1. (10 points) Prove the following generalization of subadditivity: For any events $B_i \subset A_i$ in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$

$$\mathbb{P}(\bigcup_i A_i) - \mathbb{P}(\bigcup_i B_i) \leq \sum_i (\mathbb{P}(A_i) - \mathbb{P}(B_i))$$

provided $\sum_i \mathbb{P}(A_i) < \infty$.

2. (14 points) Recall that $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$ is the $L^p$-norm of a random variable $X$, $p \in [1, \infty)$, and $\|X\|_\infty = \inf\{c \geq 0 : \mathbb{P}(|X| \leq c) = 1\}$ is the $L^\infty$-norm of $X$. Show that
   (a) the function $[1, \infty) \ni p \mapsto \|X\|_p$ is nondecreasing;
   (b) $\lim_{p \to \infty} \|X\|_p = \|X\|_\infty$.

3. (16 points) Let $X_n$ be i.i.d. exponential random variables with mean 1. Prove that

$$\frac{\max\{X_1, \ldots, X_n\}}{\ln n} \to 1 \quad \text{a.s. as } n \to \infty.$$

[Hint: First show that $\limsup_{n \to \infty} \frac{X_n}{\ln n} = 1$ a.s.]

4. (14 points) Let $X_n, n \geq 1$ be random variables and let $S_n = X_1 + \cdots + X_n$. Show that $\{n^{-1}S_n\}$ is uniformly integrable under each of the following two conditions
   (a) $X_n$ are uncorrelated and have the same mean and the same variance for every $n$;
   (b) $X_n$ are i.i.d. with finite expectation.

5. (14 points) Given a bounded and continuous function $f(x)$ on $[0, \infty)$, prove that

$$\lim_{n \to \infty} e^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!} = f(x) \quad \forall x \geq 0$$

6. (14 points) Let $\{X_n\}$ be an i.i.d. sequence such that $P\{X_1 = e^{-1}\} = P\{X_1 = e\} = 1/2$.

Prove that the random sequence

$$\left( \prod_{k=1}^{n} X_k \right)^{1/\sqrt{n}} \quad n = 1, 2, \ldots$$

converges in distribution and determine the limit distribution.

7. (12 points) Let $\tau$ be stopping times with respect to filtration $\{\mathcal{F}_n\}$. Show that a random variable $Y$ is $\mathcal{F}_\tau$-measurable if and only if $Y 1_{\{\tau = n\}}$ is $\mathcal{F}_n$-measurable for each $n$.

8. (12 points) If $\{X_n\}$ is a martingale and it is bounded either from above or below by a constant $K$, then $\{X_n\}$ is bounded in $L^1$.

9. (14 points) Let $X$ and $Y$ be integrable random variables with $E(Y | X) = X$ and $E(X | Y) = Y$.

Prove that $X = Y$ a.s.
1. (12 points) Let $A_n$ be the square $\{(x, y) : |x| \leq 1, |y| \leq 1\}$ pinned at $(0,0)$ and rotated through the angle $2\pi n\theta$. Give geometric descriptions of $\limsup_n A_n$ and $\liminf_n A_n$ when

(a) $\theta = 1/8$;
(b) $\theta$ is irrational. [Hint: $2\pi n\theta$ reduced modulo $2\pi$ are dense in $[0, 2\pi]$ if $\theta$ is irrational.]

2. (12 points) (i) Let $A_n$ be events such that $\mathbb{P}(A_n) \to 0$ and $\sum_{n=1}^{\infty} \mathbb{P}(A_n^c \cap A_{n+1}) < \infty$. Show that $\mathbb{P}(A_n \text{ i.o.}) = 0$.
(ii) Find an example of a sequence $A_n$ to which the result in (i) can be applied but the Borel-Cantelli lemma cannot.

3. (12 points) Let $\{X_n\}$ be independent random variables. Show that $\mathbb{P}(\sup_n X_n < \infty) = 1$ if and only if $\exists c < \infty$ such that $\sum_n \mathbb{P}(X_n > c) < \infty$.

4. (14 points) Let $X_n \geq 0$ be independent random variables. Show that the following are equivalent:
(i) $\sum_{n=1}^{\infty} X_n < \infty$ a.s.  (ii) $\sum_{n=1}^{\infty} \{\mathbb{P}(X_n > 1) + \mathbb{E}[X_n I_{X_n \leq 1}]\} < \infty$.

5. (12 points) Let $S_n = \sum_{i=1}^{n} X_i$, where $X_i$ are i.i.d. with $X_i \geq 0$, $\mathbb{E}X_i = 1$, and $\text{Var}(X_i) = \sigma^2 \in (0, \infty)$. Show that
$$\sqrt{S_n} - \sqrt{n} \xrightarrow{d} N(0, \sigma^2/4) \quad \text{as } n \to \infty.$$ 

6. (14 points) For given $n \geq 1$, let $X_1, \ldots, X_n$ be independent uniform on $[-n, n]$ random variables. Put
$$Y_n = \sum_{i=1}^{n} \frac{\text{sgn}(X_i)}{|X_i|^{\beta}},$$
where $\beta > 1/2$ is a constant. Show that $Y_n$ converges in distribution as $n \to \infty$ to a random variable with characteristic function $\phi(t) = \exp(-c|t|^{1/\beta})$, where $c$ is a positive constant.

7. (12 points) Let $\sigma$ and $\tau$ be stopping times with respect to filtration $\{\mathcal{F}_n\}$. Show that
(i) $\sigma \wedge \tau := \min\{\sigma, \tau\}$ is a stopping time and $\mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$;
(ii) For any integrable random variable $Y$
$$\mathbb{E}\{\mathbb{E}[Y | \mathcal{F}_\tau] | \mathcal{F}_\sigma \} = \mathbb{E}\{\mathbb{E}[Y | \mathcal{F}_\sigma] | \mathcal{F}_\tau \} = \mathbb{E}[Y | \mathcal{F}_{\sigma \wedge \tau}].$$

8. (10 points) Let $\xi, \xi_1, \xi_2, \ldots$ be random variables with $\xi_n \to \xi$ in $L^1$. Show for any increasing $\sigma$-algebras $\mathcal{F}_n$ that $\mathbb{E}[\xi_n | \mathcal{F}_n] \to \mathbb{E}[\xi | \mathcal{F}_\infty]$ in $L^1$, as $n \to \infty$.

