The mean value of cubic L-functions over function fields

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The Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \qquad \mathrm{Re}(s) > 1$$

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- The functional equation.

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

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- Trivial zeros: $-2, -4, -6, \dots$
- The Riemann Hypothesis (RH) : nontrivial zeros of $\zeta(s)$ at $\mathrm{Re}(s) = \frac{1}{2}$.

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The Lindelöf Hypothesis

The Lindelöf Hypothesis (LH) is a consequence of RH :

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$$\zeta\left(\frac{1}{2}+it\right) = O(t^{\varepsilon}).$$

This is equivalent to

$$\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt = O(T^{\varepsilon}) \qquad \forall k \in \mathbb{N}.$$

Understanding higher moments of $\zeta(s)$ gives us progressively better bounds for $\zeta(1/2+it)$.

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Let

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Moments of the Riemann zeta function - Some history

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$$I_2(X) \sim \frac{1}{2\pi^2} T(\log T)^4$$

- $k \geq 3$ remain unsolved! Keating and Snaith (2000) conjectured

$$I_k(X) \sim C_k T(\log T)^{k^2}$$

Dirichlet L-functions

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=\prod_{p\text{ prime}}\left(1-\frac{\chi(p)}{p^s}\right)^{-1}\qquad \mathrm{Re}(s)>1,$$

where χ is a primitive Dirichlet character modulo d.

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The moments at the central critical value $L\left(\frac{1}{2},\chi\right)$ are associated with nonvanishing results.

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The moments at the central critical value $L\left(\frac{1}{2},\chi\right)$ are associated with nonvanishing results.
There is a general philosophy that non-vanishing of L-functions at the critical point should be explained by arithmetic reasons.

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Moments ζ and L (a very vague argument)

What is the relation between

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^k dt \text{ and } \sum_{\substack{\chi \bmod d \\ d\leq T}}^{*} L\left(\frac{1}{2},\chi\right)^k ?$$

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$$L\left(\frac{1}{2},\chi\right) \leftrightarrow \sum \frac{\chi(n)}{n^{\frac{1}{2}}}$$

(Warning: the right-hand sides do not make any sense.)

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$$\zeta\left(\frac{1}{2}+it\right) \leftrightarrow \sum \frac{n^{-it}}{n^{\frac{1}{2}}}$$

(Warning: the right-hand sides do not make any sense.)
Notice: $\varphi_t(n) = n^{-it}$ is "like" a character. Integrating is summing over a continuous index.

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Moments of primitive quadratic Dirichlet L-functions

Keating and Snaith (2000) conjectured that

$$\sum_{\substack{\chi \bmod d \\ \chi^2=1,\, d\leq T}}^{*} L\left(\frac{1}{2},\chi\right)^k \sim C_k' T(\log T)^{\frac{k(k+1)}{2}}.$$

The $*$ indicates primitive characters.

- $k=1$ Jutila (1981)
- $k=2$ Jutila (1981), Soundararajan (secondary main term, 2000)
- $k=3$ Soundararajan (2000), Diaconu, Goldfeld, Hoffstein
- $k=4$ Soundararajan and Young (under GRH, 2010)

Non-vanishing results

By Cauchy–Schwartz,

$$\sum\nolimits^{*}_{\substack{\chi \bmod d \\ \chi^2=1,\, d\leq T}} L\left(\tfrac{1}{2},\chi\right) \ll$$

$$\left(\sum\nolimits^{*}_{\substack{\chi \bmod d \\ \chi^2=1,\, d\leq T}} L\left(\tfrac{1}{2},\chi\right)^2\right)^{1/2} \left(\#\left\{\chi_d \text{ primitive }, d \leq T, L\left(\tfrac{1}{2},\chi\right) \neq 0\right\}\right)^{1/2}$$

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$$\gg \frac{T}{\log T} \sim T^{1-\varepsilon}.$$

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Soundararajan (2000) developed a finer method to prove that at least 7/8 of the $L\left(\frac12,\chi\right)$ do not vanish.

