Solutions of the Markov equation over polynomial rings

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Markov equation

Introduction

- In his work on Diophantine approximation, Andrei Markov studied the equation

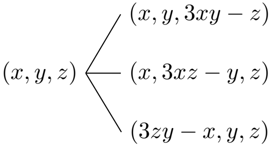
$$x^2 + y^2 + z^2 = 3xyz.$$

- The structure of the integral solutions of the *Markov equation* is very interesting.

Markov equation

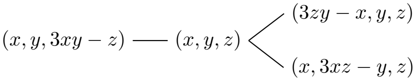
Automorphisms

$$x^2 + y^2 + z^2 = 3xyz$$

- $(x_1, x_2, x_3) \longrightarrow (x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)})$
- $(x, y, z) \longrightarrow (-x, -y, z)$
- A *Markov triple* (x, y, z) is a solution of the Markov equation that satisfies $0 < x \leq y \leq z$.
- 
  (x, y, z)
  $(x, y, 3xy - z)$
  $(x, 3xz - y, z)$
  $(3zy - x, y, z)$

Markov Tree

$$x^2 + y^2 + z^2 = 3xyz$$



$(x, y, 3xy - z)$

(x, y, z)

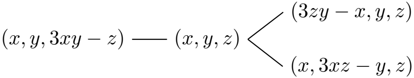
$(3zy - x, y, z)$

$(x, 3xz - y, z)$

(1,1,1)

Markov Tree

$$x^2 + y^2 + z^2 = 3xyz$$

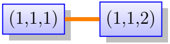


$(x, y, 3xy - z)$

(x, y, z)

$(3zy - x, y, z)$

$(x, 3xz - y, z)$

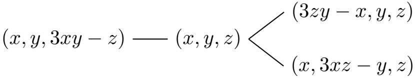


(1,1,1)

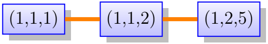
(1,1,2)

Markov Tree

$$x^2 + y^2 + z^2 = 3xyz$$



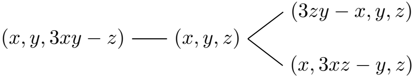
$(x, y, 3xy - z)$ — (x, y, z) — $(3zy - x, y, z)$, $(x, 3xz - y, z)$



(1,1,1) — (1,1,2) — (1,2,5)

Markov Tree

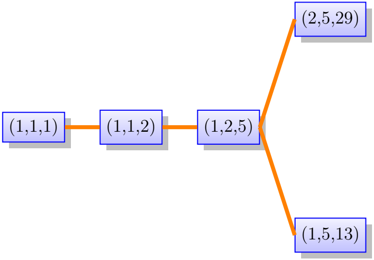
$$x^2 + y^2 + z^2 = 3xyz$$



$(x, y, 3xy - z)$ — (x, y, z)

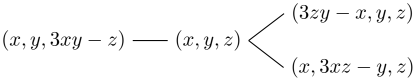
$(3zy - x, y, z)$

$(x, 3xz - y, z)$



Markov Tree

$$x^2 + y^2 + z^2 = 3xyz$$

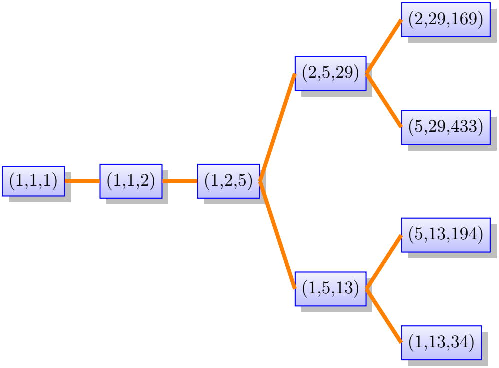


$(x, y, 3xy - z)$

(x, y, z)

$(3zy - x, y, z)$

$(x, 3xz - y, z)$



(1,1,1)

(1,1,2)

(1,2,5)

(2,5,29)

(1,5,13)

(2,29,169)

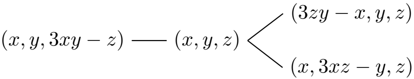
(5,29,433)