9. (12 points) Let $X_n$ and $Y_n$ be nonnegative integrable random variables adapted to increasing $\sigma$-algebras $\mathcal{F}_n$. Suppose $\mathbb{E}[X_{n+1} | \mathcal{F}_n] \leq X_n + Y_n$, with $\mathbb{E}[\sum_n Y_n] < \infty$. Prove that $X_n$ converges a.s. to a finite limit.
[Hint: Verify that $W_n = X_n - \sum_{i=1}^{n-1} Y_i$ is a supermartingale.]
Stochastics Preliminary Exam

Friday, January 4, 2019
9:00–13:00

This exam has 2 pages and 9 questions.

Unless otherwise mentioned, the events, sub σ-algebras, and random variables specified in each question are defined on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

**Question 1.** Let \(\{X_n\}_{n \geq 1}\) be a sequence of i.i.d. random variables with distribution \(\mu\). For \(A \in \mathcal{B}(\mathbb{R})\) with \(\mu(A) \in (0, 1)\), define

\[
\tau = \inf\{k \geq 1; X_k \in A\}.
\]

(1) Prove that \(\mathbb{P}(\tau < \infty) = 1\).

(2) Prove that \(X_\tau\) has distribution given by

\[
\mathbb{P}(X_\tau \in H) = \frac{\mu(H \cap A)}{\mu(A)}, \quad H \in \mathcal{B}(\mathbb{R}).
\]

**Question 2.** For every \(n \geq 1\), let \(X_n\) denote the maximum of \(n\) independent exponentially distributed random variables \(e_1, e_2, \ldots, e_n\), each with mean 1.

(1) Find the explicit form of \(\mathbb{P}(X_n < x)\) for all \(x \in \mathbb{R}\).

(2) Use (1) to show that

\[
X_n - \ln n \xrightarrow{\text{(d)}}_{n \to \infty} G,
\]

where \(G\) has a distribution function given by \(\mathbb{P}(G \leq x) = e^{-e^{-x}}, \ x \in \mathbb{R}\).

**Question 3.** Let \(\{X_n\}_{n \geq 1}\) be an arbitrary sequence of random variables. Show that the series \(\sum_{n=1}^{\infty} a_n X_n\) converges absolutely a.s. for some constants \(a_n\)'s.

**Question 4.** Use an appropriate random variable to construct \(A_n\)'s such that

\[
\mathbb{P}\left(\limsup_{n \to \infty} A_n\right) = 1 \quad \& \quad \sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty.
\]

**Question 5.** Let \(\{X_n\}_{n \geq 1}\) be a sequence of independent random variables such that \(Y_n = a_n \sum_{j=1}^{n} X_j\) converges in probability to a random variable \(Y_\infty\) for some constants \(a_n \to 0\). Show that \(Y_\infty\) is almost surely equal to a constant.
Question 6. Given any $\lambda \in (0, \infty)$, consider a random variable $X_\lambda$ taking values in $\mathbb{Z}_+$ with
\[ \mathbb{P}(X_\lambda = k) = \frac{e^{-\lambda} \lambda^k}{k!}. \]

(1) Show that $\mathbb{E}[e^{i\theta X_\lambda}] = e^{\lambda(e^{i\theta} - 1)}$ for all $\theta \in \mathbb{R}$.

(2) Show that $\mathbb{E}[X_\lambda] = \lambda$ and $\text{Var}(X_\lambda) = \lambda$.

(3) Use (1) and (2) to show that
\[ \lim_{N \to \infty} e^{-N} \left( 1 + N + \frac{N^2}{2} + \cdots + \frac{N^N}{N!} \right) = \frac{1}{2}. \]

Question 7. Let $\{X_n\}_{n \geq 1}$ be a sequence of independent random variables such that $\mathbb{E}[X_n] = 0$ for all $n \geq 1$,
\[ \sum_{n=1}^{\infty} \mathbb{E}\left[ |X_n| \mathbb{I}_{\{|X_n| > 1\}} \right] < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \mathbb{E}\left[ X_n^2 \mathbb{I}_{\{|X_n| \leq 1\}} \right] < \infty. \]
Prove that $\sum_{n=1}^{\infty} X_n$ converges a.s.

Question 8. Let $\{X_n\}_{n \geq 0}$ be a nonnegative $(\mathcal{F}_n)$-martingale and set
\[ \tau = \inf\{n \geq 0; X_n = 0\}. \]
Show that, for every $k \geq 0$, $X_{\tau+k} = 0$ a.s. on $\{\tau < \infty\}$.

Question 9. Let $\{X_n\}_{n \geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{E}[X_1]^p < \infty$ for some $p \in (0, \infty)$. Define
\[ Y_n = \frac{X_n}{n^{1/p}} \mathbb{I}_{\{|X_n| \leq n^{1/p}\}}, \quad \forall \ n \geq 1. \]
Show that for all $\alpha \in (p, \infty)$,
\[ \sum_{n=1}^{\infty} \mathbb{E}[|Y_n|^\alpha] \leq \frac{\alpha}{\alpha - p} (\mathbb{E}[|X_1|^p] + 1). \]

Hint: Define $E_j = \{(j-1)^{1/p} < |X_1| \leq j^{1/p}\}$ and show that
\[ \sum_{n=1}^{\infty} \mathbb{E}[|Y_n|^\alpha] \leq \sum_{j=1}^{\infty} \left( \frac{1}{j} - \frac{1}{\alpha/p + 1} \right) \int_{E_j} (|X_1|^p + 1) d\mathbb{P}. \]
Stochastics Preliminary Exam

Friday, August 17, 2018
9:00–13:00

This exam has 2 pages and 9 questions.

Unless otherwise mentioned, the events, sub $\sigma$-algebras, and random variables specified in each question are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

**Question 1**. Show that for any sequence of random variables $\{X_n\}$, there exists a sequence of strictly positive constants $\{a_n\}$ such that $a_nX_n$ converges to zero in probability.

