x = 1$, so:

$$\lim_{x \to 1} \frac{x^3 - x}{x^2 - x + 1} = \frac{1^3 - 1}{1^2 - 1 + 1} = \frac{0}{1} = 0.$$

□

(b) [5 points] $\lim_{x \to -\infty} \frac{x^3 - 1}{x^2 - 100x + 1000}$

Solution.

$$\begin{aligned} \lim_{x \to -\infty} \frac{x^3 - 1}{x^2 - 100x + 1000} &= \lim_{x \to -\infty} \frac{x^3(1 - 1/x^3)}{x^2(1 - 100/x + 1000/x^2)} \\ &= \lim_{x \to -\infty} x \cdot \frac{1 - 1/x^3}{1 - 100/x + 1000} \\ &= -\infty \end{aligned}$$

□

(c) [10 points] $\lim_{x \to 1} \frac{x^2 - 2x + 1}{|x-1|}$ [**Hint:** You might need side limits.]

Solution.

$$\begin{aligned} \lim_{x \to 1^+} \frac{x^2 - 2x + 1}{|x-1|} &= \lim_{x \to 1^+} \frac{(x-1)^2}{x-1} \\ &= \lim_{x \to 1^+} x - 1 = 0. \end{aligned}$$

$$\begin{aligned} \lim_{x \to 1^-} \frac{x^2 - 2x + 1}{|x-1|} &= \lim_{x \to 1^-} \frac{(x-1)^2}{-(x-1)} \\ &= \lim_{x \to 1^-} -(x-1) = 0. \end{aligned}$$

Since the side limits are equal, we have:

$$\lim_{x \to 1} \frac{x^2 - 2x + 1}{|x-1|} = 0.$$

□

(d) [10 points] $\lim_{x \to 1} \frac{1}{x-1} + \frac{1}{(x-1)(x-2)}$

Solution.

$$\begin{aligned} \frac{1}{x-1} + \frac{1}{(x-1)(x-2)} &= \frac{(x-2)+1}{(x-1)(x-2)} \\ &= \frac{x-1}{(x-1)(x-2)} \\ &= \frac{1}{x-2}. \end{aligned}$$

So,

$$\begin{aligned} \lim_{x \to 1} \frac{1}{x-1} + \frac{1}{x^2 - 3x + 2} &= \lim_{x \to 1} \frac{1}{x-2} \\ &= -1. \end{aligned}$$

□

2) [15 points] Compute the following derivative (using the formulas, no need for limits).

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x \cdot \sin(x) - \mathrm{e}^x}{x^2 + \sqrt{x}}\right)$$

No need to simplify! Note that you might get less partial credit if you skip steps and get the wrong answer. [No penalty if the answer is correct.]

Solution.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x \cdot \sin(x) - \mathrm{e}^x}{x^2 + \sqrt{x}}\right) \\
&\quad= \frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(x \cdot \sin(x) - \mathrm{e}^x\right)\left(x^2 + \sqrt{x}\right) - \left(x \cdot \sin(x) - \mathrm{e}^x\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 - \sqrt{x}\right)}{(x^2 + \sqrt{x})^2} \\
&\quad= \frac{\left(\frac{\mathrm{d}}{\mathrm{d}x}(x) \cdot \sin(x) + x\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) - \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x)\right)\left(x^2 + \sqrt{x}\right) - (x \cdot \sin(x) - \mathrm{e}^x)(2x - \frac{1}{2}x^{-1/2})}{(x^2 + \sqrt{x})^2} \\
&\quad= \frac{\left(\sin(x) + x\cos(x) - \mathrm{e}^x\right)\left(x^2 + \sqrt{x}\right) - (x \cdot \sin(x) - \mathrm{e}^x)(2x - \frac{1}{2}x^{-1/2})}{(x^2 + \sqrt{x})^2}
\end{aligned}$$

