Conductors and minimal discriminants of hyperelliptic curves

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What are conductors and minimal discriminants?

Degenerating family of hyperelliptic curves

Measures of degeneracy

Artin conductor

Minimal discriminant

How are these related?

\[ y^2 = (x - 1)(x^2 - t) \]

\[ y^2 = x^3 - t \]
How are conductors and minimal discriminants related?

Earlier results: (small genus, all residue characteristics)

- If $g = 1$, then $\text{Art}^+(X) = \Delta_X$. [Ogg-Saito formula]
- If $g = 2$, then Liu showed that $\text{Art}^+(X) \leq \Delta_X$. He showed that equality does not always hold.
How are conductors and minimal discriminants related?

Earlier results: (small genus, all residue characteristics)

- If \( g = 1 \), then \( \text{Art}^+ (X) = \Delta_X \). [Ogg-Saito formula]
- If \( g = 2 \), then Liu showed that \( \text{Art}^+ (X) \leq \Delta_X \). He showed that equality does not always hold.

Question: Does \( \text{Art}^+ (X) \leq \Delta_X \) hold for hyperelliptic curves of arbitrary genus \( g \)?

Today:

- Yes, if the residue characteristic is \( > 2g + 1 \). [S.]
- Combinatorial restrictions for equality when \( g \geq 2 \).
- Yes, if the residue characteristic is \( \neq 2 \). [Joint work with Obus]
1 Introduction

2 Definitions

3 Computational tools

4 Proof strategies in examples
\( R \): complete discrete valuation ring
\( K \): fraction field of \( R \)
\( k \): residue field of \( R \), algebraically closed, char \( \neq 2 \)
\( \overline{K} \): a fixed separable closure of \( K \)
\( G_K \): Galois group of \( \overline{K}/K \)
\( \nu \): valuation \( \overline{K} \rightarrow \mathbb{Q} \cup \{ \infty \} \)
\( t \): a uniformizer of \( R \), i.e., \( \nu(t) = 1 \).

**Examples:** \( \mathbb{C}[[t]] \), \( \mathbb{Z}^{\text{unr}} \)
\( X \): smooth hyperelliptic \( K \)-curve
\( g \): genus of \( X \)
Definition: The minimal discriminant $\Delta_X$ of $X/K$ is the nonnegative integer

$$
\Delta_X := \min_{\substack{f(x) \in R[x] \\ y^2 = f(x), \text{ eqn. for } X}} \nu(\text{disc}(f)).
$$

An example: $K = \mathbb{C}((t))$

$C_1: y^2 = x(x - t)(x - 2t)(x - 3t) \leadsto \nu(\text{disc}(f)) = 2\binom{4}{2}$.

$C_2: y'^2 = x'(x' - 1)(x' - 2)(x' - 3) \leadsto \nu(\text{disc}(f)) = 0$.

Here $C_1 \cong_K C_2$ via $x' = \frac{x}{t}, y' = \frac{y}{t^2} \leadsto \Delta_X = 0$. 
Fix a prime $\ell \neq \text{char } k$. For any curve $C$ over an algebraically closed field of char $\neq \ell$, let

$$\chi(C) := \sum_{i=0}^{2} (-1)^i \dim H^i_{\text{ét}}(C, \mathbb{Q}_\ell).$$

$\delta$: Swan conductor for the $G_K$ representation $H^1(X_K, \mathbb{Q}_\ell)$ (integer, $\geq 0$, measure of wild ramification).

$\mathcal{X}^{\text{min}}$: minimal proper regular $R$-model of $X$.

**Definition:** The Artin conductor $\text{Art}^+(X)$ of $X/K$ is

$$\text{Art}^+(X) := \chi(\mathcal{X}^{\text{min}}_K) - \chi(\mathcal{X}^{\text{min}}_k) + \delta.$$
Properties of the Artin Conductor

- $\text{Art}^+(X)$ is independent of $\ell$.
- $\text{Art}^+(X) \geq 0$.
  \[
  \text{Art}^+(X) = 0 \iff X_{\min} \to \text{Spec } R \text{ is smooth or } g = 1 \text{ and } (X_k)_{\text{red}} \text{ is smooth}.
  \]
- Let $n$ be the number of components of $X_{\min}^k$ and let $\epsilon$ be the tame conductor for the $G_K$ representation $H^1(X_K, \mathbb{Q}_\ell)$. Then,
  \[
  \text{Art}^+(X) = (n - 1) + \epsilon + \delta.
  \]
- When $X_{\min}$ is regular and semi-stable,
  \[
  \text{Art}^+(X) = \# \text{ singular points of } X_{\min}^k.
  \]
Theorem (S.)

Let $K$ be the fraction field of a Henselian discrete valuation ring. Let $X$ be a smooth hyperelliptic curve over $K$ of genus $g \geq 1$. Assume that the residue characteristic is $> 2g + 1$.

Then,

$$\text{Art}^+(X) \leq \Delta_X.$$
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Remark: Suffices to find ONE proper regular model $\mathcal{X}$ such that

$$\text{Art}^+(\mathcal{X}) \leq \Delta_\mathcal{X}.$$
Explicit regular models when char $k \neq 2$

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Two reasons for non regular Weierstrass models:

- Components of div $f \subset \mathbb{P}^1_R$ intersect.
  (Example: $K = \mathbb{C}((t))$, $y^2 = x(x - t)(x - 1)$.)

- Components of div $f \subset \mathbb{P}^1_R$ are not regular curves.
  (Example: $K = \mathbb{C}((t))$, $y^2 = x^3 - t^2$.)
Explicit regular models when char \( k \neq 2 \)

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\[
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\]

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- Components of \( \text{div} \, f \subset \mathbb{P}^1_R \) are not regular curves.
  
  (Example: \( K = \mathbb{C}((t)), \ y^2 = x^3 - t^2 \).)

Solution: Blow-up \( \mathbb{P}^1_R \) first \textit{before} taking a double cover.
Lemma

Let $\text{Bl}\ \mathbb{P}^1_R$ be an arithmetic surface birational to $\mathbb{P}^1_R$. Let $f$ be an element of the function field of $\mathbb{P}^1_R$. Assume that the odd multiplicity components of the divisor of $f$ on $\text{Bl}\ \mathbb{P}^1_R$ are disjoint and regular. Then, the normalization of $\text{Bl}\ \mathbb{P}^1_R$ in $K(x, \sqrt{f(x)})$ is a proper regular model for the hyperelliptic curve given by $y^2 = f(x)$. 

Explicit regular model:

Let $y^2 = f(x)$ be an equation for $X$ with $f(x) \in R[x]$ and $\Delta_X = \Delta_f$. Let $\text{Bl}\ \mathbb{P}^1_R$ be the (minimal) blowup of $\mathbb{P}^1_R$ satisfying the conditions above and $X_f$ the associated proper regular model of $X$. 

Explicit regular models when char $k \neq 2$
Lemma

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Then, the normalization of $\text{Bl } \mathbb{P}^1_R$ in $K(x, \sqrt{f(x)})$ is a proper regular model for the hyperelliptic curve given by $y^2 = f(x)$.

Explicit regular model:

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Explicit regular models when char $k \neq 2$
Riemann-Hurwitz formula: If $\mathcal{X} \to \mathcal{Y}$ is a double cover of arithmetic surfaces, branched over the divisor $B$, then,

$$\text{Art}^+(\mathcal{X}) = [2\chi(\mathcal{Y}_k) - \chi(B_k)] - [2\chi(\mathcal{Y}_\overline{K}) - \chi(B_{\overline{K}})] + \delta.$$ 

Inclusion-exclusion/additivity for $\chi$ (good for induction!).
Computational tools

- Riemann-Hurwitz formula: If $\mathcal{X} \to \mathcal{Y}$ is a double cover of arithmetic surfaces, branched over the divisor $B$, then,

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Additional tools:

<table>
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<th>char $k &gt; 2g + 1$</th>
<th>char $k \neq 2$</th>
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<td>$\to \delta$</td>
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<td>Roots of $f$</td>
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<td>Metric tree of $f$</td>
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<td>Abhyankar’s Inversion formula</td>
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Key inductive inequality:

$$\Delta_f - \Delta_{f_{\text{new}}} = n(n - 1) \geq 2 = \text{Art}^+(\mathcal{X}_f) - \text{Art}^+(\mathcal{X}_{f_{\text{new}}}) \quad (\because n \geq 2).$$
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Roots of $f \sim \text{Metric tree of } f$

$t^{2/3} + t^{5/6}, t^{2/3} - t^{5/6},$
$
\omega t^{2/3} - \omega^2 t^{5/6}, \omega t^{2/3} + \omega^2 t^{5/6},$
$
\omega^2 t^{2/3} + \omega t^{5/6}, \omega^2 t^{2/3} - \omega t^{5/6}$
Inductive process on metric trees using Abhyankar’s inversion formula

(In the example below, \( a = 2, b = 3 \).)

\( b \) identical subtrees \( \rightsquigarrow \) \( a \) identical subtrees.

distance \( a/b \) from \( \eta \) \( \rightsquigarrow \) distance \( (b/a) - 1 \) from \( \eta \).

New subtree metric \( = (\text{Old subtree metric}) \cdot b/a \).
Proof in an easy example, \( K = \mathbb{C}((t)) \)

\[
f(x) = x(x - 1 - t)(x - 1 - 2t)(x - 1 - 3t)(x - 1 - 4t)
\]

\[
f^{\text{new}}(x) = (x - 1)(x - 2)(x - 3)(x - 4)
\]

\[
\text{Art}^+(X_f) - \text{Art}^+(X_{f^{\text{new}}}) = 2.
\]

\[
\Delta_f - \Delta_{f^{\text{new}}} = 2\binom{4}{2} = 12.
\]

\[
\text{Art}^+(X_{f^{\text{new}}}) = \Delta_{f^{\text{new}}} = 0.
\]
Examples where $\delta \neq 0$

Let $K = \hat{\mathbb{Q}}_{p}^{\text{unr}}$, $p$ odd.

\[
y^2 = x^p - p
\]

Weierstrass model is regular!

\[
\text{Art} + (X) = 2 \chi(Y) - 2 \chi(Y_K) - \delta = p - 1 + \delta = \Delta_K(p_{1/p})/K + 1 = \Delta_f - 2 \nu p_{2/p}.
\]
Examples where $\delta \neq 0$

Let $K = \widehat{\mathbb{Q}}_p^\text{unr}$, $p$ odd.

\[ y^2 = x^p - p \]

- Weierstrass model is regular!
- $\text{Art}^+(X) = [2\chi(\mathcal{Y}_k) - 2\chi(\mathcal{Y}_K)] - [\chi(B_k) - \chi(B_K)] + \delta = p - 1 + \delta$
- $\delta = \Delta_{K(p^{1/p})/K} - [K(p^{1/p}) : K] + 1 = \Delta_f - p + 1$.

\[ y^2 = x^p - p^2 \]
Examples where $\delta \neq 0$

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$y^2 = x^p - p^2$

- $\delta = \Delta_{K(p^{2/p}/K) - [K(p^{2/p}/K) : K]} + 1$
  - $= \Delta_{K(p^{1/p}/K) - [K(p^{1/p}/K) : K]} + 1$
  - $= \Delta_f - 2(\nu_p(p^{2/p}) - \nu_p(p^{1/p}))(p^2) - p + 1$
  - $= \Delta_f - 2(p - 1)$. 
Examples where $\delta \neq 0$

Let $K = \overline{\mathbb{Q}_p}^{unr}$, $p$ odd.

\[ y^2 = x^p - p \]

- Weierstrass model is regular!
- $\text{Art}^+(X) = [2\chi(\mathcal{Y}_k) - 2\chi(\mathcal{Y}_{K^1_k})] - [\chi(B_k) - \chi(B_{K^1})] + \delta = p - 1 + \delta$
- $\delta = \Delta_{K(p^1/p)/K} - [K(p^1/p) : K] + 1 = \Delta_f - p + 1$.

\[ y^2 = x^p - p^2 \]

- Weierstrass model is not regular! Need $(p - 1)/2$ blowups of $\mathbb{P}^1_R$.
- $\delta = \Delta_{K(p^2/p)/K} - [K(p^2/p) : K] + 1$
  $= \Delta_{K(p^1/p)/K} - [K(p^1/p) : K] + 1$
  $= \Delta_f - 2\left(\nu_p(p^2/p) - \nu_p(p^1/p)\right)(\frac{p}{2}) - p + 1$
  $= \Delta_f - 2(p - 1)$.
Examples where $\delta \neq 0$

Let $K = \hat{Q}_p^{unr}$, $p$ odd.

$y^2 = x^p - p$

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- $\text{Art}^+(X) = [2\chi(Y_k) - 2\chi(Y_K)] - [\chi(B_k) - \chi(B_K)] + \delta = p - 1 + \delta$
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$y^2 = x^p - p^2$

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- $\delta = \Delta_{K(p^{2/p})/K} - [K(p^{2/p}) : K] + 1$
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Finally . . .

Thank you!