(5,13,194)

(1,13,34)

Markov Tree

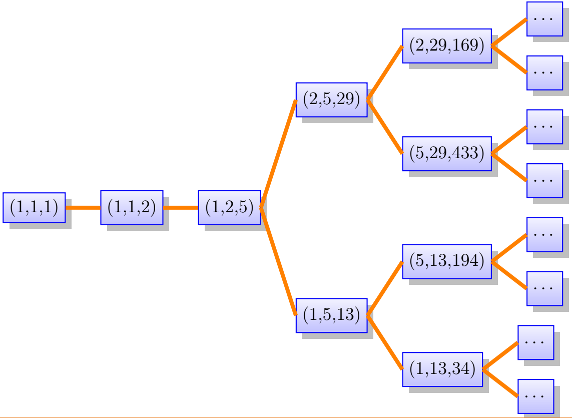
$$x^2 + y^2 + z^2 = 3xyz$$



$(3zy - x, y, z)$

$(x, y, 3xy - z)$ — (x, y, z)

$(x, 3xz - y, z)$



(1,1,1)

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(2,29,169)

(5,29,433)

(5,13,194)

(1,13,34)

. . .

Markov numbers

Definition and properties

Theorem (Markov)

All Markov triples lie on the Markov tree.

- The entries of all Markov triples form the sequence of *Markov numbers*

$$1, 2, 5, 13, 29, 34, 89, 169, \ldots \quad \text{(A002559 in the OEIS)}$$

- Every Markov number m_n is the maximum of some Markov triple (x, y, m_n).

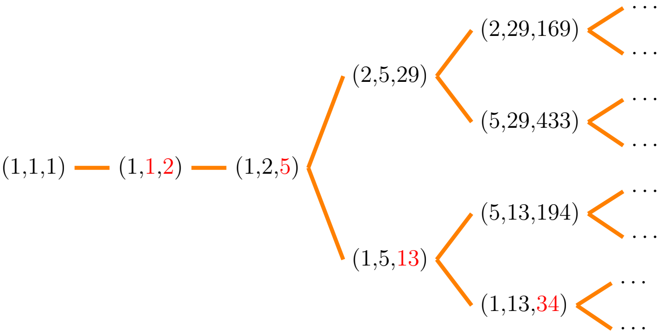
Conjecture (Frobenius)

Given a Markov number m*, there is exactly one Markov triple of the form* (x, y, m)*.*

Markov numbers
Subsequence

All odd-indexed Fibonacci number is a Markov number:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233... (A000045)


(1,1,1)
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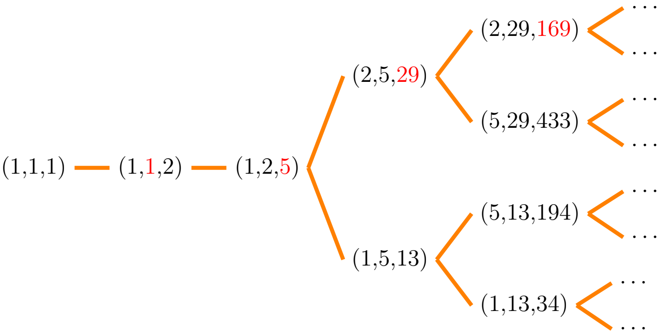
. . .
. . .
(1,5,13)
(5,13,194)
. . .
. . .
(1,13,34)
. . .
. . .

Markov numbers

Subsequence

All odd-indexed Pell number is a Markov number:

0, 1, 2, 5, 12, 29, 70, 169, 408, 985, ... (A000129)



(1,1,1)

(1,1,2)

(1,2,5)

(2,5,29)

(2,29,169) ...

(5,29,433) ...

(1,5,13)

(5,13,194) ...

(1,13,34) ...

Related work

- The equation $x^2 + y^2 + z^2 = axyz$ has positive integral solutions if and only if $a = 1, 3$.
- The positive integral solutions of

  $$x_1^2 + \cdots + x_n^2 = ax_1 \cdots x_n$$

  are also "finitely generated".
- Over orders in a number field, the set of solutions

  $$x_1^2 + \cdots + x_n^2 = ax_1 \cdots x_n$$

  may or may not be "finitely generated". See work by Silverman and Baragar.
- Several authors have investigated the solutions of Markov equation and its variations over finite fields.

