1) [15 points] Assuming that the system $A\mathbf{x} = \mathbf{b}$ has the reduced row echelon form of its augmented matrix as below, find the solution(s) of the system (or stated that there is no solution if that is the case). No need to show work.

(a) $[A|\mathbf{b}] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 3 & 0 & 5 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Solution. No solution. [Fourth row gives $0 = 1$.]

□

(b) $[A|\mathbf{b}] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$

Solution. $(x_1, x_2, x_3, x_4) = (2, 5, 0, 3)$

□

(c) $[A|\mathbf{b}] \sim \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 2 & 0 & 1 & 5 \\ 0 & 1 & 0 & -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Solution. $(x_1, x_2, x_3, x_4, x_5, x_6) = (5 - t - 2s, 1 - 2t + s, r, s, 3 - t, t)$

□

2) [40 points] Quickies! (You should be able to answer these quickly and with no, or very little, calculations!) No need to show work!

(a) $$\begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 \\ 5 & 2 & -3 & 3 & 0 \\ 5 & 0 & 0 & -2 & 5 \end{vmatrix} =$$

Solution. $= 2 \cdot 1 \cdot (-1) \cdot 3 \cdot 5 = -30$. [Determinant of lower triangular matrix.] $\square$

(b) $$\begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & -3 & 1 & 4 \\ 2 & 4 & 2 & 2 \\ -1 & 2 & 4 & 5 \end{vmatrix} =$$

Solution. $= 0$. [Third row is a multiple of the first.] $\square$

(c) $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} =$$

Solution.

$$\frac{1}{1 \cdot 4 - 2 \cdot 3} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

$\square$

(d) If A is an $n \times n$ matrix with $\det(A) \neq 0$, then how many solutions can the system $A\mathbf{x} = \mathbf{b}$ possibly have?

Solution. The only possibility is exactly one solution. $\square$

(e) If A is an $n \times n$ matrix which is not invertible, then how many solutions can the *homogeneous* system $A\mathbf{x} = \mathbf{0}$ possibly have?

Solution. The only possibility is infinitely many solutions. $\square$

(f) $\left(\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} - 3 \cdot \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\right)^{\mathrm{T}} =$

Solution.

$$= \left(\begin{bmatrix} 1-3\cdot 1 & -3-3\cdot(-1) \\ 2-3\cdot 0 & 1-3\cdot 2 \end{bmatrix}\right)^{\mathrm{T}} = \left(\begin{bmatrix} -2 & 0 \\ 2 & -5 \end{bmatrix}\right)^{\mathrm{T}} = \begin{bmatrix} -2 & 2 \\ 0 & -5 \end{bmatrix}$$

□

(g) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} =$

Solution.

$$= \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{bmatrix}.$$

□

(h) Find the $(2,1)$-entry of the *inverse* of the matrix A below, knowing that $\det(A) = -14$. [**Hint:** You do not need to compute the whole inverse! This is quick if you use cofactors.]

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & -1 \\ 4 & 0 & 2 \end{bmatrix}$$

Solution. Using cofactors [remember to use the $(1,2)$-cofactor instead of $(2,1)$ due to the transpose]:

$$\frac{(-1)^{1+2}}{-14} \begin{vmatrix} 0 & -1 \\ 4 & 2 \end{vmatrix} = \frac{2}{7}.$$

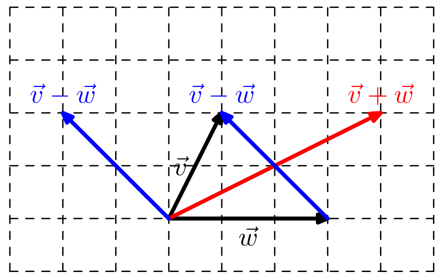
□

(i) If A and B are given below, what is the $(3,1)$ entry of $A \cdot B$?

$$A = \begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix}$$

Solution. $1 \cdot 1 + 1 \cdot 2 = 3.$ □

(j) If $\vec{v}$ and $\vec{w}$ are the vectors given below, draw the vectors $\vec{v}+\vec{w}$ and $\vec{v}-\vec{w}$ on the grid. [Label each one appropriately!]



Solution. Solutions on the figure. [See two possibilities for the difference.] ☐

3) [20 points] Let $\mathbf{v} = (1, -1, 3)$ and $\mathbf{w} = (2, 2, 1)$. Compute:

(a) $\|\mathbf{v}\|$ [length of $\mathbf{v}$]

Solution. $\|\mathbf{v}\| = \sqrt{1^2 + (-1)^2 + 3^2} = \sqrt{11}$. ☐

(b) $\mathbf{v} \cdot \mathbf{w}$ [dot product]

Solution. $\mathbf{v} \cdot \mathbf{w} = 1 \cdot 2 + (-1) \cdot 2 + 3 \cdot 1 = 3$. ☐

(c) the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$

Solution.

$$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \, \|\mathbf{w}\|} = \frac{3}{\sqrt{11} \cdot \sqrt{2^2 + 2^2 + 1^2}} = \frac{1}{\sqrt{11}} = \frac{\sqrt{11}}{11}.$$

☐

(d) the orthogonal projection of $\mathbf{v}$ on $\mathbf{w}$

Solution.

$$\mathrm{proj}_{\mathbf{w}}(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|}\mathbf{w} = \frac{3}{9}(2, 2, 1) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right).$$

☐

(e) the component of $\mathbf{v}$ orthogonal to $\mathbf{w}$

Solution.

$$\mathbf{v} - \mathrm{proj}_{\mathbf{w}}(\mathbf{v}) = (1, -1, 3) - \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) = \left(\frac{1}{3}, -\frac{5}{3}, \frac{8}{3}\right).$$

☐

4) [25 points] Compute the reduced row echelon form, inverse (if it exits – if not say so and explain why) and determinant of the matrix A below. [**Hint:** You can compute all of them *together*, while finding the reduced row echelon form of A.]

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

Solution.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & -1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & 1 & 2 & 0 \\ 0 & 3 & -1 & 0 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & -1/3 & 0 \\ 0 & 1 & 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right]$$

So, the reduced echelon form of the matrix is the [3 by 3] identity matrix, the inverse is

$$\begin{bmatrix} 1/3 & -1/3 & 0 \\ 1/3 & 2/3 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

and the determinant is $3 \cdot (-1)^3 = -3$. [The determinant can be found by checking we need to multiply 1 by (-1) on steps 1, 2 and 5 and multiply by 3 on step 6.] $\square$