Complex Moments and the distribution of Values of $L(1, \chi_D)$ over Function Fields with Applications to Class Numbers

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Recent Developments in Number Theory
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The norm of $f \neq 0 \in \mathbb{A}$: $|f| = \#(\mathbb{A}/f\mathbb{A}) = q^{\deg(f)}$.
Prime elements correspond to monic irreducible polynomials.

Set Up II

The zeta function over $\mathbb{A}$: Let $s \in \mathbb{C}$, $\mathfrak{Re}(s)>1$.

$$\zeta_{\mathbb{A}}(s) = \sum_{\substack{f \in \mathbb{A} \\ f \text{ monic}}} \frac{1}{|f|^s} = \prod_{\substack{P \text{ irreducible} \\ P \text{ monic}}} \left(1 - \frac{1}{|P|^s}\right)^{-1}.$$

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Let $\chi : (\mathbb{A}/f\mathbb{A})^\times \to \mathbb{C}^\times$, extend the definition to $\mathbb{A}$:

$$L(s,\chi) = \sum_{g \text{ monic}} \frac{\chi(g)}{|g|^s} = \prod_{\substack{P \text{ irreducible}\\ P \text{ monic}}} \left(1 - \frac{\chi(P)}{|P|^s}\right)^{-1}.$$

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Orthogonality relations for χ give us that $L(s,\chi)$ is a polynomial of $\deg(f) - 1$.

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Taking D square free gives $\mathcal{O}_K$ is a Dedekind domain ie:
$\text{Pic}\mathcal{O}_K = \mathcal{C}l(K)$.

Hyperelliptic Curves over $\mathbb{F}_q$

Let $D \in \mathbb{A}$ be a monic squarefree polynomial with degree $2g + 1$. Consider

$$C_D : y^2 = D(T).$$

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$$P_{C_D}(u) = L(s, \chi_D) = \sum_{f \text{ monic}} \frac{\chi_D(f)}{|f|^s}, \text{ where } \chi_D(f) = \left(\frac{D}{f}\right)_2.$$

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Artin proved

$$L(1, \chi_D) = \frac{\sqrt{q}}{\sqrt{|D|}} h_D = q^{-g} h_D.$$

What we know: Mean Value of h_D

Theorem (Rosen & Hoffstein (1992))

Let $M > 0$ be a fixed odd number then

$$\frac{1}{q^M} \sum_{\substack{D \text{ monic} \\ \deg(D)=M}} h_D = \frac{\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}(3)} q^{\frac{M-1}{2}} - q^{-1}.$$

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Directly analogous to Gauß conjectures about the growth of class numbers with discriminant $\pm 4k$:

$$\frac{1}{N} \sum_{1 \le k \le N} h_{-4k} \sim \frac{4\pi}{21\zeta(3)} N^{\frac{1}{2}},$$

and

$$\frac{1}{N} \sum_{1 \le k \le N} R_{4k} h_{4k} \sim \frac{4\pi^2}{21\zeta(3)} N^{\frac{1}{2}}.$$

Mean Value of h_D: perspective shift

Fix q and define

$$\mathcal{H}_{2g+1} = \{D \in \mathbb{A} : D \text{ monic}, \deg(D) = 2g+1 \text{ and } D\,\square\text{-free}\}$$

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Let $q \equiv 1 \pmod 4$ *then*

$$\frac{1}{\#\mathcal{H}_{2g+1}} \sum_{D \in \mathcal{H}_{2g+1}} h_D \sim \zeta_{\mathbb{A}}(2) \prod_{\substack{P \text{ irreducible} \\ P \text{ monic}}} \left(1 - \frac{1}{(|P|+1)|P|^2}\right) q^g$$

as $g \to \infty$.

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Let $M = 2g+1$ be a fixed odd number then Rosen and Hoffstein give

$$\frac{1}{q^{2g+1}} \sum_{\substack{D \text{ monic} \\ \deg(D) = 2g+1}} h_D = \frac{\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}(3)} q^g - q^{-1}.$$

Mean Value of h_D: perspective shift

Fix q and define

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Artin proves:

$$L(1, \chi_D) = \frac{q-1}{\sqrt{|D|}} h_D R_D,$$

with R_D the regulator of $\mathbb{A}[\sqrt{D(T)}]$.