Markov equation over polynomial rings

Rational numbers vs Function Fields

Close your eyes and think of the few slides from Matilde Lalín's and Allysa Lumley's talks.

- Goal: Find all "positive" solutions to the Markov equation equation over the polynomial ring $K[t]$, where K is a field.
- Every non-zero element of $K[t]$ can be written as

$$\alpha f(t)$$

where $f(t)$ is monic and $\alpha \in K^*$.
(Every element $n \in \mathbb{Z}^*$ satisfies $n = (\pm 1)\tilde{n}$, with $\tilde{n} > 0$.)

Markov equation over polynomials
Monic solutions

Let K be a field of characteristic $\neq 2$.

- For A monic,
  $$x^2 + y^2 + z^2 = Axyz$$
  has a monic solution if and only if $A = 1$.
- $$x^2 + y^2 + z^2 = xyz$$
  has a monic solution if and only if $i = \sqrt{-1} \in K$.
- If $\beta = 2i$ then
  $$(t, t + \beta, t^2 + \beta t - 2)$$
  is a monic solution.

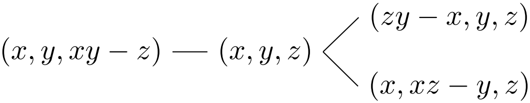
Markov equation

New solutions from old solutions

$$x^2 + y^2 + z^2 = xyz$$

- $(x_1, x_2, x_3) \longrightarrow (x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)})$
- $$\begin{tikzcd} & (zy - x, y, z) \\ (x, y, z) \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & (x, xz - y, z) \\ & (x, y, xy - z) \end{tikzcd}$$
- $(x, y, z) \longrightarrow (x(f), y(f), z(f))$, where f is any monic polynomial.

Markov tree of monic solutions



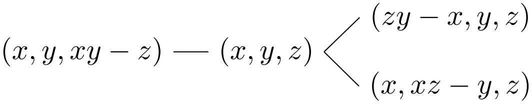
$$(x, y, xy - z) \;—\; (x, y, z)$$

$$(zy - x, y, z)$$

$$(x, xz - y, z)$$

$\boxed{(t, t + \beta, t^2 + \beta t - 2)}$

Markov tree of monic solutions

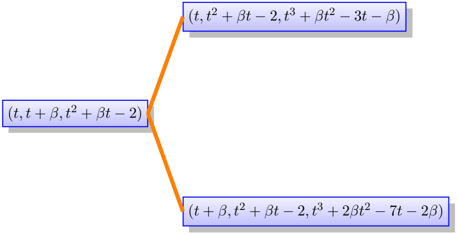


$(x, y, xy - z)$

(x, y, z)