**Question 2**. Let $\{X_n\}$ be a sequence of independent Gaussian random variables with $\mathbb{E}[X_n] = 0$ for all $n$. Find the probability of the following event:

$$\limsup_{n \to \infty} \{X_n X_{n+1} > 0\}.$$

**Question 3**. Let $\{A_n\}$ be a sequence of events such that $\lim_{n \to \infty} \mathbb{P}(A_n)$ exists. Prove that the following two properties of $\{A_n\}$ are equivalent:

1. $$\lim_{n \to \infty} \mathbb{P}(A_n \cap E) = \lim_{n \to \infty} \mathbb{P}(A_n) \cdot \mathbb{P}(E), \quad \forall \ E \in \mathcal{F}.$$  
2. $$\lim_{n \to \infty} \int_{A_n} X d\mathbb{P} = \lim_{n \to \infty} \mathbb{P}(A_n) \cdot \int_{\Omega} X d\mathbb{P}, \quad \forall \ X \in L_1(\mathbb{P}).$$

**Question 4**. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. nonnegative random variables such that for some $p_0 \in (0, \infty)$,

$$\mathbb{E}[X_1^p] < \infty, \forall \ p \in (0, p_0] \quad \& \quad \mathbb{E}[X_1^p] = \infty, \forall \ p \in (p_0, \infty).$$

Prove that, for $p$ ranging over $(0, \infty)$, we have

$$\limsup_{n \to \infty} \frac{X_n}{n^{1/p}} \begin{cases} = \infty \text{ a.s.} & \forall \ p \in (p_0, \infty); \\ < \infty \text{ a.s.} & \forall \ p \in (0, p_0]. \end{cases}$$

**Question 5**. Utilize series of the form $\sum_n 1/n^p$ to construct independent, nonnegative random variables $X_n$ such that $\sum_n X_n$ converges a.s. but $\sum_n \mathbb{E}[X_n]$ diverges.
Question 6. Let $\mathcal{P}_0$ denote the set of probability measures $\mu$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and define a subset $\mathcal{P}_1$ of $\mathcal{P}_0$ by

$$\mathcal{P}_1 = \left\{ \mu \in \mathcal{P}_0 \mid \int_{\mathbb{R}} x \mu(dx) = 0 \quad \& \quad \int_{\mathbb{R}} x^2 \mu(dx) = 1 \right\}.$$ 

Define a map $T : \mathcal{P}_1 \rightarrow \mathcal{P}_0$ by

$$T \mu(\Gamma) = \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{\Gamma} \left( \frac{x+y}{\sqrt{2}} \right) \mu(dx) \mu(dy), \quad \Gamma \in \mathcal{B}(\mathbb{R}).$$

1. Give a probabilistic interpretation to $T \mu$ by explaining how it can be identified as the distribution of a function of suitable random variables identically distributed as $\mu$.

2. Show that $T \mathcal{P}_1 \subseteq \mathcal{P}_1$.

3. Prove by the central limit theorem that the following fixed point equation admits a unique solution:

$$\mu = T \mu = \lim_{n \to \infty} T^n \mu$$

and the unique solution is given by $\mathcal{N}(0,1)$.

Question 7. Let $\{X_n\}$ be an $(\mathcal{F}_n)$-martingale such that

$$\sup_{\omega \in \Omega} |X_1(\omega)| \leq K \quad \& \quad \sup_{n \in \mathbb{N}, \omega \in \Omega} |X_n(\omega) - X_{n-1}(\omega)| \leq K$$

for some finite constant $K$.

1. Prove that $X_\tau$ is integrable for any $(\mathcal{F}_n)$-stopping time $\tau$ with $\mathbb{E}[\tau] < \infty$.

2. Use (1) to show that $\mathbb{E}[X_\tau] = \mathbb{E}[X_1]$ for all stopping times with the same property.

Question 8. Let $\{Y_n\}$ be a sequence of i.i.d. random variables such that $Y_n$ takes values in $\{1/2, 1, 3/2\}$ with probability $1/3$ each. Define $X_n = \prod_{i=1}^n Y_i$. Prove the following properties:

1. $\{X_n\}$ is a martingale.

2. $\lim_{n \to \infty} X_n = 0$ a.s. and then explain why $\mathbb{E}[\prod_{i=1}^\infty Y_i] < \prod_{i=1}^\infty \mathbb{E}[Y_i]$.

Question 9. Let $X \geq 0$ be a random variable with distribution function $F(t)$ such that $F(t) < 1$ for all $t \in \mathbb{R}$ and, for some $\eta \in (1, \infty)$,

$$\lim_{t \to \infty} \frac{1 - F(\eta t)}{1 - F(t)} = 0.$$ 

Show that $\mathbb{E}[X^m] < \infty$ for any $m \in (0, \infty)$.

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End of questions

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2
Stochastics Preliminary Exam
Friday, January 6, 2017
09:00-13:00

Name: ____________________________

Instructions:

1. Answer all questions and show all work. You will not receive credit if you do not justify your answers where it is necessary.

2. There are 9 questions worth a total of 100 points.

3. Remember to sign the Honor Pledge.

4. Before submitting your answers, staple everything together and use the front page of the exam with your name on it as the cover page of your final submission.

5. Good luck!

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the University Honor Code.

Pledge: ____________________________

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1. (12 points) Answer the following questions:
   a. Let \( \{X_n\}_{n \geq 1} \) be independent \( \mathcal{N}(0, 1) \) random variables. Show that \( \limsup_{n \to \infty} \frac{|X_n|}{\sqrt{\log n}} = \sqrt{2} \) a.s. 
      (Hint: \( 1 - \Phi(x) \sim x^{-1} \phi(x) \), where \( \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \), and \( \phi(x) \) the corresponding density.)
   b. Show that 
      \[
      \mathbb{P}(X_n > a_n \text{ i.o.}) = \begin{cases} 
      0, & \text{if } \sum \mathbb{P}(X_1 > a_n) < \infty \\
      1, & \text{if } \sum \mathbb{P}(X_1 > a_n) = \infty
      \end{cases}
      \]

2. (10 points) Let \( X \) and \( Y \) be independent variables following the exponential distribution with parameters \( \lambda \) and \( \mu \) respectively. Let \( U = \min\{X, Y\} \), \( V = \max\{X, Y\} \), and \( W = U - V \). Show that \( U \) and \( W \) are independent.