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Complex moments of $L(1, \chi_D)$, $D \in \mathcal{H}_n$

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$$d_z(P^a) = \frac{\Gamma(z+a)}{\Gamma(z)a!},$$

and extend it to all monic polynomials multiplicatively.

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Theorem (L, (2018))

Let n be a positive integer, and $z \in \mathbb{C}$ be such that $|z| \leq \frac{n}{260 \log(n) \ln \log(n)}$. Then

$$\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} L(1, \chi_D)^z = \sum_{f \text{ monic}} \frac{d_z(f^2)}{|f|^2} \prod_{P|f} \left(1 + \frac{1}{|P|}\right)^{-1} \left(1 + O\left(\frac{1}{n^{11}}\right)\right).$$

Applications Part I

Corollary (L, (2018))

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$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} h_D^z = q^{gz} \sum_{f \text{ monic}} \frac{d_z(f^2)}{|f|^2} \prod_{P|f} \left(1 + \frac{1}{|P|}\right)^{-1} \times \left(1 + O\left(\frac{1}{g^{11}}\right)\right),$$

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$$\frac{1}{|\mathcal{H}_{2g+2}|} \sum_{D \in \mathcal{H}_{2g+2}} (h_D R_D)^z = \left(\frac{q^{g+1}}{q-1}\right)^z \sum_{f \text{ monic}} \frac{d_z(f^2)}{|f|^2} \prod_{P|f} \left(1 + \frac{1}{|P|}\right)^{-1} \times \left(1 + O\left(\frac{1}{g^{11}}\right)\right).$$

Distribution of Values of $L(1, \chi_D)$

Theorem (L, (2018))

Let n be large. Uniformly in $1 \leq \tau \leq \log n - 2\log_2 n - \log_3 n$ *we have*

$$\frac{1}{|\mathcal{H}_n|}|\{D \in \mathcal{H}_n : L(1,\chi_D) > e^{\gamma}\tau\}| = \exp\left(-C_1(q,\tau)\frac{q^{\tau - C_0(q,\tau)}}{\tau}\left(1 + O\left(\frac{\log\tau}{\tau}\right)\right)\right),$$

and similar estimates hold for the number of discriminants $D \in \mathcal{H}_n$ *such that* $L(1,\chi_D) < \frac{\zeta_{\mathbb{A}}(2)}{e^{\gamma}\tau}$.

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the argument of C_0 and C_1 is bounded between 1 and q. In particular, C_1 is a multiplicatively periodic function with multiplicator q.

Applications Part II

Corollary (L, (2018))

Let g be large and $1 \le \tau \le \log g - 3\log\log g$. *The number of discriminants* $D \in \mathcal{H}_{2g+1}$ *such that* $h_D > e^{\gamma}\tau q^g$ *equals*

$$|\mathcal{H}_{2g+1}| \cdot \exp\left(-C_1(q,\tau)\frac{q^{\tau - C_0(q,\tau)}}{\tau}\left(1 + O\left(\frac{\log\tau}{\tau}\right)\right)\right),$$

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We also obtain distribution values for $h_D R_D$ with $D \in \mathcal{H}_{2g+2}$.

Context for these results

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b) class numbers of the Chowla family (a special family of real quadratic fields) by Dahl-Lamzouri 2016.
Granville and Soundararajan prove the proportion of imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$ with $|d| \leq x$ such that

$$h(-d) \geq \frac{\sqrt{d}}{\pi} e^{\gamma}\tau$$

for $1 \leq \tau \leq \log_2 x - 3\log_3 x$ equals

$$\exp\left(-\frac{e^{\tau - C_0}}{\tau}\left(1 + O\left(\frac{1}{\tau}\right)\right)\right),$$

where $C_0 = 0.8187\ldots$

Context for these results

Similar results have been obtained for
a) class numbers of imaginary quadratic fields by Granville-Soundararajan 2003
b) class numbers of the Chowla family (a special family of real quadratic fields) by Dahl-Lamzouri 2016.
Granville and Soundararajan prove the proportion of imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$ with $|d| \leq x$ such that

$$h(-d) \geq \frac{\sqrt{d}}{\pi} e^{\gamma} \tau$$

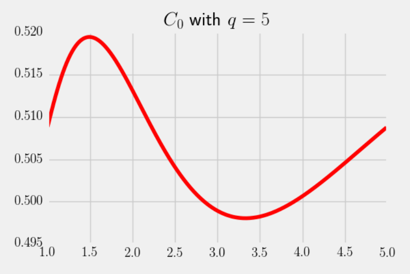
for $1 \leq \tau \leq \log_2 x - 3\log_3 x$ equals

