a-Numbers of Curves in Artin-Schreier Covers

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The Riemann-Hurwitz Formula

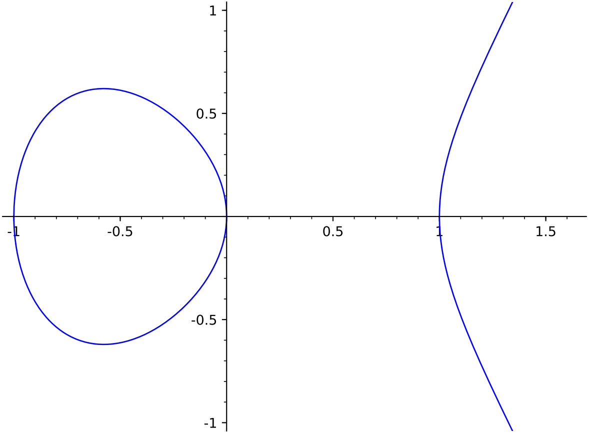
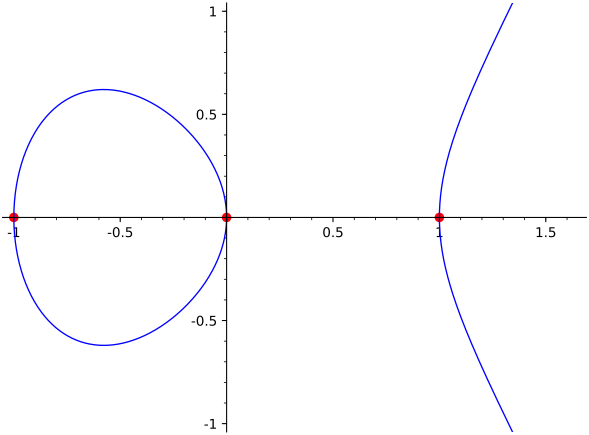


Figure: The elliptic curve $y^2 = x^3 - x$

The Riemann-Hurwitz Formula



Degree 2 cover of $\mathbf{P}^1$, Ramified above 0, 1, -1, ∞. Genus 1

Riemann-Hurwitz in Characteristic p

Consider the curve X defined by $y^5 - y = x^3$ defined over $k = \mathbf{F}_5$.

- Artin-Schreier extension of function fields:

$$k(X) = k(x)[y]/(y^5 - y - x^3)$$

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- $d = 3$ is the unique break in the lower ramification filtration above infinity. Also the order of the pole of x^3 at infinity.
- Riemann-Hurwitz says that

$$2g_X - 2 = 5(2g_{\mathbf{P}^1_k} - 2) + (d+1)(5-1)$$

i.e. that $g_X = 4$.

Invariants in Characteristic p

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  known as the *Cartier operator*.
- Easy to calculate with the Cartier operator:

$$V_X \left(\sum_i a_i t^i \frac{dt}{t} \right) = \sum_j a_{pj}^{1/p} t^j \frac{dt}{t}.$$

Other Invariants in Characteristic p

Based on the Cartier operator V_X, decompose

$$H^0(X, \Omega^1_X) = H^0(X, \Omega^1_X)^{\mathrm{bij}} \oplus H^0(X, \Omega^1_X)^{\mathrm{nilp}}$$

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Definition

The p-rank of X, denoted f_X, is $\dim_k H^0(X, \Omega^1_X)^{\text{bij}}$.

Definition

The a-number of X, denoted a_X, is $\dim_k \ker V_X$.

Simple Examples

Example

An ordinary elliptic curve has p-rank 1 and a-number 0.

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$y^5 - y = x^3$ over $\mathbf{F}_5$: genus 4, p-rank 0, a-number 4.

Analogs of Riemann-Hurwitz

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Two $\mathbf{Z}/5\mathbf{Z}$-covers of $\mathbf{P}^1$ with the same ramification information:

Example

$y^5 - y = x^3$ over $\mathbf{F}_5$: genus 4, p-rank 0, a-number 4.

Example

$y^5 - y = x^3 + x^2$ over $\mathbf{F}_5$: genus 4, p-rank 0, a-number 3.

Invariants of the Jacobian

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- $f_X = \dim_{\mathbf{F}_p} \mathrm{Hom}_{\overline{k}}(\mu_p, \mathrm{Jac}(X)[p])$
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The connection between $\mathrm{Jac}(X)[p]$ and these invariants comes from relating the Dieudonné module and de Rham cohomology.

Invariants of the Jacobian

If E is an elliptic curve, $E \simeq \mathrm{Jac}(E)$.

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If E is an ordinary, p-rank is one and a-number is zero, while

$$E[p] = \mu_p \times \mathbf{Z}/p\mathbf{Z}.$$

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Example

If E is a supersingular, p-rank is 0 and a-number is one, while

$$0 \to \alpha_p \to E[p] \to \alpha_p \to 0.$$

The Igusa Tower

Let X_n be nth Igusa curve in characteristic p: moduli space of elliptic curves with level p^n Igusa structure.

They form a $\mathbf{Z}_p$-tower

$$\ldots \to X_3 \to X_2 \to X_1.$$

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Using ramification information worked out by Katz and Mazur:

- $g_{X_n} = c_1 p^{2n} + c_2 p^n + c_3$ (Riemann-Hurwitz)
- $f_{X_n} = c_4(p^n - 1)$ (Deuring-Shafarevich)
- $\frac{1}{2} + O(p^{-1}) \leq \frac{a_{X_n}}{g_{X_n}} \leq \frac{2}{3} + O(p^{-1})$ (our results)

Motivation: $\mathbf{Z}_p$-towers and Iwasawa Theory

Consider a "nice" $\mathbf{Z}_p$-tower of curves

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Question

Is the growth of a_{X_n} regular? (for genus stable towers?)