Moments with cubic characters - $\mathbb{Q}$ (non-Kummer case)

Let $p \equiv 1 \pmod 3$.

$$\chi_p(a) \text{ is defined using } a^{\frac{p-1}{3}} \pmod p.$$

We get two characters, χ_p and $\overline{\chi_p}$.

$$\#\{\chi \pmod d : \chi^3 = 1, d \leq T\} \sim CT$$

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Baier & Young (2010) proved,

$$\sum_{\substack{\chi \pmod d \\ \chi^3=1,\, d\leq T}}^{*} L\left(\frac{1}{2}, \chi\right) \sim cT + O\left(T^{\frac{37}{38}+\varepsilon}\right),$$

This implies a nonvanishing result of $\gg T^{\frac{6}{7}-\varepsilon}$.

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Moments with cubic characters - $\mathbb{Q}(\xi_3)$ (Kummer case)

Over $\mathbb{Q}(\xi_3)$, for each prime $\mathfrak{p}$, we have $\chi_{\mathfrak{p}}$ and $\overline{\chi_{\mathfrak{p}}}$, where

$$\chi_{\mathfrak{p}}(a) \text{ is defined using } a^{\frac{N(\mathfrak{p})-1}{3}} \pmod{\mathfrak{p}}.$$

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Luo (2004)

$$\sum^*_{\substack{\chi \pmod d \\ d\in\mathbb{Z}[\xi_3],\, d\equiv 1 \pmod 9 \\ d\,\square-\text{free},\, N(d)\leq T}} L\left(\frac{1}{2}, \chi\right) \sim cT + O\left(T^{\frac{21}{22}+\varepsilon}\right),$$

This considers only a thin subset of the cubic characters, as it does not consider $\overline{\chi_{\mathfrak{p}}}$.

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This implies a nonvanishing result of $\gg T^{1-\varepsilon}$.

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Function fields

Let q power of a prime, $\mathbb{F}_q$ finite field with q elements.

Number Fields		Function Fields
$\mathbb{Q}$	$\leftrightarrow$	$\mathbb{F}_q(X)$
$\mathbb{Z}$	$\leftrightarrow$	$\mathbb{F}_q[X]$
p positive prime	$\leftrightarrow$	$P(X)$ monic irreducible polynomial
$\lvert n\rvert = \lvert\mathbb{Z}/n\mathbb{Z}\rvert = n \in \mathbb{N}$	$\leftrightarrow$	$\lvert F(X)\rvert = \lvert\mathbb{F}_q[X]/(F(X))\rvert = q^{\deg F}$
$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$	$\leftrightarrow$	$\zeta_q(s) = \sum_{\substack{F \in \mathbb{F}_q[X] \\ F \text{ monic}}} \frac{1}{\lvert F\rvert^s}$
Riemann Hypothesis ☹	$\leftrightarrow$	Riemann Hypothesis ☺

Dirichlet characters over function fields

A Dirichlet character is a

$$\chi : (\mathbb{F}_q[X]/(D(X)))^* \to \mathbb{C}^*$$

extended to $\mathbb{F}_q[X]$ by periodicity and with the condition that

$$\chi(A) = 0 \text{ when } (A, D) \neq 1.$$

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The Dirichlet L-function is

$$L(s,\chi) = \sum_{f \text{ monic}} \frac{\chi(f)}{|f|^s} = \sum_{n=0}^{\infty} \frac{1}{q^{ns}} \sum_{\deg f = n} \chi(f).$$

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Since $\sum_{\deg f=n} \chi(f) = 0$ for $n \geq \deg D$, we get a finite sum with finitely many zeros:

$$L(s,\chi) = \sum_{\deg f < \deg D} \frac{\chi(f)}{|f|^s}.$$

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Quadratic Dirichlet L-function over function fields

Assume $q \equiv 1 \bmod 4$. Let $D(X) \in \mathbb{F}_q[X]$, $\square$-free, $\deg D = 2g+1$. Write

$$\left(\frac{D}{f}\right) = \chi_D(f),$$

$$L(s,\chi_D) = \sum_{f \text{ monic}} \frac{\chi_D(f)}{|f|^s} = \prod_{\substack{P \text{ irreducible} \\ \text{monic} \\ P \nmid D}} \left(1 - \frac{\chi_D(P)}{|P|^s}\right)^{-1}$$

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Make the change $u = q^{-s}$.

$$\mathcal{L}(u,\chi_D) = \sum_{f \text{ monic}} \chi_D(f) u^{\deg f} = \sum_{n=0}^{\infty} u^n \sum_{\substack{f \text{ monic} \\ \deg f = n}} \chi_D(f).$$

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$$Z_{C_D}(u) = \frac{\mathcal{L}(u, \chi_D)}{(1-u)(1-qu)}.$$

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The Weil conjectures imply

- $\mathcal{L}(u, \chi_D)$ is a polynomial of degree $2g$.
- Functional equation relating $\mathcal{L}(u, \chi)$ and $\mathcal{L}\left(\frac{1}{qu}, \overline{\chi}\right)$.
- The Riemann Hypothesis holds, the zeros are at $|u| = \frac{1}{\sqrt{q}}$.

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Moments of quadratic L-functions over function fields

- $q \to \infty$ equidistribution results of Katz and Sarnak.
- $g \to \infty$ Andrade and Keating (2014) conjectured

$$\sum_{\substack{\deg D = 2g+1 \\ D \text{ monic}}} \mathcal{L}\left(\frac{1}{\sqrt{q}}, \chi_D\right)^k = q^{2g+1} P_k(2g+1) + o\left(q^{2g+1}\right),$$

where P_k is a polynomial of degree $\frac{k(k+1)}{2}$.

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Li (2018) vanishing $\gg T^{\frac{1}{3}-\varepsilon}$.

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Cubic characters over function fields (Kummer case)

Let $q \equiv 1 \pmod 3$ odd and fix

$$\Omega: \text{ roots of unity in } \mathbb{C}^* \to \text{ roots of 1 in } \mathbb{F}_q^*.$$

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$$f^{\frac{q^{\deg(P)}-1}{3}} \equiv \Omega(\alpha) \pmod P \qquad \chi_P(f) := \alpha$$

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and for $P \mid f$,

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$$\Omega: \text{ roots of unity in } \mathbb{C}^* \to \text{ roots of 1 in } \mathbb{F}_q^*.$$

For a monic irreducible P, and $f \in \mathbb{F}_q[X]$ such that $P \nmid f$, define

$$f^{\frac{q^{\deg(P)}-1}{3}} \equiv \Omega(\alpha) \pmod P \qquad \chi_P(f) := \alpha$$

and for $P \mid f$,

$$\chi_P(f) = 0.$$

If $Q = P_1^{e_1} \cdots P_k^{e_k}$, the Jacobi symbol is

$$\chi_Q(f) = \prod_{i=1}^{k} \chi_{P_i}(f)^{e_i}.$$

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The mean value of cubic Dirichlet L-functions over function fields (Kummer case)

Theorem (David, Florea, L. (2019+))

Let q be an odd prime power such that $q \equiv 1 \pmod 3$. Let χ_3 be a fixed cubic character on $\mathbb{F}_q^*$

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g \\ \chi|_{\mathbb{F}_q^*}=\chi_3}} L(1/2,\chi) = C_1 g q^{g+1} + C_2 q^{g+1} + O\left(q^{g\frac{1+\sqrt{7}}{4}+\varepsilon g}\right),$$

where C_1 and C_2 are certain constants and $g = \deg(\text{Cond}(\chi)) - 1$.

$$\#\{\chi : \text{genus}(\chi) = g\} = B_1 g q^{g+1} + B_2 q^{g+1} + O\left(q^{\left(\frac{1}{2}+\varepsilon\right)g}\right)$$

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$$\frac{1+\sqrt{7}}{4} \approx 0.9114378\cdots > 0.875 = \frac{7}{8}.$$

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The Weil Zeta function (Kummer case)

$q \equiv 1 \pmod 6$. Consider

$$C_{F_1,F_2} : Y^3 = F_1(X)F_2(X)^2,$$

where F_1, F_2 are $\square$-free and relatively prime.

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$$d_1 + d_2 = g + 1$$

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Some features

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g \\ \chi|_{\mathbb{F}_q^*}=\chi_3}} L(1/2,\chi) = \sum_{d_1+d_2=g+1} \sum_{\substack{F_1,F_2\in\mathbb{F}_q[T] \\ (F_1,F_2)=1 \\ \deg(F_i)=d_i, F_i \square-\text{free}}} L(1/2,\chi_{F_1F_2^2})$$

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- The shape of the main term is strange.
- The sieve is complicated.
- Difficulty estimating the error when one d_i is small.
- This implies a nonvanishing result of $\gg q^{g(1-\varepsilon)}$.

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Cubic characters over function fields (non-Kummer case)

Let $q \equiv 2 \pmod 3$ odd and fix

$$\Omega : \text{ roots of unity in } \mathbb{C}^* \to \text{ roots of 1 in } \mathbb{F}_{q^2}^*.$$

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Write $P = \pi\tilde{\pi}$ in $\mathbb{F}_{q^2}[T]$ and get $\chi_\pi, \chi_{\tilde{\pi}}$ characters in $\mathbb{F}_{q^2}[T]$. Then

$$\chi_P = \chi_\pi|_{\mathbb{F}_q[T]}.$$

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Extend as before...

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Our results imply nonvanishing of $\gg q^{g(1-\varepsilon)}$.

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$$\mathcal{L}^*(u, \chi_{F_1F_2^2}) = \frac{\mathcal{L}(u, \chi_{F_1F_2^2})}{1-u}$$

$$d_1 + d_2 = g + 2$$

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A better way to count for the non-Kummer case

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g}} L(1/2,\chi) = \sum_{d_1+d_2=g+2} \sum_{\substack{F_1,F_2\in\mathbb{F}_q[T] \\ (F_1,F_2)=1 \\ \deg(F_i)=d_i, F_i\square-\text{free} \\ P|F_i \Rightarrow \deg(P)\text{even}}} L(1/2,\chi_{F_1F_2^2})$$

$$= \sum_{\substack{F\in\mathbb{F}_{q^2}[T],\square-\text{free} \\ P|F\Rightarrow P\not\in\mathbb{F}_q[T] \\ \deg(F)=(g+2)/2}} L(1/2,\chi_F)$$

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Ideas in the proof - Approximate functional equation

$$\mathcal{L}^*(u,\chi) = \omega(\chi)(\sqrt{q}u)^g \mathcal{L}^*\left(\frac{1}{qu}, \overline{\chi}\right),$$

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$$+ \frac{1}{1-\sqrt{q}} \sum_{f \in \mathcal{M}_{q,A+1}} \frac{\chi(f)}{q^{\deg(f)/2}} + \frac{\omega(\chi)}{1-\sqrt{q}} \sum_{f \in \mathcal{M}_{q,g-A}} \frac{\overline{\chi(f)}}{q^{\deg(f)/2}}.$$

Ideas in the proof - Main term

If $f = \cube$,

$$\chi_F(f) = \begin{cases} 1 & (F, f) = 1 \\ 0 & (F, f) \neq 1 \end{cases}$$

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$$\sum_{f=\square} = \sum_{\deg(f)\leq A/3} \frac{a(F)}{|f|^{3/2}},$$

$$a(F) = \#\{\chi\,:\,\chi^3=1,\ \text{primitive},\ \mathrm{Cond}(\chi)=F\}$$

with $(F,f)=1$.

Ideas in the proof - Main term

$$a(F) = \#\{\chi : \chi^3 = 1, \text{ primitive}, \operatorname{Cond}(\chi) = F\}$$

Let

$$\mathcal{G}(u) = \sum_F a(F) u^{\deg(F)}$$

By Perron's formula,

$$\sum_{\deg(F)=d} a(F) = \frac{1}{2\pi i} \oint_{|u|=q^{-2}} \frac{\mathcal{G}(u)}{u^d} \frac{du}{u}$$

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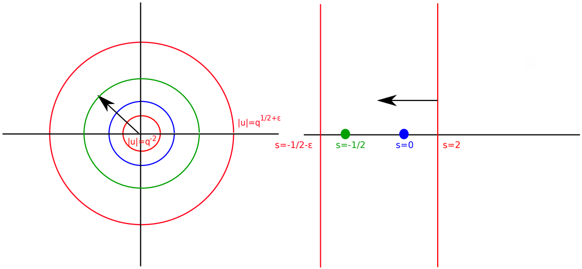

non-Kummer:

$$\mathcal{G}(u) = \prod_{\substack{2|\deg(P) \\ P\nmid f}} \left(1 + 2u^{\deg(P)}\right)$$

$$\sum_{f=\square} = Mq^{g} + Nq^{g-\frac{A}{6}} + O\left(q^{g-\frac{A}{2}}\right)$$

(A sieve for Kummer)

Ideas in the proof - The sum over the non-cubes

$$\sum_{f \neq \text{cube}} \ll q^{\frac{A+g}{2}+\varepsilon g}$$ uses the Lindelöf bound.

For $\mathrm{Re}(s) \geq 1/2$ and all $\varepsilon > 0$,

$$|L(s,\chi_F)| \ll |F|^{\varepsilon}$$

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Ideas in the poof - The dual sum

$$\mathcal{L}^*(u,\chi) = \omega(\chi)(\sqrt{q}u)^g \mathcal{L}^*\left(\frac{1}{qu}, \overline{\chi}\right),$$

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$$\omega(\chi) = q^{-g/2-1} G(\chi)$$

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Cubic Gauss sum!

Ideas in the proof - Gauss sums

If $a \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$, define,

$$e(\alpha) = e^{\frac{2\pi i \operatorname{tr}(a_1)}{p}}, \text{ if } \alpha = \cdots + \frac{a_1}{t} + \cdots$$

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"almost" multiplicative.

- If $(F_1, F_2) = 1$,

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- $$P \nmid V \Rightarrow |G(V, P)| = \sqrt{|P|}.$$

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Ideas in the proof - the generating series for the Gauss sums

Hoffstein (1992) and Patterson (2007) studied

$$\Psi_f(u) = \sum_F G(f, F) u^{\deg F}$$

(from the context of metaplectic Eisenstein series).

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- There is a functional equation

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- Only poles with $u^3 = \frac{1}{q^4}$.

$$\rho(f) = \mathrm{Res}_{u^3=\frac{1}{q^4}} \Psi_f(u)$$

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Ideas in the proof - the generating series for the Gauss sums

Hoffstein (1992) and Patterson (2007) studied

$$\Psi_f(u) = \sum_F G(f,F) u^{\deg F}$$

(from the context of metaplectic Eisenstein series).

- There is a functional equation

$$\Psi_f(u) \longleftrightarrow \Psi_f\left(\frac{1}{q^2 u}\right) \qquad (s \leftrightarrow 2-s)$$

- Only poles with $u^3 = \frac{1}{q^4}$.

$$\rho(f) = \mathrm{Res}_{u^3=\frac{1}{q^4}} \Psi_f(u)$$

- Identities allow us to compute $\rho(f)$ by factorizing f.

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Ideas in the proof - the sum of the Gauss sums

Use Perron formula

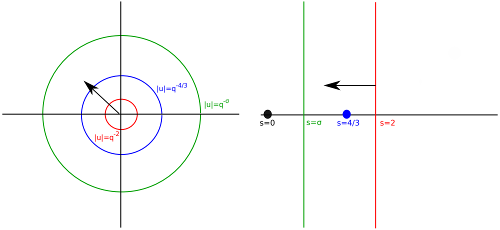
$$\sum_{\substack{F \text{ monic} \\ \deg F = d}} G(f,F) = \frac{1}{2\pi i} \oint_{|u|=q^{-2}} \frac{\Psi_f(u)}{u^d} \frac{du}{u}$$

Ideas in the proof - the sum of the Gauss sums

Use Perron formula

$$\sum_{\substack{F \text{ monic} \\ \deg F = d}} G(f,F) = \frac{1}{2\pi i} \oint_{|u|=q^{-2}} \frac{\Psi_f(u)}{u^d} \frac{du}{u}$$

Using Phragmén–Lindelöf to estimate the error term.



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Ideas in the proof - the sum of the Gauss sums

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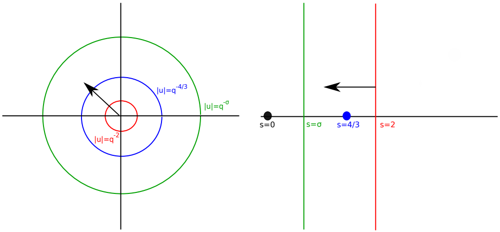


$$= \rho(f) q^{\frac{4d}{3}} + O\left(q^{d\sigma} |f|^{\left(\frac{3}{2}-\sigma\right)\frac{1}{2}}\right).$$