3. (12 points) Let \( X_1, X_2, \ldots, X_n \) be independent standard normal random variables. Find the distribution of the following random variables:
   (a) \( X_1^2 \)
   (b) \( \sum_{i=1}^{n} X_i^2 \)

4. (12 points) Answer the following questions:
   a. State (with details) one theorem which establishes the continuity of the expectation, \( \mathbb{E} \).
   b. Let \( X_n \) be a sequence of random variables satisfying \( X_n \leq Y \) a.s. for some \( Y \) with \( \mathbb{E}|Y| \leq \infty \). Show that 
      \[
      \mathbb{E}(\limsup_{n \to \infty} X_n) \geq \limsup_{n \to \infty} \mathbb{E}X_n.
      \]
   c. Let \( f_n(x) = \frac{\cos(x/n)}{x^2} \), for \( x \geq 1 \). Show that \( \int_1^{\infty} f_n(x) \, dx \to 1 \) as \( n \to \infty \).

5. (10 points) Let \( X_1, X_2, \ldots, X_n \) be i.i.d. whose common characteristic function \( \phi \) satisfies \( \phi'(0) = i\mu \). Show that \( \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{p} \mu \) as \( n \to \infty \).

6. (10 points) Let \( \{X_j\}_{j \geq 1} \) be i.i.d. \( \mathcal{N}(1, 3) \) random variables. Show that 
   \[
   \lim_{n \to \infty} \frac{X_1 + X_2 + \cdots + X_n}{X_1^2 + X_2^2 + \cdots + X_n^2} = \frac{1}{4} \text{ a.s.}
   \]

7. (12 points) If \( \tau_1 \) and \( \tau_2 \) are stopping times with respect to filtration \( \mathcal{F} \), show that \( \tau_1 + \tau_2, \max\{\tau_1, \tau_2\} \) and \( \min\{\tau_1, \tau_2\} \) are stopping times also.

8. (12 points) Let \( Y_n \) be a submartingale and let \( S \) and \( T \) be stopping times satisfying \( 0 \leq S \leq T \leq N \) for some deterministic \( N \). Show that \( \mathbb{E}(Y_0) \leq \mathbb{E}(Y_S) \leq \mathbb{E}(Y_T) \leq \mathbb{E}(Y_N) \).

9. (10 points) Let \( \{X_r, r \geq 1\} \) be independent Poisson variables with respective parameters \( \{\lambda_r, r \geq 1\} \). Show that \( \sum_{r=1}^{\infty} X_r \) converges or diverges almost surely according as \( \sum_{r=1}^{\infty} \lambda_r \) converges or diverges.

End of exam.
Stochastics Preliminary Exam
Monday, August 8, 2016
09:00-13:00

Name: ________________________________

Instructions:

1. Answer all questions and show all work. You will not receive credit if you do not justify your answers where it is necessary.

2. There are 10 questions worth a total of 100 points.

3. Remember to sign the Honor Pledge.

4. Before submitting your answers, staple everything together and use the front page of the exam with your name on it as the cover page of your final submission.

5. Good luck!

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the University Honor Code.

Pledge: ________________________________

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1. (10 points) If $X_1, X_2, \ldots, X_n$ a sequence of independent identically distributed random variables with pdf $f$, find the probability density function of the order statistics, $X_{(1)} = \min(X_1, X_2, \ldots, X_n)$, and $X_{(n)} = \max(X_1, X_2, \ldots, X_n)$, respectively.

2. (10 points) Let $X_r, r \geq 1$ be independent, non-negative and identically distributed with infinite mean. Show that $\limsup_{r \to \infty} \frac{\bar{X}}{r} = \infty$ a.s.

3. (10 points) Answer the following questions:
   (a) Let $X_1, X_2, \ldots, X_n$ be independent exponential variables with parameter $\lambda$. Show that $S_n = \sum_{i=1}^{n} X_i$ has the $\text{Gamma}(n, \lambda)$ distribution.
   (b) If $X, Y$ are two independent random variables distributed according to $\text{Gamma}(m, \lambda)$ and $\text{Gamma}(n, \lambda)$, respectively, then compute the distribution of $X + Y$.

4. (10 points) Let $X_1, X_2, \ldots$ be independent $N(0,1)$ variables. Find the characteristic functions of the following random variables:
   (a) $X_i^2$
   (b) $\sum_{i=1}^{n} X_i^2$

5. (10 points) Provide a (counter)example to each of the following claim.
   (a) There exist sequences of random variables which converge a.s. but not in mean.
   (b) There exist sequences of random variables which converge in mean but not a.s.
   (c) If a sequence converges in probability then it does not necessarily converge in mean.
   (d) If $r > s \geq 1$, and a sequence converges in $L^s$, then it does not necessarily converge in $L^r$.

6. (10 points) Let $X_1, X_2, \ldots$ be a sequence of independent non-negative random variables and let $N(t) = \max\{n : \sum_{i=1}^{n} X_i \leq t\}$. Show that $N(t) + 1$ is a stopping time with respect to a suitable filtration to be specified.

7. (10 points) Let $X_1, X_2, \ldots$ be independent identically distributed random variables with common density function $f$. Suppose that it is known that $f(\cdot)$ is either $p(\cdot)$ or $q(\cdot)$, where $p, q$ are given (different) densities. The statistical problem is to decide which of the two is the true. For this we consider the likelihood ratio

$$Y_n = \frac{p(X_1)q(X_2)\cdots q(X_n)}{p(X_1)p(X_2)\cdots p(X_n)},$$

and we adopt the strategy that $f = p$ if $Y_n \geq a$ or $f = q$ otherwise. Let $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$ the filtration generated by $X_1, X_2, \ldots, X_n$. Is the likelihood ratio, $Y_n$, a martingale? If yes, under what constraint?

8. (10 points) Let $X_n, Y_n$ some random variables. Answer the following questions:
   (a) Suppose that $X_n$ converges in distribution to $X$ and $Y_n$ in probability to some constant $c$. Show that the product $X_nY_n$ converges in distribution to $cX$
   (b) Suppose that $X_n$ converges in distribution to 0 and $Y_n$ in probability to $Y$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be such that $g(x, y)$ is a continuous function of $y$ for all $x$, and $g(x, y)$ is continuous at $x = 0$ for all $y$. Show that $g(X_n, Y_n)$ converges in probability to $g(0, Y)$.

