Newton Polygon Stratification of the Torelli Locus in PEL-type Shimura Varieties

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May 29, 2019

Torelli Locus in $\mathcal{A}_g$

The Torelli map which sends a smooth curve to its Jacobian together with the induced polarization maps the moduli of smooth curves of genus g to the moduli of p.p.a.v. of dimension g. By Torelli theorem, this map is injective and we refer to the image of this map as the Torelli locus in $\mathcal{A}_g$.

$$\mathcal{T} : \mathcal{M}_g \hookrightarrow \mathcal{A}_g/\mathbb{Z}.$$

As we know,

$$\dim \mathcal{M}_g = 3g - 3 \text{ and } \dim \mathcal{A}_g = \frac{g(g+1)}{2}$$

So the Torelli locus is very small when g gets large.

Question (The Schottky problem)

Which p.p.a.v. is the Jacobian of a smooth projective curve?

Newton Polygon for Abelain Varieties over Finite Fields

Let $\mathbb{F}_q$ be a finite field of characteristic p and $q = p^n$. Let A be an abelian variety defined over $\mathbb{F}_q$ of dim g. Define polynomial

$$P(x) = \prod_{i=1}^{2g}(1 - \alpha_i x) = a_0 + a_1 x + \ldots + a_{2g} x^{2g}$$

where α_i's are eigenvalues of Frob_q acting on the ℓ-adic Tate module of A.

We define the Newton polygon of A as the lower convex hull of the points

$$\left(i, \frac{v_p(a_i)}{n}\right).$$

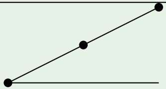
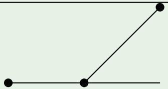
Example and properties of Newton polygons

Basic properties of Newton polygons:

- break at points with integer coefficients
- start at $(0, 0)$ and end at $(2g, g)$
- slope λ appears with the same multiplicity as $1 - \lambda$

Newton polygons with these properties are called symmetric admissible Newton polygons of height $2g$.

Example (reductions of elliptic curve $E : y^2 = x^3 + 1$)

field	$\mathbb{F}_5$	$\mathbb{F}_7$
$P(x)$	$1 + 5x^2$	$1 + 4x + 7x^2$
Newton polygon	 supersingular	 ordinary

Newton polygon stratification of $\mathcal{A}_g$

There is a partial ordering on the set of symmetric admissible Newton polygons by comparing whether one lies above the other. With this ordering, the polygon with only slops 0 and 1 (ordinary) is the lowest and the polygon with only slope $1/2$ (supersingular) is the highest.

Theorem (Oort)

Let v be a symmetric admissible Newton polygon of height $2g$. The stratum $\mathcal{A}_g[v]$ is nonempty.
Moreover, the codimension of $\mathcal{A}_g[v]$ in $\mathcal{A}_g$ is given by the length of any path from the ordinary polygon to v with respect to the partial order.

In particular, the codimension of the supersingular locus in $\mathcal{A}_g$ is $\frac{g(g+1)}{2} - \lfloor \frac{g^2}{4} \rfloor$ which grows quadratically in terms of g.

Newton polygons in the Torelli locus

Question (Oort)

Which Newton polygons occur for Jacobians of smooth curves?
Or what is the locus $\mathcal{A}_g[v] \cap \mathcal{M}_g$?

This question is hard in general, but there are two easy cases:

- When $g = 1, 2, 3$, all symmetric admissible Newton polygons occur for Jacobians of smooth curves
  Key: $\dim \mathcal{A}_g = \dim \mathcal{M}_g$
- For any g, the ordinary Newton polygons occur for Jacobians of smooth curves (Faber–van der Geer)
  Key: ordinary Newton polygon is the most generic and its locus is open and dense in $\mathcal{A}_g$

Question

Can we replace $\mathcal{M}_g$ and $\mathcal{A}_g$ and by smaller spaces where we can generalize the two cases above?

Special families of cyclic covers of $\mathbb{P}^1$

Fix integers $m \geq 2$ and $N \geq 4$, together with an N-tuple $a = (a_1, \ldots, a_N)$. Then a is an inertia type for m and (m, N, a) is a monodromy datum if

- $a_i \neq 0 \bmod m$,
- $\gcd(m, a_1, \ldots, a_N) = 1$,
- $a_1 + \ldots + a_N = 0 \bmod m$.

The equation $y^m = \prod_{i=1}^N (x - t_i)^{a_i}$ defines a smooth projective curve C. The function x on C yields a map $C \to \mathbb{P}^1$, and there is a μ_m-action on C over $\mathbb{P}^1$ given by $\zeta(x, y) = (x, \zeta y)$ for all $\zeta \in \mu_m$.

Varying the branch points in $\mathrm{Conf}^N(\mathbb{P}^1)$, we obtain an $N-3$-dimensional subvariety $\mathcal{Z}^\circ(m, N, a) \subset \mathcal{M}_g$.

Special families of cyclic covers of $\mathbb{P}^1$

Let $\mathcal{Z}(m, N, a)$ be the closure of $\mathcal{Z}^\circ(m, N, a)$ in $\mathcal{A}_g$. Then $\mathcal{Z}(m, N, a)$ naturally lies in a Shimura subvariety of PEL-type. We denote the largest irreducible component of this Shimura variety containing $\mathcal{Z}(m, N, a)$ by $\mathcal{S}(m, N, a)$.