□

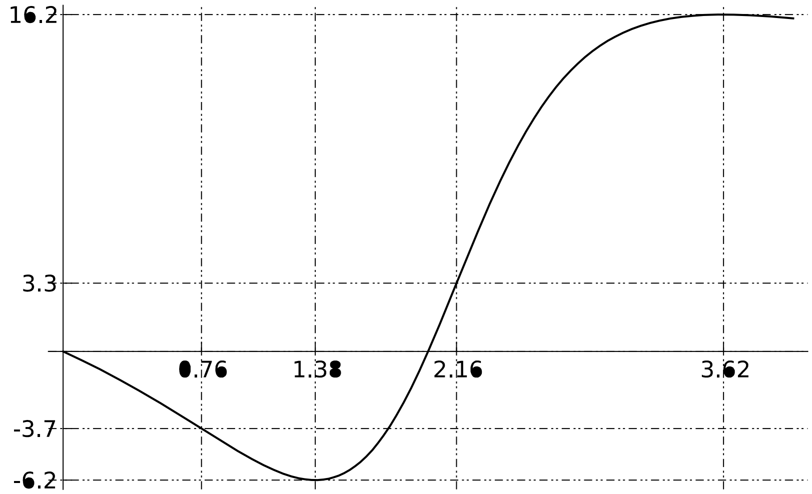
3) [15 points] Compute the derivative of $f(x) = \sqrt{2x+1}$. **You cannot use any formulas we haven't seen in class! You must use limits!**

Solution.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \cdot \frac{\sqrt{2(x+h)+1} + \sqrt{2x+1}}{\sqrt{2(x+h)+1} + \sqrt{2x+1}} \\
&= \lim_{h\to 0} \frac{(2(x+h)+1) - (2x+1)}{h \cdot (\sqrt{2(x+h)+1} + \sqrt{2x+1})} \\
&= \lim_{h\to 0} \frac{2h}{h \cdot (\sqrt{2(x+h)+1} + \sqrt{2x+1})} \\
&= \lim_{h\to 0} \frac{2}{\sqrt{2(x+h)+1} + \sqrt{2x+1}} \\
&= \frac{2}{2\sqrt{2x+1}} \\
&= \frac{1}{\sqrt{2x+1}}.
\end{aligned}$$

□

4) [20 points] The graph of the position s of particle moving along a straight line with respect to time t for $t \in [0, 4]$ is given below.



Determine [for $t \in [0, 4]$]:

(a) Fill the table below with "+" if the corresponding function (velocity and acceleration of the particle) is positive in the respective interval and with "−" if the function is negative.

Solution. Since the velocity is the derivative, the velocity is negative when the graph is decreasing and positive when it is increasing.

Since the acceleration is the second derivative, the acceleration is negative when the graph is concave down and positive when it is concave up.

So:

	$(0, 0.76)$	$(0.76, 1.38)$	$(1.38, 2.16)$	$(2.16, 3.62)$	$(3.62, 4)$
velocity	$-$	$-$	$+$	$+$	$-$
acceleration	$-$	$+$	$+$	$-$	$-$

□

(b) When was the velocity maximal and when was it minimal?

Solution. The velocity was negative at $t = 0$ and decreased until $t = 0.76$. After that it increases only until $t = 2.16$, when starts to decrease and keeps decreasing until the end. But at $t = 4$ the slope is larger than at $t = 0.76$. So, it is minimal at $t = 0.76$.

The velocity increases from $t = 0.76$ until $t = 2.16$. Since the slope at $t = 2.16$ is larger than the slope at $t = 0$, the velocity is maximal at $t = 2.16$.

□

5) [20 points] Sketch the graph of a function $f(x)$ which satisfies all of the following conditions:

- $f(0) = f'(-2) = f'(1) = f'(9) = 0$;
- $\lim_{x\to\infty} = 0$, $\lim_{x\to 6} = -\infty$;
- $f'(x) < 0$ on the intervals $(-\infty, -2)$, $(1, 6)$, and $(9, \infty)$;
- $f'(x) > 0$ on the intervals $(-2, 1)$ and $(6, 9)$;
- $f''(x) > 0$ on the intervals $(-\infty, 0)$ and $(12, \infty)$;
- $f''(x) < 0$ on the intervals $(0, 6)$ and $(6, 12)$.

Solution. This is Problem 43 from the review from Chapter 2, on pg. 178. See the solution on pg. A93 from the text.

$\square$