Continued...
9. (10 points) Let $X_r, 1 \leq r \leq n$ be independent and identically distributed with mean $\mu$ and finite variance $\sigma^2$. Let $\bar{X} = \frac{1}{n} \sum_{r=1}^{n} X_r$. Show that
\[
\frac{\sum_{r=1}^{n} (X_r - \mu)}{\sqrt{\sum_{r=1}^{n} (X_r - \bar{X})^2}}
\]
converges in distribution to $N(0, 1)$.

10. (10 points) Let $X_1, X_2, \ldots$ be independent identically distributed random variables with

\[
X_n = \begin{cases} 
1 & \text{with probability } \frac{1}{2^n} \\
0 & \text{with probability } 1 - \frac{1}{n} \\
-1 & \text{with probability } \frac{1}{2^n}
\end{cases}
\]

Let $Y_1 = X_1$ and for $n \geq 2$

\[
Y_n = \begin{cases} 
X_n & \text{if } Y_{n-1} = 0 \\
ny_{n-1}X_n & \text{if } Y_{n-1} \neq 0
\end{cases}
\]

Show that

(a) $Y_n$ is a martingale with respect to $\mathcal{F}_n = \sigma(Y_1, Y_2, \ldots, Y_n)$
(b) $Y_n$ does not converge almost surely. Does it converge in any way?
(c) The martingale convergence theorem does not apply.
Instructions:

- There are a total of nine (9) problems. An answer without an explanation may receive no credit.

- Throughout the exam, \((\Omega, \mathcal{F}, P)\) is a fixed probability space and given random variables are assumed to be defined on this probability space and take values in \(\mathbb{R}\). The expression \(1_A\) denotes the indicator function of a set \(A\).

1. (10 points) Let \(X\) be a random variable and let \(\sigma(X)\) denote the \(\sigma\)-algebra generated by \(X\). Show that a map \(Y : \Omega \to \mathbb{R}\) is \(\sigma(X)\)-measurable if and only if there exists a Borel measurable function \(f : \mathbb{R} \to \mathbb{R}\) such that \(Y = f(X)\).

2. (12 points) Let \(X\) and \(Y\) be independent Poisson random variables with respective parameters \(\lambda\) and \(\mu\).
   
   (a) Show that \(X + Y\) has a Poisson distribution with parameter \(\lambda + \mu\).
   
   (b) According to Problem 1, the conditional expectation \(E[X|X+Y]\) can be written as a function of \(X + Y\); i.e. there exists \(f : \mathbb{R} \to \mathbb{R}\) such that
   
   \[E[X|X + Y] = f(X + Y).\]
   
   Specify the function \(f\).

3. (10 points) Let \((X_n)_{n \geq 1}\) be a sequence of random variables and \(X\) be a random variable. Show that \(X_n \to X\) in probability as \(n \to \infty\) if and only if
   
   \[
   \lim_{n \to \infty} E[|X_n - X| \wedge 1] = 0,
   \]
   
   where \(a \wedge b := \min\{a, b\}\).

4. (10 points) Let \((X_n)_{n \geq 1}\) be a sequence of i.i.d. random variables with \(P(X_n = 1) = P(X_n = -1) = 1/2\). What can be concluded about the probability
   
   \[
P\left(\sum_{n=1}^{\infty} \frac{X_n}{n} \text{ converges} \right)\? \]

5. (10 points) Let \(X\) be an integrable random variable. Let \(\Lambda\) be the family of all sub-\(\sigma\)-algebras of \(\mathcal{F}\). Let \(Y_G := E[X|G]\) for \(G \in \Lambda\). Show that the family \((Y_G)_{G \in \Lambda}\) is uniformly integrable.
6. (10 points) Using the central limit theorem, prove that
\[
\lim_{n \to \infty} \left( \frac{\sqrt{n}}{a} \sin \frac{a}{\sqrt{n}} \right)^n = e^{-a^2/6},
\]
where \(a\) is a nonzero real constant.

**Hint:** Consider an i.i.d. sequence of uniform random variables on \([-1, 1]\).

7. (14 points) Let \((X_n)_{n \geq 1}\) be a sequence of i.i.d. random variables with common density function \(f(x) = e^{-x}1_{(0, \infty)}(x), x \in \mathbb{R}\). For a fixed \(\theta < 1\), let
\[
Y_n := (1 - \theta)^n e^{\theta S_n}
\]
for \(n \geq 1\), where \(S_n = \sum_{k=1}^n X_k\).

(a) Show that there exists \(Y_\infty \in L^1\) such that \(Y_n \to Y_\infty \) a.s.

(b) Suppose further that \(\theta \neq 0\). Find the a.s. limit of the sequence \((\frac{1}{n} \log Y_n)\), and deduce that \(Y_\infty = 0 \) a.s.

8. (10 points) Let \((X_n)_{n \geq 1}\) be a sequence of random variables. Let \(f : \mathbb{R} \to \mathbb{R}\) be a nonnegative Borel measurable function such that \(f(x) \to \infty\) as \(|x| \to \infty\) and \(\sup_{n \geq 1} E[f(X_n)] < \infty\). Show that \((X_n)\) is tight.

9. (14 points) Let \((A_n)_{n \geq 1}\) be events in \(\mathcal{F}\) satisfying the following two conditions:

- \(\lim_{n \to \infty} \frac{\sum_{j,k=1}^n P(A_j \cap A_k)}{\left(\sum_{j=1}^n P(A_j)\right)^2} = 1\).
- \(\sum_{n=1}^\infty P(A_n) = \infty\).

(a) Show that
\[
\lim_{n \to \infty} \inf P\left(\left|\sum_{k=1}^n 1_{A_k} - \sum_{k=1}^n P(A_k)\right| > \frac{1}{2} \sum_{k=1}^n P(A_k)\right) = 0.
\]

(b) Show that there is a subsequence \(\{n_m\}_{m=1}^\infty\) of \(N\) such that with probability 1,
\[
\sum_{k=1}^{n_m} 1_{A_k} \geq \frac{1}{2} \sum_{k=1}^{n_m} P(A_k) \quad \text{for sufficiently large } m.
\]

**Hint:** Consider the sequence \(c_n := P\left(\sum_{k=1}^n 1_{A_k} < \frac{1}{2} \sum_{k=1}^n P(A_k)\right)\).

(c) Show that \(P(\lim \sup_{n \to \infty} A_n) = 1\).