$(zy - x, y, z)$

$(x, xz - y, z)$

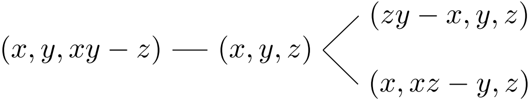


$(t, t + \beta, t^2 + \beta t - 2)$

$(t, t^2 + \beta t - 2, t^3 + \beta t^2 - 3t - \beta)$

$(t + \beta, t^2 + \beta t - 2, t^3 + 2\beta t^2 - 7t - 2\beta)$

Markov tree of monic solutions

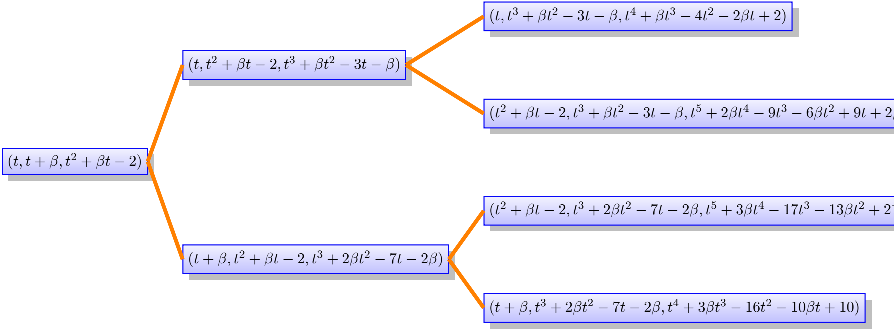


$$(x, y, xy - z)$$

$$(x, y, z)$$

$$(zy - x, y, z)$$

$$(x, xz - y, z)$$



$$(t, t + \beta, t^2 + \beta t - 2)$$

$$(t, t^2 + \beta t - 2, t^3 + \beta t^2 - 3t - \beta)$$

$$(t, t^3 + \beta t^2 - 3t - \beta, t^4 + \beta t^3 - 4t^2 - 2\beta t + 2)$$

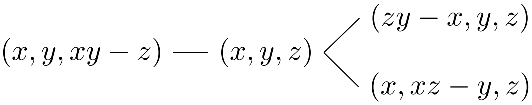
$$(t^2 + \beta t - 2, t^3 + \beta t^2 - 3t - \beta, t^5 + 2\beta t^4 - 9t^3 - 6\beta t^2 + 9t + 2$$

$$(t + \beta, t^2 + \beta t - 2, t^3 + 2\beta t^2 - 7t - 2\beta)$$

$$(t^2 + \beta t - 2, t^3 + 2\beta t^2 - 7t - 2\beta, t^5 + 3\beta t^4 - 17t^3 - 13\beta t^2 + 2$$

$$(t + \beta, t^3 + 2\beta t^2 - 7t - 2\beta, t^4 + 3\beta t^3 - 16t^2 - 10\beta t + 10)$$

Markov tree of monic solutions

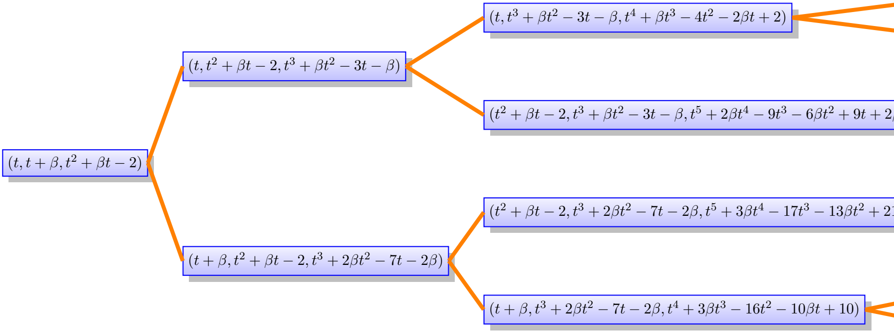


$(x, y, xy - z)$

(x, y, z)

$(zy - x, y, z)$

$(x, xz - y, z)$



$(t, t + \beta, t^2 + \beta t - 2)$

$(t, t^2 + \beta t - 2, t^3 + \beta t^2 - 3t - \beta)$

$(t, t^3 + \beta t^2 - 3t - \beta, t^4 + \beta t^3 - 4t^2 - 2\beta t + 2)$

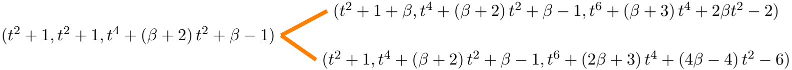
$(t^2 + \beta t - 2, t^3 + \beta t^2 - 3t - \beta, t^5 + 2\beta t^4 - 9t^3 - 6\beta t^2 + 9t + 2$

$(t + \beta, t^2 + \beta t - 2, t^3 + 2\beta t^2 - 7t - 2\beta)$

$(t^2 + \beta t - 2, t^3 + 2\beta t^2 - 7t - 2\beta, t^5 + 3\beta t^4 - 17t^3 - 13\beta t^2 + 2$

$(t + \beta, t^3 + 2\beta t^2 - 7t - 2\beta, t^4 + 3\beta t^3 - 16t^2 - 10\beta t + 10)$

$$(x, y, z) \longrightarrow (x(t^2+1), y(t^2+1), z(t^2+1))$$



$$(t^2+1, t^2+1, t^4+(\beta+2)\,t^2+\beta-1)$$

$$(t^2+1+\beta, t^4+(\beta+2)\,t^2+\beta-1, t^6+(\beta+3)\,t^4+2\beta t^2-2)$$

$$(t^2+1, t^4+(\beta+2)\,t^2+\beta-1, t^6+(2\beta+3)\,t^4+(4\beta-4)\,t^2-6)$$

Monic solutions of the Markov equation

Theorem (C., Kelly, VanFossen)

Up to permutations, all monic solutions of

$$x^2 + y^2 + z^2 = xyz$$

over K lie on a tree with root

$$(f, f + \beta, f^2 + \beta f - 2),$$

where f is a monic polynomial and $\beta = 2i$.

Remark

Up to permutation and composition, the monic polynomial solutions of the Markov equation "branch out" of the solution

$$(t, t + \beta, t^2 + \beta t - 2)$$

Monic solutions of the Markov equation

Theorem (C., Kelly, VanFossen)

Up to permutations, all monic polynomial solutions of

$$x^2 + y^2 + z^2 = xyz$$

lie on a tree with root

$$(f, f+\beta, f^2 + \beta f - 2),$$

where f *is a monic polynomial and* $\beta = 2i$*.*

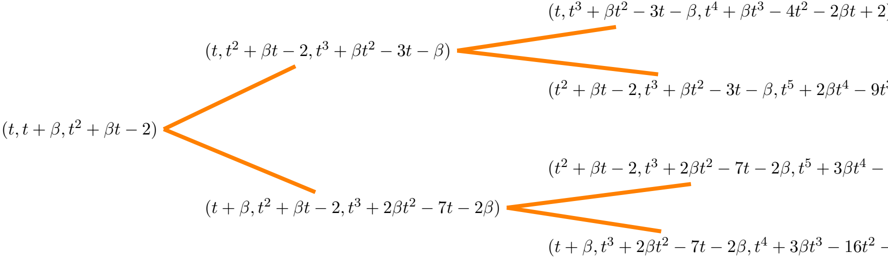
“The sketchiest of all proofs”

$$\begin{tikzcd}
 & & (zy - x, y, z) \\
(x, y, xy - z) \arrow[r, dash] & (x, y, z) \arrow[ur, dash] \arrow[dr, dash] & \\
 & & (x, xz - y, z)
\end{tikzcd}$$

Markov Polynomials

Definition

A *Markov polynomial* is a monic polynomial on the "main Markov tree". They are polynomials defined over $\mathbb{Z}[i]$.



$(t, t+\beta, t^2+\beta t-2)$

$(t, t^2+\beta t-2, t^3+\beta t^2-3t-\beta)$

$(t, t^3+\beta t^2-3t-\beta, t^4+\beta t^3-4t^2-2\beta t+2$

$(t^2+\beta t-2, t^3+\beta t^2-3t-\beta, t^5+2\beta t^4-9t$

$(t+\beta, t^2+\beta t-2, t^3+2\beta t^2-7t-2\beta)$

$(t^2+\beta t-2, t^3+2\beta t^2-7t-2\beta, t^5+3\beta t^4-$

$(t+\beta, t^3+2\beta t^2-7t-2\beta, t^4+3\beta t^3-16t^2-$

Markov Polynomials

$$\begin{aligned}
M_0 &= t \\
M_1 &= t + 2i \\
M_2 &= t^2 + 2it - 2 \\
M_3 &= t^3 + 2it^2 - 3t - 2i \\
M_4 &= t^3 + 4it^2 - 7t - 4i \\
M_5 &= t^4 + 2it^3 - 4t^2 - 4it + 2 \\
M_6 &= t^4 + 6it^3 - 16t^2 - 20it + 10 \\
M_7 &= t^5 + 2it^4 - 5t^3 - 6it^2 + 5t + 2i \\
M_8 &= t^5 + 4it^4 - 9t^3 - 12it^2 + 9t + 4i \\
M_9 &= t^5 + 6it^4 - 17t^3 - 26it^2 + 21t + 6i \\
M_{10} &= t^5 + 8it^4 - 29t^3 - 56it^2 + 57t + 24i
\end{aligned}$$

Unanswered questions – Episode I

Unanswered questions

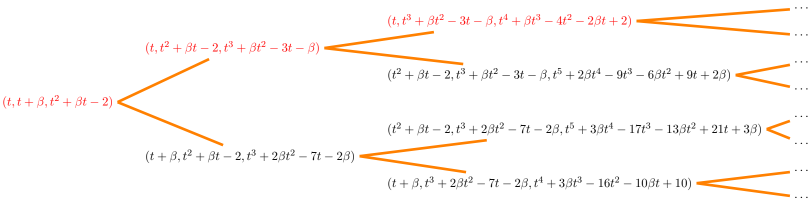
- Does the uniqueness conjecture hold for Markov polynomials?
  That is, does a Markov polynomial z determine the triple (x, y, z)? We can prove that if z or $z \pm 2$ are a power of an irreducible polynomial then (x, y, z) is unique.

- How many irreducible or power of irreducible Markov polynomials are there?
  Except for the first two, the first 500 Markov polynomials are reducible over $\mathbb{Z}[i]$.

Markov-Fibonacci polynomials

Markov polynomial vs Fibonacci numbers

Do (t, f_{n-1}, f_n) behave like "Fibonacci numbers"?



$(t, t+\beta, t^2+\beta t-2)$

$(t, t^2+\beta t-2, t^3+\beta t^2-3t-\beta)$

$(t+\beta, t^2+\beta t-2, t^3+2\beta t^2-7t-2\beta)$

$(t, t^3+\beta t^2-3t-\beta, t^4+\beta t^3-4t^2-2\beta t+2)$

$(t^2+\beta t-2, t^3+\beta t^2-3t-\beta, t^5+2\beta t^4-9t^3-6\beta t^2+9t+2\beta)$

$(t^2+\beta t-2, t^3+2\beta t^2-7t-2\beta, t^5+3\beta t^4-17t^3-13\beta t^2+21t+3\beta)$

$(t+\beta, t^3+2\beta t^2-7t-2\beta, t^4+3\beta t^3-16t^2-10\beta t+10)$

$\cdots$

Definition

They are called the Markov-Fibonacci polynomials.

Markov-Fibonacci polynomials

- The polynomials f_n satisfy the recurrence relation

  $$f_n = tf_{n-1} - f_{n-2}$$

  with $f_0 = 2$ and $f_1 = t + 2i$.

- They satisfy

  $$f_n(i) \cdot i^{-n} = F_{n+2}$$
  $$f_n(2i) \cdot i^{-n} = 2P_{n+1}$$

Markov-Fibonacci polynomials and Chebyshev polynomials

We have

$$f_{n+1}(t) = (-t + \beta)U_n(t/2) + 2U_{n+1}(t/2),$$

where $U_n(t)$ be the n-th Chebyshev polynomial of the second kind, that is,

$$U_{n-1}(\cos\theta) \cdot \sin\theta = \sin n\theta.$$

Divisibility properties

- Recall: The Fibonacci numbers satisfy:

$$F_n \mid F_m \text{ if and only if } m \equiv 0 \mod n.$$

- Do Markov-Fibonacci polynomials satisfy any divisibility property?
  Not for general K.
  If K has characteristic zero then $f_n \mid f_m$ if and only if $m = n$.

Divisibility properties over Finite fields

Theorem

Suppose K is a finite field of odd characteristic containing $i = \sqrt{-1}$.

For every n, there exists a positive integer r such that $f_n \mid f_m$ if and only if $m \equiv n \mod r$.