By a result of Moonen, there are exactly 20 positive-dimensional families of cyclic covers of the projective line for which the Torelli image of $\mathcal{Z}^\circ(m, N, a)$ is open and dense in $\mathcal{S}(m, N, a)$. i.e.

$$\dim \mathcal{Z}^\circ(m, N, a) = \dim \mathcal{S}(m, N, a).$$

Monodromy datum for Moonen families

label	g	m	N	a	label	g	m	N	a
1	1	2	4	(1,1,1,1)	11	4	5	4	(1,3,3,3)
2	2	2	6	(1,1,1,1,1,1)	12	4	6	4	(1,1,1,3)
3	2	3	4	(1,1,2,2)	13	4	6	4	(1,1,2,2)
4	2	4	4	(1,2,2,3)	14	4	6	5	(1,2,2,3,3)
5	2	6	4	(2,3,3,4)	15	5	8	4	(2,4,5,5)
6	3	3	5	(1,1,1,1,2)	16	6	5	5	(2,2,2,2,2)
7	3	4	4	(1,1,1,1)	17	6	7	4	(2,4,4,4)
8	3	4	5	(1,1,2,2,2)	18	6	10	4	(3,5,6,6)
9	3	6	4	(1,3,4,4)	19	7	9	4	(3,5,5,5)
10	4	3	6	(1,1,1,1,1,1)	20	7	12	4	(4,6,7,7)

Newton polygons in Moonen families

Theorem (L.–Mantovan–Pries–Tang, 2018)

For p sufficiently large, all Newton polygons in $\mathcal{S}(m, N, a)$ occur in $\mathcal{Z}^\circ(m, N, a)$. i.e. For any Newton polygon v in $\mathcal{S}(m, N, a)$, there exists a smooth curve in family with monodromy datum (m, N, a) over $\overline{\mathbb{F}}_p$ whose Newton polygon is v.

Example (Moonen family 11)

p mod 5	1	2,3	4
μ-ordinary	$(0,1)^4$	$(1/4, 3/4)$	$(0,1)^2 + (1/2,1/2)^2$
basic	$(0,1)^2 + (1/2,1/2)^2$	$((1/2,1/2)^4)^\dagger$	$((1/2,1/2)^4)^\dagger$

Note that in the case of the Newton polygons denoted by † we further assume p sufficiently large.

Question (Oort)

If Newton polygons v_1, v_2 both occur for smooth curves, do we know this for $v_1 + v_2$?

It is in general difficult to answer this question but some results can be deduced using clutching morphisms.

Pries proves that if a Newton polygon v occurs on $\mathcal{M}_g$ with the expected codimension, then the Newton polygon $v + (0,1)^e$ occurs on $\mathcal{M}_{e+g}$ with the expected codimension.

Clutching method on special families of cyclic covers of $\mathbb{P}^1$

Theorem (L.–Mantovan–Pries–Tang, 2018)

Let $(m, N_1, (a_1, \ldots, a_{N_1})), (m, N_2, (b_1, \ldots, b_{N_2}))$ be a pair of monodromy datums corresponding to families $\mathcal{Z}_1^\circ$ and $\mathcal{Z}_2^\circ$. Let $\mathcal{Z}_3^\circ$ be the family with monodromy datum $(m, N_1 + N_2, (a_1, \ldots, a_{N_1}, b_1, \ldots, b_{N_2}))$. Then if

- $\mathcal{Z}_1^\circ[\mu_1], \mathcal{Z}_2^\circ[\mu_2] \neq \emptyset$ *(where μ_i is the μ-ordinary polygon for Z_i°),*
- *and* $\mu_3 = \mu_1 + \mu_2 + (0,1)^{m-1}$
  (We give an equivalent condition on the monodromy datums).

Then $\mathcal{Z}_3^\circ[\mu_3] \neq \emptyset$.

Remark

Adding one more condition, we can replace one of the μ_i by any Newton polygon occurring in $\mathcal{Z}_i^\circ$.

Examples

Corollary

Let $\mathcal{Z}$ be one of the Moonen families and μ be the μ-ordinary Newton polygon for a prime p. Then there exists a smooth curve over $\overline{\mathbb{F}}_p$ with Newton polygon $\mu^n + (0,1)^{(n-1)(m-1)}$ for any $n \in \mathbb{Z}_{\geq 1}$.

Example (Moonen family 11)

For any prime $p \equiv 2, 3 \mod 5$ and any $n \in \mathbb{N}_{>0}$, there exists a smooth curve over $\overline{\mathbb{F}}_p$ whose Newton polygon is $(1/4, 3/4)^n + (0,1)^{4n-4}$.

Example (Moonen family 6)

For any odd prime $p \equiv 2 \mod 3$ and any $n \in \mathbb{N}_{>0}$, there exists a smooth curve over $\overline{\mathbb{F}}_p$ whose Newton polygon is $(1/2, 1/2)^n + (0,1)^{2n}$.

Unlikely intersection

Definition

We say that the Torelli locus has an unlikely intersection with the Newton polygon stratum $\mathcal{A}_g[v]$ in $\mathcal{A}_g$ if there exists a smooth curve of genus g with Newton polygon , and $\dim \mathcal{M}_g < \operatorname{codim}(\mathcal{A}_g[v], \mathcal{A}_g)$.

When $g \geq 9$, $\mathcal{M}_g$ and $\mathcal{A}_g[ss]$ form unlikely intersection inside $\mathcal{A}_g$ if

$$\mathcal{A}_g[ss] \neq \emptyset.$$

Theorem (L.–Mantovan–Pries–Tang, 2018)

Let $\mathcal{Z}$ be one of the Moonen families and μ be the μ-ordinary Newton polygon for a prime p. When μ is not ordinary, for n sufficiently large, the Torelli locus has an unlikely intersection with $\mathcal{A}_g[\mu^n + (0,1)^{(n-1)(m-1)}]$.

Acknowledgment

This project began at the Women in Numbers 4 workshop at the Banff International Research Station. We would like the thank the organizers Jennifer Balakrishnan, Chantal David, Michelle Manes and Bianca Viray.