— END OF EXAM —
Instructions:

• There are a total of eight (8) problems. An answer without an explanation may receive no credit.

• Throughout the exam, \((\Omega,\mathcal{F},P)\) is a fixed probability space and given random variables are assumed to be defined on this probability space and take values in \(\mathbb{R}\). The expression \(1_A\) denotes the indicator function of a set \(A\).

1. (16 points)
   
   (a) Suppose that \(X\) is an integrable random variable and \(\{A_n\}_{n \geq 1}\) is a sequence of events in \(\mathcal{F}\) such that \(P(A_n) \to 0\) as \(n \to \infty\). Show that \(E[X1_{A_n}] \to 0\) as \(n \to \infty\).

   (b) Suppose that \(\{X_n\}_{n \geq 1}\) is a sequence of random variables such that \(X_n \to X\) in probability for some random variable \(X\). Suppose further that there exists a random variable \(Y \in L^p\) for some \(p \geq 1\) such that \(|X_n| \leq Y\) a.s. for all \(n \geq 1\). Prove that \(X_n \to X\) in \(L^p\).

2. (12 points) Let \(\{X_n\}_{n \geq 1}\) be a sequence of i.i.d. random variables with common density \(f(x) = xe^{-x^2/2}1_{(0,\infty)}(x), x \in \mathbb{R}\). Show that

\[
\limsup_{n \to \infty} \frac{X_n}{\sqrt{2\log n}} = 1 \text{ a.s.}
\]

3. (14 points)
   
   (a) Show that a random variable \(X\) is symmetric (i.e. \(X =^d -X\)) if and only if its characteristic function is real-valued.

   (b) Show that if \(X\) and \(Y\) are i.i.d. random variables, then \(X - Y\) is symmetric.
4. (10 points) Let \( \varphi(t) \), \( t \in \mathbb{R} \), be the characteristic function of a random variable \( X \). Show that if \( \varphi(t) = 1 \) in a neighborhood of 0, then \( X = 0 \) a.s.

**Hint:** Show that \( 1 - \text{Re}[\varphi(2t)] \leq 4(1 - \text{Re}[\varphi(t)]) \) for \( t \in \mathbb{R} \).

5. (10 points) Let \( \{X_n\}_{n \geq 1} \) be a sequence of i.i.d. random variables with common mean \( \mu \) and variance \( \sigma^2 \in (0, \infty) \). Use Slutsky’s theorem to show that

\[
\sqrt{n}(e^{S_n/n} - e^{\mu}) \Rightarrow \sigma e^{\mu} \chi,
\]

where \( S_n = \sum_{k=1}^{n} X_k \) and \( \chi \sim N(0,1) \).

6. (10 points) Show that if \( \{X_n\}_{n \geq 1} \) is a sequence of random variables such that \( \sup_{n \geq 1} E[|X_n|^p] < \infty \) for some \( p > 0 \), then \( \{X_n\}_{n \geq 1} \) is tight.

7. (10 points) For a filtration \( (\mathcal{F}_n)_{n \geq 0} \) on the probability space \( (\Omega, \mathcal{F}, P) \), let \( (M_n)_{n \geq 0} \) be an \( (\mathcal{F}_n) \)-submartingale and let \( (H_n)_{n \geq 1} \) be an \( (\mathcal{F}_n) \)-predictable process such that each \( H_n \) is nonnegative and bounded. Show that the stochastic process \( (Y_n)_{n \geq 0} \) defined by

\[
Y_0(\omega) := 0, \quad Y_n(\omega) := \sum_{k=1}^{n} H_k(\omega)(M_k(\omega) - M_{k-1}(\omega)), \quad n \geq 1, \ \omega \in \Omega,
\]

is an \( (\mathcal{F}_n) \)-submartingale.

8. (18 points) Let \( \{X_n\}_{n \geq 0} \) be a sequence of i.i.d. nonnegative random variables such that \( E[X_0] = 1 \) and \( P(X_0 = 1) < 1 \). Let \( Y_n := \prod_{j=0}^{n} X_j \) for \( n \geq 0 \).

(a) Show that \( Y_n \to Y_\infty \) a.s., where \( Y_\infty \) is a finite random variable.

(b) For fixed \( \epsilon, \delta > 0 \) and \( n \geq 0 \), show that

\[
P(|Y_{n+1} - Y_n| > \epsilon \delta) \geq P(|Y_n| > \epsilon)P(|X_0 - 1| > \delta).
\]

(c) Show that \( Y_\infty = 0 \) a.s.

— END OF EXAM —

2
Name: ____________________________________________________________________________

1. (14 points) (a) Let $A_n$ be events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that

$$\mathbb{P}(\liminf_{n \to \infty} A_n) \leq \liminf_{n \to \infty} \mathbb{P}(A_n) \leq \limsup_{n \to \infty} \mathbb{P}(A_n) \leq \mathbb{P}(\limsup_{n \to \infty} A_n).$$

(b) Let $X_n$ be i.i.d. random variables such that $\mathbb{P}(X_n = 1) = p$, $\mathbb{P}(X_n = 0) = 1 - p$, with $p \in (0, 1)$. Let $s$ be any $m$-long sequence of zeros and ones and let $A_n = \{\omega : (X_n(\omega), \ldots, X_{n+m}(\omega)) = s\}$. Show that $A := \{A_n \text{ i.o.}\}$ is a tail event for $\{X_n\}$ and determine $\mathbb{P}(A)$.

2. (10 points) Let $X_n$ be random variables such that for some $a_n \in \mathbb{R}$

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n \neq a_n) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} a_n \quad \text{converges}.$$

Show that $\sum_{n=1}^{\infty} X_n$ converges a.s.

3. (10 points) Let $X$ be a random variable.