Example

Over $\mathbb{F}_{13}$, $f_1 = t + 10$ divides

$$\dots, f_{-13}, f_{-6}, f_1, f_8, f_{15}, f_{22}, \cdots$$

Example

Over $\mathbb{F}_{13}$, $f_2 = (t+4)(t+6)$ divides

$$\dots, f_{-82}, f_{-40}, f_2, f_{44}, f_{86}, f_{128}, \cdots$$

Divisibility properties over Finite fields

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Example

Over $\mathbb{F}_{13}$, $f_2 = (t+4)(t+6)$ divides

$$\dots, f_{-82}, f_{-40}, f_2, f_{44}, f_{86}, f_{128}, \dots$$

Divisibility properties

Theorem (C., Kelly, VanFossen)

Let $\alpha \in \overline{\mathbb{F}}_q$. *If* α *is a root of* f_n, *for some* $n \in \mathbb{N}$, *then there exists an integer* $r = r(\alpha, q) > 1$ *such that*

$$f_{n+kr}(\alpha) = 0$$

for all integers k.

Example

Over $\mathbb{F}_5$, $t = 1$ is a root of

$$\ldots, f_{-5}, f_{-2}, f_1, f_4, f_7, f_{10}, \cdots$$

Sketch of a proof

Theorem (C., Kelly, VanFossen)

Let $\alpha \in \overline{\mathbb{F}}_q$*. If* α *is a root of* f_n*, for some* $n \in \mathbb{N}$*, then there exists an integer* $r = r(\alpha, q) > 1$ *such that*

$$f_{n+kr}(\alpha) = 0$$

for all integers $k \geq 0$*.*

Sketch of a proof

Assume $\alpha \neq 0, \pm 2$.

- We have the generating function

$$\sum_{n=0}^{\infty} f_n(\alpha)x^n = \frac{(-\alpha+\beta)x+2}{x^2-\alpha x+1} = \frac{(-\alpha+\beta)x+2}{(x-\omega_\alpha)(x-\overline{\omega}_\alpha)}$$

  where

$$\omega_\alpha = \frac{\alpha+\sqrt{\alpha^2-4}}{2} \quad \text{and} \quad \overline{\omega}_\alpha = \frac{\alpha-\sqrt{\alpha^2-4}}{2}.$$

- If α is a root of f_n then

$$0 = (-\alpha+\beta)(\omega_\alpha^n - \overline{\omega}_\alpha^n) + 2(\omega_\alpha^{n+1} - \overline{\omega}_\alpha^{n+1})$$

Sketch of a proof

- This is equivalent to an equation $W_\alpha^n = V_\alpha$, where

$$\left(\underbrace{\left(\frac{\alpha + \sqrt{\alpha^2 - 4}}{2}\right)^2}_{W_\alpha}\right)^n = \underbrace{\frac{\alpha^2 - 8 - 4i\sqrt{\alpha^2 - 4}}{-\alpha^2}}_{V_\alpha}.$$

- Thus if $r = \operatorname{ord}(W_\alpha) < \infty$ and $m \equiv n \mod r$ then

$$f_m(\alpha) = 0.$$

Unanswered questions – Episode II

Unanswered questions

- Can we find an asymptotic formula for the number N_q of $\alpha \in \mathbb{F}_q$ that are a root of f_n, for some n?
  Over $\mathbb{F}_p$, computations show that

  $$N_p \sim cp, \quad p \longrightarrow \infty$$

  where $0.5 < c < 0.8$.

- The group generated by a unit $a + b\sqrt{\alpha^2 - 4}$ in $K(\sqrt{\alpha^2 - 4})$ can be embedded in the group of K-rational points of the Pell curve

  $$x^2 - (\alpha^2 - 4)y^2 = 1,$$

  via the map

  $$a + b\sqrt{\alpha^2 - 4} \longmapsto (a, b).$$

Unanswered questions

- Recall: If $f_n(\alpha) = 0$ then $W_\alpha = \frac{\alpha + \sqrt{\alpha^2 - 4}}{2}$ and $V_\alpha = \frac{\alpha^2 - 8 - 4i\sqrt{\alpha^2 - 4}}{-\alpha^2}$ are units such that $W_\alpha^n = V_\alpha$.
- For what values of $\alpha \in \mathbb{F}_q$, does $W_\alpha^n = V_\alpha$, for some n?
- When do two sections of the surface

  $$x^2 - (t^2 - 4)y^2 = 1$$

  have the same specialization?

Thank you!

Unanswered questions – Episode III?