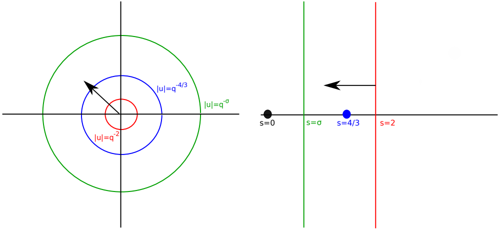
Ideas in the proof - the sum of the Gauss sums

Use Perron formula

$$\sum_{\substack{F \text{ monic} \\ \deg F = d \\ (F,f)=1}} G(f,F) = \frac{1}{2\pi i} \oint_{|u|=q^{-2}} \frac{\tilde{\Psi}_f(u)}{u^d} \frac{du}{u}$$

Extra poles at $u^3 = \frac{1}{q^2}$

Using Phragmén–Lindelöf to estimate the error term.



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Ideas in the proof - Combining with the dual sum

$$\sum_{f=\square} = Mq^{g} + Nq^{g-\frac{A}{6}} + O\left(q^{g-\frac{A}{2}}\right), \qquad \sum_{f\neq\square} \ll q^{\frac{A+g}{2}+\varepsilon g}$$

Ideas in the proof - Combining with the dual sum

$$\sum_{f=\square} = Mq^g + Nq^{g-\frac{A}{6}} + O\left(q^{g-\frac{A}{2}}\right), \qquad \sum_{f\neq\square} \ll q^{\frac{A+g}{2}+\varepsilon g}$$

- For non-Kummer:

$$S_{\mathrm{dual}} = -Nq^{g-\frac{A}{6}} + O_g\left(q^{g-\frac{A}{2}} + q^{\frac{3g}{2}-(2-\sigma)A} + q^{\frac{5g}{6}}\right)$$

Take $\sigma = 7/6$, $A = 3g/4$, then $O\left(q^{\frac{7g}{8}+\varepsilon g}\right)$.

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Ideas in the proof - Combining with the dual sum

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Take $\sigma = 7/6$, $A = 3g/4$, then $O\left(q^{\frac{7g}{8}+\varepsilon g}\right)$.

- For Kummer:

$$S_{\mathrm{dual}} = -Nq^{g-\frac{A}{6}} + O_g\left(q^{g-\frac{A}{6}} + q^{\left(\frac{23}{12}-\frac{\sigma}{2}+\varepsilon\right)g-\left(\frac{13}{12}-\frac{\sigma}{2}\right)A} + q^{\frac{3g}{2}-(2-\sigma)A} + q^{\frac{5g}{6}}\right)$$

Take $\sigma = \frac{13-2\sqrt{7}}{6}$, $A = \frac{\sqrt{7}-1}{2}g$, then $O\left(q^{g\frac{1+\sqrt{7}}{4}+\varepsilon g}\right)$.

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The results

Theorem (David, Florea, L. (2019+))

Let q be an odd prime power such that $q \equiv 1 \pmod 3$. Let χ_3 be a fixed cubic character on $\mathbb{F}_q^*$

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g \\ \chi|_{\mathbb{F}_q^*}=\chi_3}} L(1/2,\chi) = C_1 g q^{g+1} + C_2 q^{g+1} + O\left(q^{g\frac{1+\sqrt{7}}{4}+\varepsilon g}\right).$$

Theorem (David, Florea, L. (2019+))

Let q be an odd prime power such that $q \equiv 2 \pmod 3$. Then

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g}} L(1/2,\chi) = A q^{g+2} + O\left(q^{\frac{7g}{8}+\varepsilon g}\right).$$

The conjectures

Conjecture

Let q be an odd prime power such that $q \equiv 1 \pmod 3$. Let χ_3 be a fixed cubic character on $\mathbb{F}_q^*$

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g \\ \chi|_{\mathbb{F}_q^*}=\chi_3}} L(1/2,\chi) = C_1 g q^{g+1} + C_2 q^{g+1} + D_1 g q^{\frac{5g}{6}} + D_2 q^{\frac{5q}{6}} + o\left(q^{\frac{5g}{6}}\right).$$

Conjecture

Let q be an odd prime power such that $q \equiv 2 \pmod 3$. Then

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g}} L(1/2,\chi) = A q^{g+2} + B q^{\frac{5g}{6}} + o\left(q^{\frac{5g}{6}}\right).$$

Thanks for your attention!!!