(a) $X$ is independent of itself (i.e., $X$ and $X$ are independent) if and only if there is a constant $c$ such that $\mathbb{P}(X = c) = 1$. [Hint: Consider the distribution function of $X$.]

(b) $X$ is independent of $g(X)$ for some measurable function $g : \mathbb{R} \mapsto \mathbb{R}$ if and only if there is a constant $c$ such that $\mathbb{P}(g(X) = c) = 1$.

4. (12 points) Let $\{X_n\}$ be i.i.d. Normal$(0, 1)$ random variables. Show that

$$\mathbb{P}\left( \limsup_{n \to \infty} \frac{|X_n|}{\sqrt{\log n}} = \sqrt{2} \right) = 1.$$

[Hint: You may use $\int_{x=\infty}^{\infty} e^{-u^2/2} du \sim \frac{1}{\sqrt{x}} e^{-x^2/2}$ as $x \to \infty$.]

5. (12 points) Let $X_n$ be random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_n \Rightarrow X$. Suppose that $\mathbb{P}(X > b) = \delta > 0$. Show that $\mathbb{P}(X_n \geq b \text{ i.o.}) \geq \delta$. [Hint: You may use Problem 1 (a).]

6. (10 points) Let $X_j$ be i.i.d. random variables with the common distribution function $F$. Define for $x \in \mathbb{R}$

$$Y_j(x) = 1_{\{X_j \leq x\}}$$

and

$$F_n(x) = \frac{1}{n} \sum_{j=1}^{n} Y_j(x).$$

Show that $\forall x \in \mathbb{R}$,

(i) $$\lim_{n \to \infty} F_n(x) = F(x) \quad \text{a.s.}$$

and

(ii) $$\sqrt{n}(F_n(x) - F(x)) \Rightarrow Z$$

where $Z$ is normal with mean zero and variance $\sigma^2(x) = F(x)(1 - F(x))$. 
7. (10 points) Let \( \{X_n\} \) be i.i.d. random variables with \( \mu = E(X_1) \) and \( \sigma^2 = \text{Var}(X_1) < \infty \). Put \( \bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i \). Let \( h \) be a measurable function that is differentiable at \( \mu \) and \( h'(\mu) \neq 0 \). Show that
\[
\sqrt{n} \sigma^{-1}(h(\bar{X}_n) - h(\mu)) \Rightarrow h'(\mu)Z \quad \text{as } n \to \infty,
\]
where \( Z \) is a standard normal random variable. [Hint: Consider \( \frac{h(X_n) - h(\mu)}{X_n - \mu} \) as \( n \to \infty \).]

8. (10 points) Let \( (\Omega, \mathcal{F}, P) \) be a probability space and \( \mathcal{G} \) a sub-\( \sigma \)-algebra of \( \mathcal{F} \). Let \( X \) be an integrable random variable such that \( E(X \mid \mathcal{G}) \leq X \) a.s. Show that \( X = E(X \mid \mathcal{G}) \) a.s.

9. (12 points) A martingale \( \{X_n\} \) is bounded in \( L^2 \) if \( \sup_n E|X_n|^2 < \infty \). Show that a martingale \( \{X_n\} \) is bounded in \( L^2 \) if and only if \( E|X_n|^2 < \infty \) for each \( n \) and
\[
\sum_{n=1}^{\infty} E((X_{n+1} - X_n)^2) < \infty.
\]
1. (10 points) Show the following extension of subadditivity: for any events $B_i \subset A_i$

$$P \left( \bigcup_{i=1}^{n} A_i \right) - P \left( \bigcup_{i=1}^{n} B_i \right) \leq \sum_{i=1}^{n} \left[ P(A_i) - P(B_i) \right]$$

where $n \leq \infty$.

2. (12 points) For any $n \in \mathbb{N}$, let $\{X_{n,k} : 1 \leq k \leq n\}$ be i.i.d. random variables such that

$0 \leq X_{n,k} \leq C$ (same constant $C$ for all $n$ and $k$), and let $S_n = \sum_{k=1}^{n} X_{n,k}$. Show that if $\mu_n := \mathbb{E}S_n \to \infty$, then $S_n \overset{P}{\to} \infty$ (that is, $\forall M > 0\, \mathbb{P}(S_n > M) \to 1$).

(Hint: Since $\mu_n \to \infty$, it is enough to show that $\mathbb{P}(S_n \in \left(\frac{\mu_n}{2}, \frac{3\mu_n}{2}\right)) \to 1$.)

3. (12 points) Let $\{X_n\}$ be i.i.d. random variables having a Weibull distribution with density $f(x) = 3x^2 \exp(-x^3)$ for $x \geq 0$ and $f(x) = 0$ otherwise. Show that

$$\mathbb{P} \left( \limsup_{n \to \infty} \frac{X_n}{\sqrt[3]{\log n}} = 1 \right) = 1.$$ 

4. (14 points) Let $\{X_j\}$ be i.i.d. random variables and let $N$ be a Poisson($\lambda$) random variable independent of $\{X_j\}$. Let $S_n = \sum_{j=1}^{n} X_j$ and consider $S_N$.

(a) Find the characteristic function of $S_N$.

(b) Find $\mathbb{E}S_N$ and $\text{Var}(S_N)$.

(c) Let $\{\epsilon_j\}$ be i.i.d. random variables independent of $\{X_j\}$ and $N$ such that $\mathbb{P}(\epsilon_j = 1) = p$, $\mathbb{P}(\epsilon_j = 0) = 1 - p$. Show that $S'_N := \sum_{j=1}^{N} \epsilon_j X_j$ and $S''_N := \sum_{j=1}^{N} (1 - \epsilon_j) X_j$ are independent.

5. (10 points) Let $X_n$ be independent random variables such that $X_n \overset{p}{\to} X$.

(a) Show that there is a constant $c$ such that $\mathbb{P}(X = c) = 1$.

(b) If $X_n$ are iid and not equal to a constant, then no such $X$ exists.

6. (10 points) Let $X_j$ be i.i.d. and $S_n = \sum_{j=1}^{n} X_j$. Show that if $\frac{S_n}{n} \to 0$ a.s., then $\mathbb{E}|X_j| < \infty$ and also $\mathbb{E}X_j = 0$.

