Stochastics Prelim Topics List

The preliminary exam in stochastics is based on the material of the graduate probability sequence Math 523-524. The sequence covers standard topics of probability theory, starting with a brief introduction to measure theory foundations of probability and ending on the martingales theory. Martingale theory is fundamental to stochastic analysis, stochastic PDEs, mathematical finance, stochastic modeling and optimization and other applications. Therefore, there are many directions that a student can pursue after mastering material from this sequence. The current textbook is [4], other texts used in the past are in the References.

1. Foundations

1.1. Basics from Measure Theory: Operations on sets; Collections of sets, σ-algebras and their Generators.).

1.2. The Probability Space: Axioms of probability and basic formulas; The Borel sets on the real line; The Borel sets on $\mathbb{R}^n$.

2. Random Variables

2.1. Definition and Basic Properties: Functions of Random Variables; Operations on random variables; Approximation by simple random variables.

2.2. Distributions: Distribution Functions; Decomposition of Distributions; Some Standard Discrete Distributions; Some Standard Absolutely Continuous Distributions;

2.3. Random Vectors and Random Elements: Definitions and Characterization.

3. Expectation

3.1 Construction and properties of expectation of a random variable

3.2 Three theorems: Monotone Convergence Theorem, Dominated Convergence Theorem, and Fatou Lemma.

3.3 Approximation by simple random variables argument

3.4 A Change of Variables Formula

3.5 Moments, Mean, Variance

4. Independence

4.1 Independence of collections of events

4.2 Independence of random variables and of their σ-algebras

4.3 Independence of functions of independent random variables

4.4 Independence between collections of random variables

4.5 Criteria for the independence of discrete and absolutely continuous random variables

4.6 Kolmogorov's zero-one law

5. Borel-Cantelli Lemmas

5.1 Events occurring "infinitely often" or "eventually"

5.2 Borel-Cantelli Lemmas and their applications

6. Product Spaces; Fubini's Theorem

6.1 Finite and σ-finite measures

6.2 Product measure and Fubini's Theorem in connection to the independence (joint distributions, iterated expectations)

6.3 Applications: Evaluation of expectations of functions of independent random variables; Expectation in terms of distribution function.

7. Inequalities

7.1. Tail Probabilities Estimated by Moments: Markov's and Chebyshev's inequalities and the method to establish them.

7.2. Moment Inequalities: The Hölder inequality; The Minkowski inequality; Jensen's inequality.

7.3 Evaluation of the variance of a linear combination of random variables; covariance and correlation bounds.

8. Convergence

8.1. Four types of convergence of random variables: Almost Sure; in Probability; in the pth mean (in L^p); in Distribution.

8.2. Relations between the convergence almost surely and in probability. Measurability and uniqueness of limits. A subsequence principle.

8.3. Relations between the Four types of convergence of random variables.

8.5. Cauchy Convergence; Uniform Integrability; Convergence of Moments

8.6. Convergence of Sums of Sequences: Slutsky's theorem

9. The Law of Large Numbers

9.1. Preliminaries: Convergence Equivalence and Distributional Equivalence

9.2. The Weak Law of Large Numbers

9.3. The Three-Series Theorem

9.4. The Strong Law of Large Numbers

9.5. Applications

10. Characteristic Functions

10.1. Definition, basic properties, characteristic functions of some standard distributions

10.2. Uniqueness and Inversion formulas

10.3. Characteristic function of the sum of independent random variables

10.4. Taylor polynomial approximation of a characteristic function

10.5. Moments of a random variable by differentiating its characteristic function

10.6. Characteristic Function of a Random Vector (multivariate characteristic function)

10.7. Independence of random variables via multivariate characteristic function

10.8. Multivariate normal distribution

11. Convergence in distribution revisited

11.1. Convergence in distribution and Tightness

11.3. Lévy's Convergence Theorem

11.4. The Continuous Mapping Theorem

12. The Central Limit Theorem (CLT)

12.1. Method of characteristic function

12.2. CLT for i.i.d. random variables

12.3. Applications

13. Martingales

13.1. Conditional expectation: Definitions, properties, composition of conditional expectations.

13.2. Martingales: Definitions and examples; Orthogonality of martingale differences

13.3. Doob's decomposition of submartingales

13.4. Stopping times: Definition and properties; randomly stopped martingales

13.5. The optional sampling theorem

13.6. Maximal inequalities and Martingale convergence theorem

13.7. Martingale convergence for $E(Z|\mathcal{F}_n)$ and the uniform integrability

13.8. Stopped random walk and the Wald equations

References

[1] Billingsley, P. *Probability and Measure,* 3rd ed. Wiley, (1995).

[2] Durrett, R. *Probability: Theory and Examples,* 4th ed. Cambridge University Press, (2010).

[3] Fristedt, B. and Grey, L. *A Modern Approach to Probability Theory,* Birkhäuser, (1997).

[4] Gut, A. *Probability: A Graduate Course.* 2nd ed. Springer, (2013).

[5] Jacod, J. and Protter, P. *Probability Essentials,* 2nd ed. Springer, (2004).