# INTERNAL DIRECT PRODUCT

MATH 457

Here is the definition of internal direct product from the text:

**Definition 1.** Let $H_i \lhd G$ for $i \in \{1, \ldots, n\}$. [Note that we *require* that $H_i$ is normal!] Then $G$ is the *internal direct product of the $H_i$'s* if for any $g \in G$, $\exists! h_i \in H_i$ such that $g = h_1 \cdot h_2 \cdots h_n$.

Here is the properties I gave to decide if a group is isomorphic to the (external) direct product of a finite number of its subgroups:

**Definition 2.** Let $H_i \leq G$ for $i \in \{1, \ldots, n\}$. Then the sets $H_i$ *satisfy the* IDP *properties* if:

(1) $H_i \lhd G$, for all $i$;
(2) $G = H_1 \cdots H_n \stackrel{\text{def}}{=} \{h_1 \cdots h_n \ : \ h_i \in H_i\}$;
(3) if $\hat{H}_i \stackrel{\text{def}}{=} H_1 \cdots H_{i-1} \cdot H_{i+1} \cdots H_n$, then $H_i \cap \hat{H}_i = \{1\}$. [Note that if $n = 2$, then $\hat{H}_1 = H_2$ and $\hat{H}_2 = H_1$.]

We will prove that the definitions are equivalent, i.e., $G$ is the internal direct product of the $H_i$'s if and only if the $H_i$'s satisfy the IDP properties. [This is Theorem 5 below.]

We need the following lemma.

**Lemma 3.** *If $H_i \leq G$ for $i \in \{1, \ldots, n\}$ satisfy* IDP *properties, then $h_i h_j = h_j h_i$ for all $h_i \in H_i$ and $h_j \in H_j$ with $i \neq j$.*

*Proof.* Since $h_i^{-1} \in H_i \lhd G$, we have that $h_j h_i^{-1} h_j^{-1} \in H_i$. So, $h_i(h_j h_i^{-1} h_j^{-1}) \in H_i$.

Similarly, since $h_j \in H_j \lhd G$, we have that $h_i h_j h_i^{-1} \in H_j$. So, $(h_i h_j h_i^{-1}) h_j^{-1} \in H_j$.

Thus, we have that $h_i h_j h_i^{-1} h_j^{-1} \in H_i \cap H_j$. But since $i \neq j$, we have that $H_j \subseteq \hat{H}_i$, and so $H_i \cap H_j \subseteq H_i \cap \hat{H}_i$. Moreover, by property (3), we have that $H_i \cap \hat{H}_i = \{1\}$. Hence, $h_i h_j h_i^{-1} h_j^{-1} \in H_i \cap H_j \subseteq H_i \cap \hat{H}_i = \{1\}$, which implies that $h_i h_j h_i^{-1} h_j^{-1} = 1$, i.e., $h_i h_j = h_j h_i$. ∎

We then have:

**Theorem 4.** *Let* $H_1, \dots, H_n \leq G$. *Then,* $\phi : H_1 \times \cdots \times H_n \to G$ *defined by* $\phi(h_1, \dots, h_n) = h_1 \cdots h_n$ *is an isomorphism if and only if the* $H_i$*'s satisfy the IDP properties.*

*Proof.* [$\Rightarrow$:] Assume that $\phi$ [as in the statement] is an isomorphism. Let $\tilde{G} \stackrel{\text{def}}{=} H_1 \times \cdots \times H_n$ and $\tilde{H}_i \stackrel{\text{def}}{=} \{1\} \times \cdots \{1\} \times H_i \times \{1\} \times \cdots \times \{1\} \leq \tilde{G}$ [with $H_i$ in the $i$-th coordinate]. Then, clearly $\phi(\tilde{H}_i) = H_i$. Since $\tilde{H}_i \lhd \tilde{G}$ [easy exercise!], we have that $H_i \lhd G$, as $\phi$ is an isomorphism [by assumption]. [This was a problem in the exam.] Thus, IDP property (1) is proved.

Since $\phi$ is an isomorphism [and hence onto] and $\phi(\tilde{G}) = H_1 \cdots H_n$ [by definition of $\phi$ and the product of groups], we have that $G = H_1 \cdots H_n$, proving property (2).

Now, let $\hat{\tilde{H}}_i \stackrel{\text{def}}{=} H_1 \times \cdots \times H_{i-1} \times \{1\} \times H_{i+1} \times \cdots \times H_n$. Then, clearly $\phi(\hat{\tilde{H}}_i) = \hat{H}_i$ [with $\hat{H}_i$ as in Definition 2] and $\tilde{H}_i \cap \hat{\tilde{H}}_i = \{(1, \dots, 1)\}$. Thus,

$$\begin{aligned} \{1\} &= \phi(\{(1, \dots, 1)\}) \\ &= \phi(\tilde{H}_i \cap \hat{\tilde{H}}_i) && \text{[as noted above]} \\ &= \phi(\tilde{H}_i) \cap \phi(\hat{\tilde{H}}_i) && \text{[as } \phi \text{ is a \textbf{bijection} -- this is a Math 300 exercise]} \\ &= H_i \cap \hat{H}_i && \text{[as noted above]} \end{aligned}$$

Hence, property (3) is also satisfied.

[$\Leftarrow$:] Assume now that the $H_i$'s satisfy the IDP property. Then, $\phi$ is a homomorphism by Lemma 3. It is onto by property (2) [as $\phi(H_1 \times \cdots \times H_n) = H_1 \cdots H_n$ by definition of $\phi$].

Now we show that $\phi$ is injective. Suppose that $\phi(h_1, \dots, h_n) = 1$. This means that $h_1 \cdots h_n = 1$, or $h_1^{-1} = h_2 \cdots h_n$. Since the left hand side is in $H_1$ and the right hand side is in $\hat{H}_1$, property (3) tells us that $h_1 = 1$ and $h_2 \cdots h_n = 1$. Then, $h_2^{-1} = h_3 \cdots h_n$ and now the left hand side is in $H_2$ and the right hand side is in $\hat{H}_2$. As before, we obtain $h_2 = 1$ and $h_3 \cdots h_n = 1$. Inductively, we obtain that $h_i = 1$ for all $i$. Hence, $\ker \phi = \{(1, \dots, 1)\}$ and $\phi$ is injective. $\square$

Now, we can prove that equivalency of the Definitions 1 and 2:

**Theorem 5.** *Let $H_i \lhd G$ for $i \in \{1, \ldots, n\}$. [Note that we are already assuming that the $H_i$'s are normal, since it is in the conditions of* both *definitions!] We have that $G$ is the internal direct product of the $H_i$ if and only if the $H_i$'s satisfy the IDP properties.*

*Proof.* [⇒:] Assume that $G$ is the internal direct product of the $H_i$'s. Clearly properties (1) and (2) of IDP are satisfied.

Now, let $h_i \in H_i \cap \hat{H}_i$. Then, since $h_i \in \hat{H}_i$, we have, by definition, that

$$1 \cdots 1 \cdot h_i \cdot 1 \cdots 1 = h_i = x_1 \cdots x_{i-1} \cdot 1 \cdot x_{i+1} \cdots x_n,$$

where $x_j \in H_j$. By the unique representation hypothesis, we have that $h_i = 1$. Thus $H_i \cap \hat{H}_i = \{1\}$, i.e., property (3) is also satisfied.

[⇐:] Assume now that the $H_i$'s satisfy the IDP properties. [By (1), we would then get that the $H_i$'s are normal, but we are already assuming it here.] Then, by (2), every element $g \in G$ can be written as $g = h_1 \cdots h_n$ with $h_i \in H_i$. [We need to show uniqueness.]

Now assume that

$$h_1 \cdots h_n = x_1 \cdots x_n, \qquad \text{with } h_i, x_i \in H_i.$$

Thus, with $\phi$ as in the statement of Theorem 4 [which we can use since are assuming IDP properties], we have that

$$\phi(h_1, \ldots, h_n) = \phi(x_1, \ldots, x_n).$$

Since $\phi$ is an isomorphism [and hence one-to-one], we have that $h_i = x_i$ for all $i$, and hence the representation is unique. □

This gives us:

**Corollary 6.** *$G$ is the internal direct product of the subgroups $H_i$'s for $i \in \{1, \ldots, n\}$ [and hence $H_i \lhd G$* by assumption!*] if and only if $\phi : H_1 \times \cdots \times H_n \to G$ defined by $\phi(h_1, \ldots, h_n) = h_1 \cdots h_n$ is an isomorphism.*

*Proof.* By Theorem 4, we know that that $H_i$'s satisfying IDP is equivalent to $\phi$ [as in the statement] being an isomorphism. Since the former is equivalent to $G$ being the internal direct product of the subgroups $H_i$'s [by Theorem 5], the result follows. □