7. (12 points) (a) Suppose $X$ and $Y$ are i.i.d. $N(0,1)$. Show $\frac{X + Y}{\sqrt{2}} \overset{d}{=} X \overset{d}{=} Y$.

(b) Conversely: Suppose $X$ and $Y$ are i.i.d. with mean zero and variance 1, and suppose further that $\frac{X + Y}{\sqrt{2}} \overset{d}{=} X \overset{d}{=} Y$.

Show that both $X$ and $Y$ have a $N(0,1)$ distribution. (Use the Central Limit Theorem.)

8. (10 points) Let $\mathcal{A}_1 \subset \mathcal{A}_2$ be sub-$\sigma$-algebras of $\mathcal{F}$. Show that for any random variable $X \in L^2$

$$\mathbb{E}((X - \mathbb{E}(X|\mathcal{A}_2))^2) \leq \mathbb{E}((X - \mathbb{E}(X|\mathcal{A}_1))^2).$$

Discuss extremal cases of this inequality: $\mathcal{A}_1 = \{\emptyset, \Omega\}$ and $\mathcal{A}_2 = \mathcal{F}$.

9. (10 points) Let $\{X_n\}$ be a martingale such that $Y_n = \frac{X_{n+1}}{X_n} \in L^1$. Show that $\mathbb{E}Y_n = 1$ and, if $Y_n \in L^2$, then $\text{Cov}(Y_n, Y_{n+1}) = 0$. 
Stochastics Preliminary Exam
Friday, January 3, 2014

09:00-13:00

Name: ________________________________

Instructions:

1. Answer all questions and show all work. You will not receive credit if you do not justify your answers where it is necessary.

2. There are 7 questions worth a total of 100 points on 2 numbered pages. Please check the pages before start working.

3. Remember to sign the Honor Pledge.

4. Before submitting your answers, staple everything together and use the front page of the exam with your name on it as the cover page of your final submission.

5. Good luck!

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the University Honor Code.

Pledge: ________________________________
1. Let $A_r, r \geq 1$ be events such that $P(A_r) = 1$ for each $r$. Show that $P(\cap_{r=1}^{\infty} A_r) = 1$.

2. Consider two random variables $X_t$ and $Z_t$ whose distributions change with time $t$. One may observe $Z_t$ but not $X_t$ and the conditional distribution $Z_t|X_t = x_t \sim p(Z_t|x_t)$ is well-defined, where $x_t$ is the state of $X_t$ at time $t$. Denote $x_{0:t} = \{x_0, \ldots, x_t\}$ the whole history of states where the random variable $X$ visited from time 0 to time $t$. Furthermore, consider that the transition density $p(X_t|X_{0:t-1} = x_{0:t-1}) = p(X_t|X_{t-1} = x_{t-1})$ is known.

   (a) Employing Bayes theorem calculate the posterior distribution $p(X_t|Z_{0:t} = z_{0:t})$.
   (b) If $Z_t|X_t = x_t \sim N(x_t, 1)$ and $X_t|X_{t-1} = x_{t-1} \sim N(x_{t-1}, 1)$ then give a closed form of the posterior distribution $p(X_t|Z_{0:t} = z_{0:t})$.

3. (a) Consider a sequence of random variables $\{X_n\}$ which converge to some limit $X$. Define the a.s., in probability, in distribution and in $L^p$ convergence and state (without any proof) any relationship among these types of convergence.
   (b) Suppose $X_n \xrightarrow{\text{L}} X$. Show that $\text{Var}(X_n) \rightarrow \text{Var}(X)$ as $n \rightarrow \infty$.

4. (a) Let $\{X_n\}_{n \geq 1}$ be independent and identically distributed (i.i.d.) where $X_1 \sim \text{Exp}(1)$. Show that $\limsup_{n \rightarrow \infty} \frac{X_n}{\log n} = 1$ a.s.
   (b) Let $\{X_n\}_{n \geq 1}$ be i.i.d., non-negative with infinite mean. Show that $\limsup_{n \rightarrow \infty} \frac{X_n}{n} = \infty$ a.s.

5. Let $Y_n$ be a submartingale and let $S$ and $T$ be stopping times satisfying $0 \leq S \leq T \leq N$ for some deterministic $N$. Show that $E(Y_0) \leq E(Y_S) \leq E(Y_T) \leq E(Y_N)$.

6. Let $Y$ be a submartingale, $u$ a convex non-decreasing function mapping $\mathbb{R}$ to $\mathbb{R}$. Show that $\{u(Y_n) : n \geq 0\}$ is a submartinagle provided that $E\text{u}(Y_n)^+ < \infty$ for all $n$.

7. (a) Let $X_1, X_2, \cdots, X_n$ be independent random variables with characteristic functions $\phi_1, \phi_2, \cdots, \phi_n$. Match the following characteristic functions (dots) with the random variables (numbers). Place your selection in the brackets.

   - $\prod_{i=1}^{n} \phi_i(t)$: [ ]
   - $|\phi_1(t)|^2$: [ ]
   - $\sum_{j=1}^{n} p_j \phi_j(t)$, where $p_j \geq 0$ and $\sum_{i=1}^{n} p_j = 1$: [ ]
   - $(2 - \phi_1(t))^{-1}$: [ ]
   - $\int_0^{\infty} \phi_1(ut)e^{-ut}du$: [ ]

   1. $\sum_{i=1}^{N} X_i$, where $N$ is a random variable with $P(N = j) = p_j$ for $1 \leq j \leq n$, independent of $X_1, X_2, \cdots, X_n$
   2. $\sum_{i=1}^{n} X_i$
   3. $X_1 - X'_1$, where $X_1, X'_1$ are i.i.d.
   4. $Y X_1$, where $Y$ is independent of $X_1$ and follows Exp(1).
   5. $\sum_{j=1}^{M} Z_j$, where $Z_1, Z_2, \cdots$ are independent and distributed as $X_1$ and $M$ is independent of the $Z_j$ with $P(M = m) = \frac{1}{2m+1}$, for $m \geq 0$

   (b) Is the function $\phi(t) = (1 + t^4)^{-1}$ a characteristic function?

End of exam.
Stochastics Preliminary Exam

Monday, August 12, 2013

09:00-13:00

Name: _______________________

Instructions:

1. