$a$-Numbers of Curves in Artin-Schreier Covers

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The Riemann-Hurwitz Formula

Figure: The elliptic curve $y^2 = x^3 - x$
Degree 2 cover of $\mathbb{P}^1$, Ramified above 0, 1, $-1$, $\infty$. Genus 1
Consider the curve $X$ defined by $y^5 - y = x^3$ defined over $k = F_5$.

- Artin-Schreier extension of function fields:

$$k(X) = k(x)[y]/(y^5 - y - x^3)$$
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- Degree 5 map \( \pi : X \to \mathbb{P}^1_k \) ramified at infinity.

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- Riemann-Hurwitz says that

  $$2g_X - 2 = 5(2g_{\mathbb{P}^1_k} - 2) + (d + 1)(5 - 1)$$

  i.e. that $g_X = 4$. 

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Consider a smooth projective curve $X$ over a finite field $k$ of characteristic $p$.

- The genus of $X$ is $\dim_k H^0(X, \Omega^1_X)$. 

In characteristic $p$, there is additional structure not present in characteristic 0: a semilinear operator $V_X: H^0(X, \Omega^1_X) \to H^0(X, \Omega^1_X)$ known as the Cartier operator. Easy to calculate with the Cartier operator:

$$V_X(\sum_i a_i t^{i} dt) = \sum_j a_1^{i/p} t^{j/p} dt.$$
Consider a smooth projective curve $X$ over a finite field $k$ of characteristic $p$.

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Invariants in Characteristic $p$

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$$V_X \left( \sum_i a_i t^i \frac{dt}{t} \right) = \sum_j a_{pj}^{1/p} t^j \frac{dt}{t}.$$
Other Invariants in Characteristic $p$

Based on the Cartier operator $V_X$, decompose

$$H^0(X, \Omega^1_X) = H^0(X, \Omega^1_X)^{bij} \oplus H^0(X, \Omega^1_X)^{nilp}$$
Other Invariants in Characteristic $p$

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**Definition**

The $p$-rank of $X$, denoted $f_X$, is $\dim_k H^0(X, \Omega_X^1)^{\text{bij}}$.

**Definition**

The $a$-number of $X$, denoted $a_X$, is $\dim_k \ker V_X$. 
Simple Examples

**Example**
An ordinary elliptic curve has $p$-rank 1 and $a$-number 0.

**Example**
A supersingular elliptic curve has $p$-rank 0 and $a$-number 1.
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Example
\( y^5 - y = x^3 \) over \( \mathbb{F}_5 \): genus 4, \( p\)-rank 0, \( a\)-number 4.
The Deuring-Schafarevich formula computes $p$-rank for extension of degree $p$ in terms of ramification information.
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Two $\mathbb{Z}/5\mathbb{Z}$-covers of $\mathbb{P}^1$ with the same ramification information:

**Example**

$y^5 - y = x^3$ over $\mathbb{F}_5$: genus 4, $p$-rank 0, $a$-number 4.

**Example**

$y^5 - y = x^3 + x^2$ over $\mathbb{F}_5$: genus 4, $p$-rank 0, $a$-number 3.
Let $X$ be a curve of genus $g_X$, so $\text{Jac}(X)$ is an Abelian variety of dimension $g_X$. 
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- $f_X = \dim_{F_p} \text{Hom}_k(\mu_p, \text{Jac}(X)[p])$

- $a_X = \dim_{F_p} \text{Hom}_k(\alpha_p, \text{Jac}(X)[p])$
Let $X$ be a curve of genus $g_X$, so $\text{Jac}(X)$ is an Abelian variety of dimension $g_X$.

- $\text{Jac}(X)[p]$ is a group scheme of order $p^{2g_X}$.
- $f_X = \dim_{F_p} \text{Hom}_{\overline{k}}(\mu_p, \text{Jac}(X)[p])$
- $a_X = \dim_{F_p} \text{Hom}_{\overline{k}}(\alpha_p, \text{Jac}(X)[p])$

The connection between $\text{Jac}(X)[p]$ and these invariants comes from relating the Dieudonné module and de Rham cohomology.
Invariants of the Jacobian

If $E$ is an elliptic curve, $E \cong \text{Jac}(E)$.

**Example**

If $E$ is an ordinary, $p$-rank is one and $a$-number is zero, while

$$E[p] = \mu_p \times \mathbb{Z}/p\mathbb{Z}.$$
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**Example**

If $E$ is a supersingular, $p$-rank is 0 and $a$-number is one, while

$$0 \to \alpha_p \to E[p] \to \alpha_p \to 0.$$
The Igusa Tower

Let $X_n$ be $n$th Igusa curve in characteristic $p$: moduli space of elliptic curves with level $p^n$ Igusa structure.

They form a $\mathbb{Z}_p$-tower

$$\cdots \rightarrow X_3 \rightarrow X_2 \rightarrow X_1.$$
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Using ramification information worked out by Katz and Mazur:

- $g_{X_n} = c_1 p^{2n} + c_2 p^n + c_3$ (Riemann-Hurwitz)
- $f_{X_n} = c_4 (p^n - 1)$ (Deuring-Shafarevich)
- $\frac{1}{2} + O(p^{-1}) \leq \frac{a_{X_n}}{g_{X_n}} \leq \frac{2}{3} + O(p^{-1})$ (our results)
Consider a “nice” $\mathbb{Z}_p$-tower of curves

$$\ldots \rightarrow X_3 \rightarrow X_2 \rightarrow X_1.$$  

**Question**

Is the growth of $a_{X_n}$ regular? (for genus stable towers?)
Motivation: \( \mathbb{Z}_p \)-towers and Iwasawa Theory

Consider a “nice” \( \mathbb{Z}_p \)-tower of curves

\[ \ldots \to X_3 \to X_2 \to X_1. \]

Question

Is the growth of \( a_{X_n} \) regular? (for genus stable towers?)

Studying invariants of \( \text{Jac}(X_n)[p] \) like genus or \( a \)-number is a geometric analog of Iwasawa theory.
Theorem (B-Cais)

Let $\pi : Y \rightarrow X$ be a $\mathbb{Z}/p\mathbb{Z}$-cover of curves in characteristic $p$ with branch locus $S \subseteq X(\bar{k})$. For $Q \in S$ let $d_Q$ be the unique break in the lower-numbering ramification filtration at the unique point of $Y$ over $Q$. Then for any $1 \leq j \leq p - 1$, 

$$\sum_{Q \in S} p - 1 \sum_{i=1}^{j} (\lfloor i d_Q \rfloor - (p - i) \lfloor i^2 d_Q \rfloor) \leq a_Y \quad \text{and} \quad a_Y \leq pa_X + \sum_{Q \in S} p - 1 \sum_{i=1}^{j} (\lfloor i d_Q \rfloor - (p - i) \lfloor i^2 d_Q \rfloor).$$
Theorem (B-Cais)

Let \( \pi : Y \to X \) be a \( \mathbb{Z}/p\mathbb{Z} \)-cover of curves in characteristic \( p \) with branch locus \( S \subseteq X(\overline{k}) \). For \( Q \in S \) let \( d_Q \) be the unique break in the lower-numbering ramification filtration at the unique point of \( Y \) over \( Q \). Then for any \( 1 \leq j \leq p - 1 \),

\[
\sum_{Q \in S} \sum_{i=j}^{p-1} \left( \left\lfloor \frac{id_Q}{p} \right\rfloor - \left\lfloor \frac{id_Q}{p} \right\rfloor - \left( 1 - \frac{1}{p} \right) \frac{j d_Q}{p^2} \right) \leq a_Y
\]

and

\[
a_Y \leq p a_X + \sum_{Q \in S} \sum_{i=1}^{p-1} \left( \left\lfloor \frac{id_Q}{p} \right\rfloor - (p - i) \left\lfloor \frac{id_Q}{p^2} \right\rfloor \right).
\]
When $\sum_{Q \in S} d_Q = T$ is large, take $j \approx p/2$ and approximate:

- lower bound $\approx \frac{pT}{4}$
- upper bound $\approx \frac{pT}{3}$
Estimates on the Bounds

When $\sum_{Q \in S} d_Q = T$ is large, take $j \approx p/2$ and approximate:

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- upper bound $\approx \frac{pT}{3}$

In contrast,

$\text{genus} \approx \frac{pT}{2}$. 
$\pi : Y \rightarrow X$ a $\mathbb{Z}/p\mathbb{Z}$-cover ramified over $S$

**Corollary**

Suppose $p$ is odd. If $a_X = 0$ and $d_Q \mid p - 1$ for every $Q \in S$, the upper and lower bounds match, giving an explicit formula for $a_Y$.

Recovers a result of Shawn Farnell and Rachel Pries when $X = \mathbb{P}^1_k$. 

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$a$-Numbers of Curves in Artin-Schreier Covers
Invariants of Curves
Motivation
Artin-Schreier Covers

Special Cases

\( \pi : Y \rightarrow X \) a \( \mathbb{Z}/p\mathbb{Z} \)-cover ramified over \( S \)

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Recovers a result of Shawn Farnell and Rachel Pries when \( X = \mathbb{P}^1_k \).

Corollary

Suppose \( p = 2 \). If \( a_X = 0 \), the upper and lower bounds match, giving an explicit formula for \( a_Y \).

Recovers a result of Felipe Voloch.
Invariants of Curves

Motivation

Artin-Schreier Covers

When are the Bounds Sharp?

Example

Consider $y^p - y = x^d$ (cover of $\mathbb{P}^1$ ramified at infinity). The $a$-number is our upper bound.
When are the Bounds Sharp?

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Consider $y^p - y = x^d$ (cover of $\mathbb{P}^1$ ramified at infinity). The $a$-number is our upper bound.

Theorem (B-Pries)

Let $p = 3$ and $X$ be a curve with $a_X = 0$. There exists a $\mathbb{Z}/p\mathbb{Z}$-cover of $X$ with any specified $S \subset X(\overline{k})$ and $d_Q$ for $Q \in S$ with minimal $a$-number.

Relies on building basic covers of $\mathbb{P}^1$ ramified only at infinity with minimal $a$-number.
Let $k = \mathbb{F}_5$. Consider Artin-Schreier covers $\pi : Y \to \mathbb{P}^1_k$ that are ramified only above infinity with ramification invariant $d = 11$. Our bounds give:

$$10 \leq a_Y \leq 14.$$
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<table>
<thead>
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<th>$a_X$</th>
<th>Number</th>
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<td>10</td>
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<tr>
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<td>4</td>
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Figure: $a$-numbers of 10000 random covers
Let $k = \mathbb{F}_7$ and $X$ be the supersingular elliptic curve $y^2 = x^3 - x$. Consider Artin-Schreier covers $\pi : Y \to X$ that are ramified only above the point at infinity with ramification invariant $d = 6$. Our bounds give:

$$9 \leq a_Y \leq 16.$$ 

<table>
<thead>
<tr>
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</tr>
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<tbody>
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<tr>
<td>14</td>
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</tr>
</tbody>
</table>

**Figure:** $a$-numbers of all such covers
Let $\pi : Y \to X$ be ramified over $S$ and $\eta$ be the generic point of $X$.

For a differential $\omega$ on $Y$, writing $\omega = \sum_{i=0}^{p-1} \omega_i y^i$ gives

$$(\pi_* \Omega^1_Y)_{\eta} = \bigoplus_{i=0}^{p-1} (\Omega^1_X)_{\eta}, \quad \pi_* \Omega^1_Y = \bigoplus_{i=0}^{p-1} \Omega^1_X(E_i)$$

where $E_i$ are explicit divisors supported on $S$. 
Let $\pi : Y \to X$ be ramified over $S$ and $\eta$ be the generic point of $X$

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$$(\pi_* \Omega^1_Y) \cong \bigoplus_{i=0}^{p-1} \Omega^1_X(E_i),$$

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$$(\pi_* \ker V_Y)_\eta \cong \bigoplus_{i=0}^{p-1} (\ker V_X)_\eta$$

$$(\pi_* \ker V_Y) \hookrightarrow \bigoplus_{i=0}^{p-1} \ker V_X(F_* E_i).$$
An Approach to a Proof

Let \( \pi : Y \to X \) be ramified over \( S \) and \( \eta \) be the generic point of \( X \). For a differential \( \omega \) on \( Y \), writing \( \omega = \sum_{i=0}^{p-1} \omega_i y^i \) gives

\[
(\pi_* \Omega^1_Y)_\eta = \bigoplus_{i=0}^{p-1} (\Omega^1_X)_\eta, \quad \pi_* \Omega^1_Y = \bigoplus_{i=0}^{p-1} \Omega^1_X(E_i)
\]

where \( E_i \) are explicit divisors supported on \( S \).

\[
(\pi_* \ker V_Y)_\eta \cong \bigoplus_{i=0}^{p-1} (\ker V_X)_\eta \quad \varphi : (\pi_* \ker V_Y) \hookrightarrow \bigoplus_{i=0}^{p-1} \ker V_X(F_*E_i).
\]

The key is analyzing this last map using local methods.
Let $p = 5$, and consider covers $X_1, X_2$ of $\mathbb{P}^1$ given by

\[ y^5 - y = \begin{cases} \ x^{-3} & \text{or} \\ x^{-3} + x^{-2} \end{cases} \]

Bounds: $3 \leq a_{X_i} \leq 4$
An Example

Let $p = 5$, and consider covers $X_1, X_2$ of $\mathbb{P}^1$ given by

$$y^5 - y = \begin{cases} x^{-3} & \text{or} \\ x^{-3} + x^{-2} \end{cases}$$

Bounds: $3 \leq a_{X_i} \leq 4$

$$\varphi : (\pi_* \ker V_Y) \to \bigoplus_{i=0}^{p-1} \ker V_X(F_* E_i)$$

Based on the defining equation, compute that

$$\varphi^{-1}_\eta((0, 0, x^{-2} dx, 0, 0)) = \begin{cases} x^{-2} dxy^2 \\ x^{-6} dx + x^{-2} dxy^2 \end{cases}$$
Thank you.