On Graphs of Hecke Operators

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49th John H. Barrett Memorial Lectures

Let

- X be a smooth projective curve over a finite field $\mathbb{F}_q$;
- F be the function field of X;
- $\mathbb{A}$ be the adele ring of F, and $\mathcal{O}_{\mathbb{A}}$ its adelic integers;
- $G := \mathrm{GL}_n$ and $K := \mathrm{GL}_n(\mathcal{O}_{\mathbb{A}})$;

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Definition

The complex vector space $\mathcal{H}$ of all smooth, compactly supported functions $\Phi : G(\mathbb{A}) \to \mathbb{C}$ together with the convolution product

$$\Phi_1 * \Phi_2 : g \longmapsto \int_{G(\mathbb{A})} \Phi_1(gh^{-1})\Phi_2(h)dh$$

for $\Phi_1, \Phi_2 \in \mathcal{H}$ is called the *Hecke algebra* for $G(\mathbb{A})$. Its elements are called *Hecke operators*.

Let $\mathcal{H}_K$ the subalgebra of $\mathcal{H}$ of all bi-K-invariant Hecke operators, called *unramified (or spherical) Hecke algebra*.

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For x a place of F and π_x its uniformizer, let $\Phi_{x,r}$ be the characteristic function of

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Theorem (Tamagawa-Satake)

Identifying char_K *with* $1 \in \mathbb{C}$ *yields*

$$\mathcal{H}_K \cong \mathbb{C}[\Phi_{x,1}, \ldots, \Phi_{x,n}, \Phi_{x,n}^{-1}]_{x \in |X|}.$$

In particular, $\mathcal{H}_K$ *is commutative.*

The Hecke algebra $\mathcal{H}$ acts on $\mathcal{A}$ the *space of automorphic forms* by

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Proposition

Fix $\Phi \in \mathcal{H}_K$*. For all* $[g] \in G(F) \setminus G(\mathbb{A})/K$*, there is a unique set of pairwise distinct classes* $[g_1], \ldots, [g_r] \in G(F) \setminus G(\mathbb{A})/K$ *and numbers* $m_1, \ldots, m_r \in \mathbb{C}^*$ *such that*

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$$\Phi(f)(g) = \sum_{i=1}^{r} m_i f(g_i), \quad \textit{for all } f \in \mathcal{A}^K.$$

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Definition

For $\Phi = \Phi_{x,r}$ in the last proposition, let $\mathcal{V}_{x,r}([g]) := \{([g],[g_i],m_i)\}_{i=1,\ldots,r}$. We define the graph $\mathcal{G}_{x,r}$ of the Hecke operator $\Phi_{x,r}$ whose vertices are

$$\mathrm{Vert}\mathcal{G}_{x,r} = G(F) \setminus G(\mathbb{A})/K$$

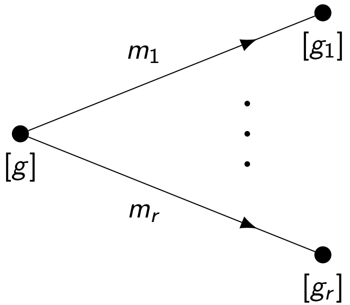
and the oriented weighted edges

$$\mathrm{Edge}\mathcal{G}_{x,r} = \bigcup_{[g]\in \mathrm{Vert}\mathcal{G}_{x,r}} \mathcal{V}_{x,r}([g]).$$

By the definition of the graph of $\Phi_{x,r}$, we have for $f \in \mathcal{A}^K$ and $[g] \in G(F) \setminus G(\mathbb{A})/K$ that

$$\Phi_{x,r}(f)(g) = \sum_{\substack{([g],[g_i],m_i) \\ \in \mathrm{Edge}\mathcal{G}_{x,r}}} m_i f(g_i).$$

Hence one can read off the action of a Hecke operator on the value of an automorphic form from the illustration of the graph:



Geometry of graphs of Hecke operators

We say that two exact sequences of coherent sheaves

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathcal{F}_1' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2' \longrightarrow 0$$

are *isomorphic with fixed $\mathcal{F}$* if there are isomorphism $\mathcal{F}_1 \to \mathcal{F}_1'$ and $\mathcal{F}_2 \to \mathcal{F}_2'$ such that the following diagram commutes.

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For $x \in |X|$, let $\mathcal{K}_x$ the skyscraper sheaf on x with stalk $\kappa(x)$ the residue field of x.