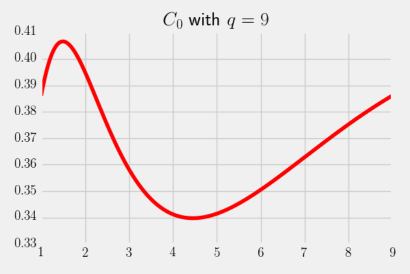
$$\exp\left(-\frac{e^{\tau - C_0}}{\tau}\left(1 + O\left(\frac{1}{\tau}\right)\right)\right),$$

where $C_0 = 0.8187\ldots$ This behaviour is typical and has been observed in multiple distribution results.

Graphs for the pathological coefficients

$C_0(t)$:





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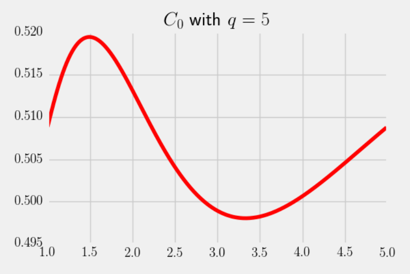
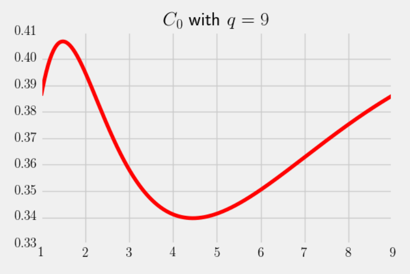
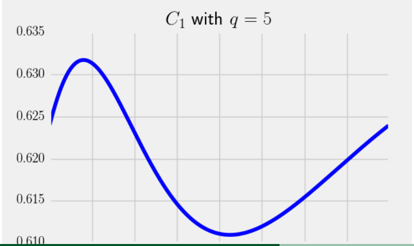
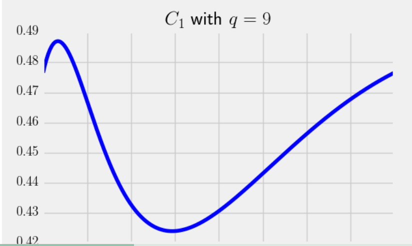




$C_1(t)$:





Extreme Values of $L(1, \chi_D)$

Conjecture (L, (2018))

Let n be large.

$$\max_{D \in \mathcal{H}_n} L(1, \chi_D) = e^{\gamma}(\log n + \log_2 n) + O(1),$$

and

$$\min_{D \in \mathcal{H}_n} L(1, \chi_D) = \zeta_{\mathbb{A}}(2) e^{-\gamma}(\log n + \log_2 n + O(1))^{-1}.$$

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We find unconditionally that

Proposition (L, (2018))

Let F be a monic polynomial, and χ be a non-trivial character on $(\mathbb{A}/F\mathbb{A})^{\times}$. For any complex number s with $\mathfrak{Re}(s) = 1$ we have

$$\frac{\zeta_{\mathbb{A}}(2)}{2e^{\gamma}}(\log_2 |F| + O(1))^{-1} \leq |L(s, \chi)| \leq 2e^{\gamma} \log_2 |F| + O(1).$$

Extreme Values of h_D

We prove unconditionally,

Theorem (L,(2018))

Let N be large. There are irreducible polynomials Q_1 and Q_2 of degree N such that

$$L(1,\chi_{Q_1}) \geq e^{\gamma}(\log_2 |Q_1| + \log_3 |Q_1|) + O(1),$$

and

$$L(1,\chi_{Q_2}) \leq \zeta_{\mathbb{A}}(2)e^{-\gamma}(\log_2 |Q_1| + \log_3 |Q_1| + O(1))^{-1}.$$

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This bound matches the conjecture.

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Let $\{\mathbb{X}(P)\}$ denote a sequence of independent random variables indexed by the irreducible (prime) elements $P \in \mathbb{A}$, taking the values $0, \pm 1$ as follows

$$\mathbb{X}(P) = \begin{cases} 0 & \text{with probability } \frac{1}{|P|+1} \\ \pm 1 & \text{with probability } \frac{|P|}{2(|P|+1)}. \end{cases}$$

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Compare the distribution of $L(1, \chi_D)$ with

$$L(1, \mathbb{X}) := \sum_{f \text{ monic}} \frac{\mathbb{X}(f)}{|f|} = \prod_{P \text{ irreducible}} \left(1 - \frac{\mathbb{X}(P)}{|P|}\right)^{-1},$$

which converges almost surely.

Connecting Moments of $L(1, \chi_D)$ to the random model

Theorem (L, (2018))

Let $z \in \mathbb{C}$ be such that $|z| \leq \frac{n}{260 \log n \log \log n}$. Then

$$\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} L(1, \chi_D)^z = \mathbb{E}(L(1, \mathbb{X})^z) \left(1 + O\left(\frac{1}{n^{11}}\right)\right),$$

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Specializing to $n = 2g + 1$ or $n = 2g + 2$ we apply the appropriate Artin formula to obtain our formulas for h_D and $h_D R_D$ respectively.

Distribution of values for $L(1, \chi_D)$

Theorem (L, (2018))

Let n be large. Uniformly in $1 \leq \tau \leq \log n - 3 \log\log n$ *we have*

$$\frac{1}{|\mathcal{H}_n|}|\{D \in \mathcal{H}_n : L(1, \chi_D) > e^{\gamma}\tau\}| = \mathbb{P}(L(1, \mathbb{X}) > e^{\gamma}\tau)\left(1 + O\left(\frac{e^{\tau}(\log n)^2 \log_2 n}{n}\right)\right).$$

Distribution of values for $L(1, \chi_D)$

Use a smooth analogue of Perron and careful saddle point analysis to obtain

$$\mathbb{P}(L(1,\mathbb{X}) > e^{\gamma}\tau) = \exp\left(-C_1(q,\tau)\frac{q^{\tau - C_0(q,\tau)}}{\tau}\left(1 + O\left(\frac{\log \tau}{\tau}\right)\right)\right)$$

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Thus obtaining the claimed result for the distribution of values of $L(1, \chi_D)$, $D \in \mathcal{H}_n$.

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Theorem

Let n be large, $1/2 < \sigma < 1$ be fixed, $B \geq 2$ a constant satisfying some condition. Then for $z \in \mathbb{C}$ with $|z| \ll n^{g(\sigma)}$ we have

$$\frac{1}{|\mathcal{H}_n|} \sum_{D \in \widetilde{\mathcal{H}}_{n,g}} L(\sigma, \chi_D)^z = \mathbb{E}(L(\sigma, \mathbb{X}))^z + O\left(\frac{\mathbb{E}(L(\sigma, \mathbb{X}))^{\mathfrak{Re}(z)}}{n^{B-(g(\sigma)+1)}}\right), \quad (1.1)$$

where $2\sigma - 1 \leq g(\sigma) \leq \sigma$ and we have for some constants $C, C' > 0$ that

$$|\mathcal{H}_n \setminus \widetilde{\mathcal{H}}_{n,g}| \ll |\mathcal{H}_n| \exp\left(C \frac{n}{\log n}(2(g(\sigma) - \sigma)\log n - C')\right).$$

Distribution of values for $L(\sigma, \chi_D)$

Theorem

Let n be large and $1/2 < \sigma < 1$ *be fixed.There exists a positive constant* $b(\sigma)$ *such that* $3 \leq \tau \leq b(\sigma)n^{1-\sigma}/(\log n)^{1/\sigma}$ *we have*

$$\frac{1}{|\mathcal{H}_n|}|\{D \in \mathcal{H}_n : \log L(\sigma, \chi_D) > \tau\}| = \exp\left(-C_1(q, \tau, \sigma)\tau^{\frac{1}{1-\sigma}}(\log \tau)^{\frac{\sigma}{1-\sigma}}\left(1 + O\left(\frac{\log_2 \tau}{(\log \tau)^{2-\frac{1}{\sigma}}}\right)\right)\right),$$

This C_1 is a function of τ, similar to the case for $L(1, \chi_D)$. We prove also that this function is multiplicatively periodic with multiplicator q^σ.

Thank you for listening! :)