Conductors and minimal discriminants of hyperelliptic curves

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Outline

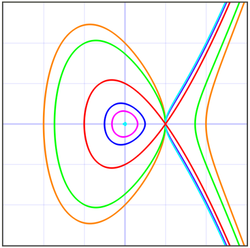
What are conductors and minimal discriminants?

Degenerating family of hyperelliptic curves → Measures of degeneracy

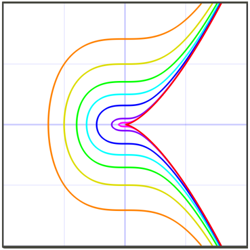
Measures of degeneracy → Artin conductor

Measures of degeneracy → Minimal discriminant

How are these related?



$y^2 = (x-1)(x^2-t)$



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How are conductors and minimal discriminants related?

Earlier results: (small genus, all residue characteristics)

- If $g = 1$, then $\mathrm{Art}^+(X) = \Delta_X$. [Ogg-Saito formula]
- If $g = 2$, then Liu showed that $\mathrm{Art}^+(X) \leq \Delta_X$. He showed that equality does not always hold.

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Question: Does $\mathrm{Art}^{+}(X) \leq \Delta_X$ hold for hyperelliptic curves of arbitrary genus g?

Today:

- Yes, if the residue characteristic is $> 2g + 1$. [S.]
  - Combinatorial restrictions for equality when $g \geq 2$.
- Yes, if the residue characteristic is $\neq 2$. [Joint work with Obus]

Outline

Notation

R: complete discrete valuation ring

K: fraction field of R

k: residue field of R, algebraically closed, char $\neq 2$

$\overline{K}$: a fixed separable closure of K

G_K: Galois group of $\overline{K}/K$

ν: valuation $\overline{K} \to \mathbb{Q} \cup \{\infty\}$

t: a uniformizer of R, *i.e.*, $\nu(t) = 1$.

Examples: $\mathbb{C}[[t]], \widehat{\mathbb{Z}_p^{\mathrm{unr}}}$

X: smooth hyperelliptic K-curve

g: genus of X

Definition: The minimal discriminant Δ_X of X/K is the nonnegative integer

$$\Delta_X := \min_{\substack{f(x)\in R[x] \\ y^2=f(x),\ \text{eqn. for X}}} \underbrace{\nu(\mathrm{disc}(f))}_{\in R}.$$

An example: $K = \mathbb{C}((t))$

$$C_1 : y^2 = x(x-t)(x-2t)(x-3t) \quad \rightsquigarrow \quad \nu(\mathrm{disc}(f)) = 2\binom{4}{2}.$$

$$C_2 : y'^2 = x'(x'-1)(x'-2)(x'-3) \quad \rightsquigarrow \quad \nu(\mathrm{disc}(f)) = 0.$$

Here $C_1 \cong_K C_2$ via $x' = \frac{x}{t}, y' = \frac{y}{t^2} \rightsquigarrow \Delta_X = 0$.

Fix a prime $\ell \neq \mathrm{char}\, k$. For any curve C over an algebraically closed field of char $\neq \ell$, let

$$\chi(C) := \sum_{i=0}^{2} (-1)^i \dim H^i_{\acute{e}t}(C, \mathbb{Q}_\ell).$$

δ: Swan conductor for the G_K representation $H^1(X_{\overline{K}}, \mathbb{Q}_\ell)$ (integer, ≥ 0, measure of wild ramification).

$\mathcal{X}^{\mathrm{min}}$: minimal proper regular R-model of X.