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Studying invariants of $\mathrm{Jac}(X_n)[p]$ like genus or a-number is a geometric analog of Iwasawa theory.

Bounds for a-Numbers

Theorem (B-Cais)

Let $\pi : Y \to X$ be a $\mathbf{Z}/p\mathbf{Z}$-cover of curves in characteristic p with branch locus $S \subseteq X(\overline{k})$. For $Q \in S$ let d_Q be the unique break in the lower-numbering ramification filtration at the unique point of Y over Q. Then for any $1 \leq j \leq p-1$,

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$$\sum_{Q \in S} \sum_{i=j}^{p-1} \left(\left\lfloor \frac{id_Q}{p} \right\rfloor - \left\lfloor \frac{id_Q}{p} - \left(1 - \frac{1}{p}\right) \frac{jd_Q}{p} \right\rfloor \right) \leq a_Y$$

$$\text{and} \quad a_Y \leq pa_X + \sum_{Q \in S} \sum_{i=1}^{p-1} \left(\left\lfloor \frac{id_Q}{p} \right\rfloor - (p-i) \left\lfloor \frac{id_Q}{p^2} \right\rfloor \right).$$

Estimates on the Bounds

When $\sum_{Q \in S} d_Q = T$ is large, take $j \approx p/2$ and approximate:

$$\text{lower bound} \approx \frac{pT}{4} \quad \text{and} \quad \text{upper bound} \approx \frac{pT}{3}$$

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In contrast,

$$\text{genus} \approx \frac{pT}{2}.$$

Special Cases

$\pi : Y \to X$ a $\mathbf{Z}/p\mathbf{Z}$-cover ramified over S

Corollary

Suppose p is odd. If $a_X = 0$ and $d_Q | p - 1$ for every $Q \in S$, the upper and lower bounds match, giving an explicit formula for a_Y.

Recovers a result of Shawn Farnell and Rachel Pries when $X = \mathbf{P}^1_k$.

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Corollary

Suppose $p = 2$. If $a_X = 0$, the upper and lower bounds match, giving an explicit formula for a_Y.

Recovers a result of Felipe Voloch.

When are the Bounds Sharp?

Example

Consider $y^p - y = x^d$ (cover of $\mathbf{P}^1$ ramified at infinity).
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Theorem (B-Pries)

Let $p = 3$ and X be a curve with $a_X = 0$. There exists a $\mathbf{Z}/p\mathbf{Z}$-cover of X with any specified $S \subset X(\overline{k})$ and d_Q for $Q \in S$ with minimal a-number.

Relies on building basic covers of $\mathbf{P}^1$ ramified only at infinity with minimal a-number.

Distribution of a-numbers

Let $k = \mathbf{F}_5$. Consider Artin-Schreier covers $\pi : Y \to \mathbf{P}^1_k$ that are ramified only above infinity with ramification invariant $d = 11$. Our bounds give:

$$10 \leq a_Y \leq 14.$$

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a_X	Number
10	8021
11	1901
12	64
13	10
14	4

Figure: a-numbers of 10000 random covers

Distribution of a-numbers

Let $k = \mathbf{F}_7$ and X be the supersingular elliptic curve $y^2 = x^3 - x$. Consider Artin-Schreier covers $\pi : Y \to X$ that are ramified only above the point at infinity with ramification invariant $d = 6$. Our bounds give:

$$9 \leq a_Y \leq 16.$$

a_X	Number
10	86436
11	11760
12	2562
13	0
14	84

Figure: a-numbers of all such covers

An Approach to a Proof

Let $\pi : Y \to X$ be ramified over S and η be the generic point of X

For a differential ω on Y, writing $\omega = \sum_{i=0}^{p-1} \omega_i y^i$ gives

$$(\pi_* \Omega^1_Y)_\eta = \bigoplus_{i=0}^{p-1} (\Omega^1_X)_\eta, \qquad \pi_* \Omega^1_Y = \bigoplus_{i=0}^{p-1} \Omega^1_X(E_i)$$

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$$(\pi_* \ker V_Y)_\eta \simeq \bigoplus_{i=0}^{p-1} (\ker V_X)_\eta \qquad \varphi : (\pi_* \ker V_Y) \hookrightarrow \bigoplus_{i=0}^{p-1} \ker V_X(F_* E_i).$$

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The key is analyzing this last map using local methods.

An Example

Let $p = 5$, and consider covers X_1, X_2 of $\mathbf{P}^1$ given by

$$y^5 - y = \begin{cases} x^{-3} & \text{or} \\ x^{-3} + x^{-2} \end{cases}$$

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$$\text{Bounds: } 3 \leq a_{X_i} \leq 4$$

$$\varphi : (\pi_* \ker V_Y) \hookrightarrow \bigoplus_{i=0}^{p-1} \ker V_X(F_* E_i)$$

Based on the defining equation, compute that

$$\varphi_\eta^{-1}((0, 0, x^{-2}dx, 0, 0)) = \begin{cases} x^{-2}dxy^2 \\ x^{-6}dx + x^{-2}dxy^2 \end{cases}$$

Thank you.