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Fix $\mathcal{E} \in \mathrm{Bun}_n X$. For $r \in \{1, \ldots, n\}$, and $\mathcal{E}' \in \mathrm{Bun}_n X$ we define $m_{x,r}(\mathcal{E}, \mathcal{E}')$ as the number of isomorphism classes of exact sequences

$$0 \longrightarrow \mathcal{E}'' \longrightarrow \mathcal{E} \longrightarrow \mathcal{K}_x^{\oplus r} \longrightarrow 0$$

with fixed $\mathcal{E}$ and with $\mathcal{E}'' \cong \mathcal{E}'$.

Geometry of graphs of Hecke operators

Definition

Let $x \in |X|$. For a vector bundle $\mathcal{E} \in \mathrm{Bun}_n X$ we define

$$\mathcal{V}_{x,r}(\mathcal{E}) := \{(\mathcal{E}, \mathcal{E}', m) | m = m_{x,r}(\mathcal{E}, \mathcal{E}') \neq 0\},$$

and we call $\mathcal{E}'$ a $\Phi_{x,r}$-neighbor of $\mathcal{E}$ if $m_{x,r}(\mathcal{E}, \mathcal{E}') \neq 0$, and $m_{x,r}(\mathcal{E}, \mathcal{E}')$ its multiplicity.

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Theorem

Let $x \in |X|$. The graph $\mathcal{G}_{x,r}$ of $\Phi_{x,r}$ is described in geometric terms as

$$\mathrm{Vert}\, \mathcal{G}_{x,r} = \mathrm{Bun}_n X \quad \textit{and} \quad \mathrm{Edge}\, \mathcal{G}_{x,r} = \coprod_{\mathcal{E} \in \mathrm{Bun}_n X} \mathcal{V}_{x,r}(\mathcal{E}).$$

Moreover, the multiplicities of the edges originating in $\mathcal{E}$ sum up to $\#\mathrm{Gr}(n-r, n)(\kappa(x))$.

The graph of $\Phi_{x,1}$, for a closed point x in $\mathbb{P}^1$ of degree $|x|$ and $n = 1$.

A neighbor of $\mathcal{O}_{\mathbb{P}^1}(d)$ is given in terms of exact sequences

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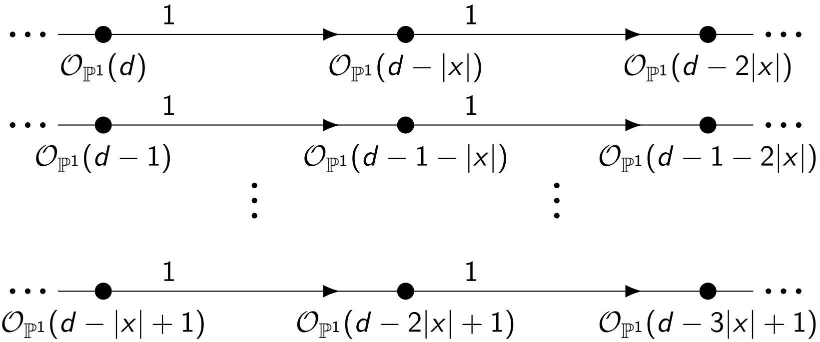
Therefore, $\mathcal{E}' = \mathcal{O}_{\mathbb{P}^1}(d - |x|)$ and $m_{x,1}(\mathcal{O}_{\mathbb{P}^1}(d), \mathcal{O}_{\mathbb{P}^1}(d - |x|)) = 1$.



The Hall algebra of X

Fix a square root v of q^{-1}. Let $\mathbf{H}_X$ be the $\mathbb{C}-$vector space

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Hall numbers

$$h^{\mathcal{H}}_{\mathcal{F},\mathcal{G}} := \frac{\#\{0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow \mathcal{F} \longrightarrow 0\}}{\#\mathrm{Aut}(\mathcal{F})\#\mathrm{Aut}(\mathcal{G})},$$

which is finite since $\mathrm{Coh}(X)$ is a finitary category.

The Hall algebra of X

The Euler form

For $\mathcal{F}, \mathcal{G}$ coherent sheaves on X, we define

$$\langle \mathcal{F}, \mathcal{G} \rangle := \sum_i (-1)^i \dim_{\mathbb{F}_q} \mathrm{Ext}^i(\mathcal{F}, \mathcal{G}).$$

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The following product defines on $\mathbf{H}_X$ the structure of an associative algebra

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Definition

We call $\mathbf{H}_X$ *the Hall algebra of X*.

The Hall algebra of X

Lemma

Let $x \in |X|$, $\mathcal{E}, \mathcal{E}' \in \mathrm{Bun}_n X$ *and* $r \in \{1, \ldots, n\}$. *Then*

$$h^{\mathcal{E}}_{\mathcal{K}_x^{\oplus r}, \mathcal{E}'} = m_{x,r}(\mathcal{E}, \mathcal{E}').$$

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Remark

The graph of Hecke operator $\mathcal{G}_{x,r}$ *is completely determined by*

$$\mathcal{K}_x^{\oplus r} * \mathcal{E} = v^{n|x|r} \sum_{\mathcal{F}} h^{\mathcal{F}}_{\mathcal{K}_x^{\oplus r} \mathcal{E}} \mathcal{F}$$

where $\mathcal{E}$ *runs through* $\mathrm{Bun}_n X$.

The Hall algebra of $\mathbb{P}^1_{\mathbb{F}_q}$

Example

For $X = \mathbb{P}^1_{\mathbb{F}_q}$, Baumann and Kassel (1999) prove that

$$\mathcal{K}_x * \mathcal{O}_{\mathbb{P}^1}(d) = \mathcal{O}_{\mathbb{P}^1}(d + |x|) + q^{|x|}\, \mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{K}_x.$$

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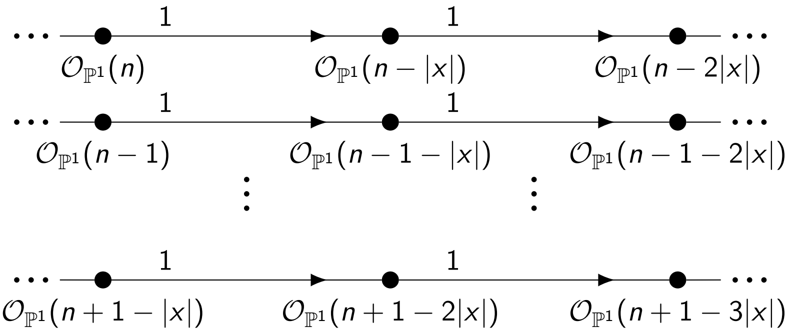
This reproves our earlier example:



The algorithm for elliptic curves

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The Algorithm

Let X be an elliptic curve. Using structure results of Burban, Schiffmann and Fratila we developed an algorithm to calculate the products

$$\mathcal{K}_x^{\oplus r} * \mathcal{E} = v^{\mathrm{rk}(\mathcal{E})|x|r} \sum_{\mathcal{F}} h_{\mathcal{K}_x^{\oplus r}\mathcal{E}}^{\mathcal{F}} \, \mathcal{F}$$

in the Hall algebra of an elliptic curve. Hence we are able to describe explicitly the graphs $\mathcal{G}_{x,r}$ for all $\mathrm{rk}(\mathcal{E}) = n \in \mathbb{Z}_{\geq 1}$, $0 \leq r \leq n$ and $x \in |X|$.

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Theorem (Atiyah)

The following hold.

(i) *The Harder-Narasinham filtration of any coherent sheaf splits. In particular, any indecomposable coherent sheaf is semistable.*

(ii) *The set of stable sheaves of slope μ is the class of simple objects of C_μ.*

(iii) *The choice of any rational point $x_0 \in X(\mathbb{F}_q)$, induces an exact equivalence of abelian categories $\epsilon_{\nu,\mu} : \mathsf{C}_\mu \to \mathsf{C}_\nu$, for any $\mu, \nu \in \mathbb{Q} \cup \{\infty\}$.*

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$$T_{(0,m),x} := \begin{cases} 0 & \text{if} \quad |x| \nmid m \\ \frac{[m]|x|}{m} \sum_{|\lambda|=m/|x|} n(l(\lambda)-1)\, \mathcal{K}_x^{(\lambda)} & \text{if} \quad |x| \mid m. \end{cases}$$

where $[m] := (v^m - v^{-m})/(v - v^{-1})$, and $n(l) = \prod_{i=1}^{l}(1 - v^{-2i|x|})$.

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For any $\mu \in \mathbb{Q} \cup \{\infty\}$, we consider the subspace $\mathbf{H}_X^{(\mu)} \subset \mathbf{H}_X$ that is spanned by classes $\{\mathcal{F} | \mathcal{F} \in \mathsf{C}_\mu\}$. Since the category C_μ is stable under extensions, $\mathbf{H}_X^{(\mu)}$ is a subalgebra of $\mathbf{H}_X$.

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For every $\mathbf{v} \in \mathbf{Z}$ with slope μ, we define

$$T_{\mathbf{v},x} := \epsilon_{\mu,\infty}(T_{(0,m),x}).$$

The Hall algebra of an elliptic curve

For a character ρ of the Picard group $\mathrm{Pic}^0 X_n$ where $X_n := X \times_{\mathrm{Spec}\mathbb{F}_q} \mathrm{Spec}\mathbb{F}_{q^n}$, we define the twisted average of the elements $T_{\mathbf{v},x}$ as

$$T_{\mathbf{v}}^{\rho} = \sum_{x' \in |X|} \rho(x') T_{\mathbf{v},x'}$$

The algorithm for elliptic curves

Let $x \in X$ be a closed point, $\mathcal{E} \in \mathrm{Bun}_n X$ and r an integer such that $1 \leq r \leq n$.

Theorem (Base change)

Let $\mathcal{E} \in \mathrm{Bun}_n X$, then

$$\mathcal{E} = \sum_{\widetilde{\rho}_{ij_k}} a_{ij_k} \; T_{\mathbf{v}_{1j_1}}^{\widetilde{\rho}_{1j_1}} \cdots T_{\mathbf{v}_{1j_{k_j}}}^{\widetilde{\rho}_{1j_{k_j}}} \cdots T_{\mathbf{v}_{sj_1}}^{\widetilde{\rho}_{sj_1}} \cdots T_{\mathbf{v}_{sj_{k_j}}}^{\widetilde{\rho}_{sj_{k_j}}},$$

for some $a_{ij_k} \in \mathbb{C}$ where $\widetilde{\rho}_{ij_k}$ runs through the elements in $\widetilde{\mathrm{Pic}^0(X_{\gamma(\mathbf{v}_{ij_k})})}$ and the path given by $\left(\mathbf{v}_{1j_1}, \ldots, \mathbf{v}_{1j_{k_j}}, \ldots, \mathbf{v}_{sj_1}, \ldots, \mathbf{v}_{sj_{k_j}}\right)$ is a convex path which defines the same polygonal line as $\mathbf{p}(\mathcal{E})$.

The algorithm for elliptic curves

Theorem (Order by slopes)

Let $x \in |X|$, $r \geq 1$ an integer and $\mathcal{E} \in \mathrm{Bun}_n X$, then

$$\mathcal{K}_x^{\oplus r}\ \mathcal{E} = \sum_{i=1}^{m} a_i\ T_{\mathbf{v}_{i_1}}^{\widetilde{\rho}_{i_1}} \cdots T_{\mathbf{v}_{i_\ell}}^{\widetilde{\rho}_{i_\ell}}$$

for some $a_i \in \mathbb{C}$ where $\widetilde{\rho}_{i_j}$ runs through the elements in $\widetilde{\mathrm{Pic}^0(X_{\gamma(\mathbf{v}_{i_j})})}$, and where $\mathbf{v}_{i_j} \in \mathbf{Z}$ are such that $\left(\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_\ell}\right)$ defines a convex path in $\Delta_{(\mathrm{rk}(\mathcal{E}),\deg(\mathcal{E})),(0,r|x|)}$ for all $i = 1, \ldots, m$.

The algorithm for elliptic curves

Theorem

Let $x \in |X|$, $r \geq 1$ *an integer and* $\mathcal{E} \in \mathrm{Bun}_n X$, *then*

$$\mathcal{K}_x^{\oplus r}\, \mathcal{E} = \sum_{i=1}^{m} a_i \;\, T_{\mathbf{v}_{i_1}, x_{i_1}} \oplus \cdots \oplus T_{\mathbf{v}_{i_\ell}, x_{i_\ell}}$$

for some $v^{n|x|r} a_i \in \mathbb{Z}$, *where* $x_{i_j} \in |X|$, *and* $\mathbf{v}_{i_j} \in \mathbf{Z}$ *are such that* $(\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_\ell})$ *defines a convex path in* $\Delta_{(\mathrm{rk}(\mathcal{E}), \deg(\mathcal{E})), (0, r|x|)}$ *for all* $i = 1, \ldots, m$.

Applying the algorithm

Theorem

Let X be an elliptic curve. Fix an integer $n \geq 1$. Let $x \in |X|$ with $|x| = 1$. Let $\mathcal{E}' = \mathcal{E}_{(y,1)}^{(n,d)}$ with $n|d$ be a stable vector bundle on X. Then $m_{x,1}(\mathcal{E}, \mathcal{E}') \neq 0$ if and only if $\mathcal{E} \cong \mathcal{E}_{(y',1)}^{(n,d+1)}$ where $y' = x + y$. Graphically

holds in $\mathcal{G}_{x,1}$ for every degree one closed point $x \in X$.

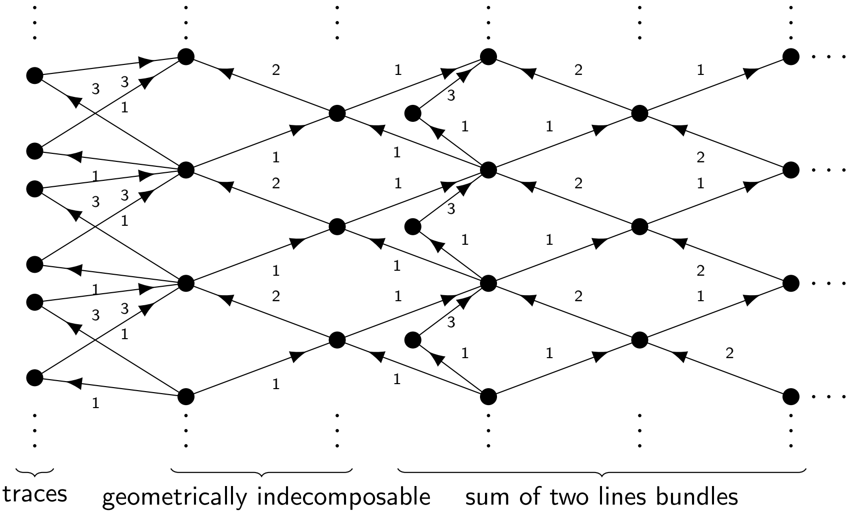
Applying the algorithm

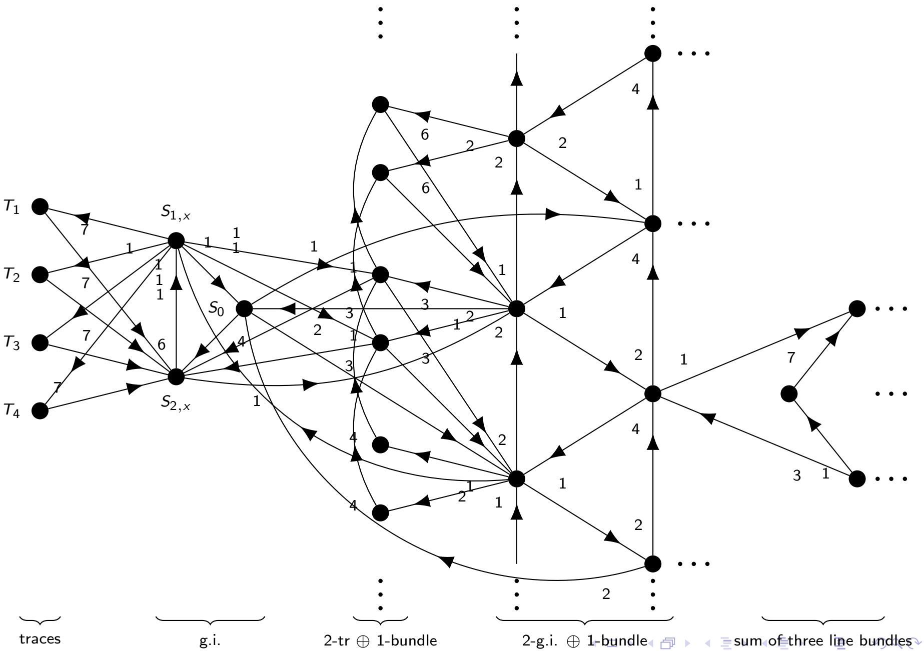
Example

There is up to isomorphism one elliptic curve X over $\mathbb{F}_2$ with only one rational point x, namely:

- X is defined by the Weierstrass equation $Y^2 + Y = X^3 + X + 1$,

The graph $\mathcal{G}_{x,1}$ for $n = 2, 3$ of X is given as in the following figure.


traces
geometrically indecomposable
sum of two lines bundles



T_1
T_2
T_3
T_4
$S_{1,x}$
$S_{2,x}$
S_0
traces
g.i.
2-tr ⊕ 1-bundle
2-g.i. ⊕ 1-bundle
sum of three line bundles

Thank you