Definition: The Artin conductor $\mathrm{Art}^+(X)$ of X/K is

$$\mathrm{Art}^+(X) := \chi(\mathcal{X}^{\mathrm{min}}_k) - \chi(\mathcal{X}^{\mathrm{min}}_{\overline{K}}) + \delta.$$

Properties of the Artin Conductor

- $\mathrm{Art}^+(X)$ is independent of ℓ.
- $\mathrm{Art}^+(X) \geq 0$.
  $\mathrm{Art}^+(X) = 0 \iff \mathcal{X}^{\min} \to \operatorname{Spec} R$ is smooth or $g = 1$ and $(\mathcal{X}_k)_{\mathrm{red}}$ is smooth.
- Let n be the number of components of $\mathcal{X}_k^{\min}$ and let ϵ be the tame conductor for the G_K representation $H^1(X_{\overline{K}}, \mathbb{Q}_\ell)$. Then,

$$\mathrm{Art}^+(X) = (n-1) + \epsilon + \delta.$$

- When $\mathcal{X}^{\min}$ is regular and semi-stable,

$$\mathrm{Art}^+(X) = \#\text{ singular points of } \mathcal{X}_k^{\min}.$$

Theorem (S.)

Let K be the fraction field of a Henselian discrete valuation ring. Let X be a smooth hyperelliptic curve over K of genus $g \geq 1$. Assume that the residue characteristic is $> 2g+1$. Then,

$$\mathrm{Art}^{+}(X) \leq \Delta_X .$$

Outline

Explicit regular models when char $k \neq 2$

Remark: Suffices to find ONE proper regular model $\mathcal{X}$ such that

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Two reasons for non regular Weierstrass models:

- Components of $\operatorname{div} f \subset \mathbb{P}^1_R$ intersect.
  (Example: $K = \mathbb{C}((t))$, $\quad y^2 = x(x-t)(x-1)$.)
- Components of $\operatorname{div} f \subset \mathbb{P}^1_R$ are not regular curves.
  (Example: $K = \mathbb{C}((t))$, $\quad y^2 = x^3 - t^2$.)

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Solution: Blow-up $\mathbb{P}^1_R$ first *before* taking a double cover.

Lemma

Let $\mathrm{Bl}\ \mathbb{P}^1_R$ be an arithmetic surface birational to $\mathbb{P}^1_R$.
Let f be an element of the function field of $\mathbb{P}^1_R$.
Assume that the odd multiplicity components of the divisor of f on $\mathrm{Bl}\ \mathbb{P}^1_R$ are disjoint and regular.
Then, the normalization of $\mathrm{Bl}\ \mathbb{P}^1_R$ in $K(x, \sqrt{f(x)})$ is a proper regular model for the hyperelliptic curve given by $y^2 = f(x)$.

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Explicit regular model:
Let $y^2 = f(x)$ be an equation for X with $f(x) \in R[x]$ and $\Delta_X = \Delta_f$.
Let $\mathrm{Bl}\ \mathbb{P}^1_R$ be the (minimal) blowup of $\mathbb{P}^1_R$ satisfying the conditions above and $\mathcal{X}_f$ the associated proper regular model of X.

- Riemann-Hurwitz formula: If $\mathcal{X} \to \mathcal{Y}$ is a double cover of arithmetic surfaces, branched over the divisor B, then,

$$\mathrm{Art}^+(\mathcal{X}) = [2\chi(\mathcal{Y}_k) - \chi(B_k)] - [2\chi(\mathcal{Y}_{\overline{K}}) - \chi(B_{\overline{K}})] + \delta.$$

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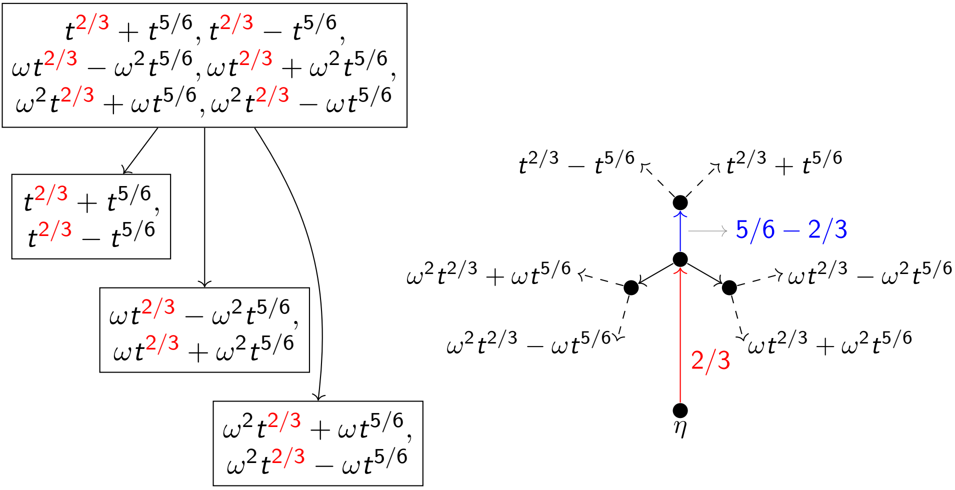
Additional tools:

char $k > 2g+1 \rightsquigarrow \delta = 0$	char $k \neq 2$
• Roots of $f \rightsquigarrow$ Metric tree of f	• Explicit formula for δ
• Induction on the metric tree	• Maclane valuations and approximate roots
• Abhyankar's Inversion formula	

Key inductive inequality:

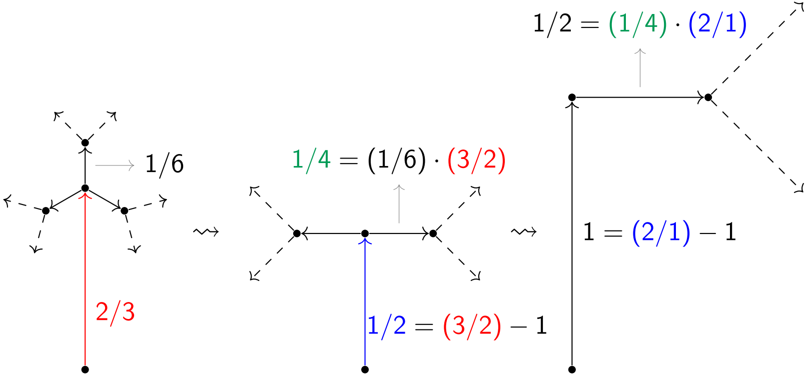
$$\boxed{\Delta_f - \Delta_{f^{\mathrm{new}}} = n(n-1) \geq 2 = \mathrm{Art}^{+}(\mathcal{X}_f) - \mathrm{Art}^{+}(\mathcal{X}_{f^{\mathrm{new}}}) \quad (\because n \geq 2).}$$

Roots of f $\rightsquigarrow$ Metric tree of f



Inductive process on metric trees using Abhyankar's inversion formula

(In the example below, $a = 2, b = 3$.)
b identical subtrees $\rightsquigarrow$ a identical subtrees.
distance a/b from η $\rightsquigarrow$ distance $(b/a) - 1$ from η.
New subtree metric $=$ (Old subtree metric) $\cdot$ b/a.



Proof in an easy example, $K = \mathbb{C}((t))$

$$f(x) = x(x - 1 - t)(x - 1 - 2t)(x - 1 - 3t)(x - 1 - 4t)$$

$$\downarrow$$

$$f^{\mathrm{new}}(x) = (x - 1)(x - 2)(x - 3)(x - 4)$$

$$\mathrm{Art}^{+}(\mathcal{X}_f) - \mathrm{Art}^{+}(\mathcal{X}_{f^{\mathrm{new}}}) = 2.$$

$$\Delta_f - \Delta_{f^{\mathrm{new}}} = 2\binom{4}{2} = 12.$$

$$\mathrm{Art}^{+}(\mathcal{X}_{f^{\mathrm{new}}}) = \Delta_{f^{\mathrm{new}}} = 0.$$

Examples where $\delta \neq 0$

Let $K = \widehat{\mathbb{Q}_p^{\mathrm{unr}}}$, p odd.

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- $\mathrm{Art}^+(X) = [2\chi(\mathcal{Y}_k) - 2\chi(\mathcal{Y}_{\overline{K}})] - [\chi(B_k) - \chi(B_{\overline{K}})] + \delta = p - 1 + \delta$
- $\delta = \Delta_{K(p^{1/p})/K} - [K(p^{1/p}) : K] + 1 = \Delta_f - p + 1.$

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Thank you!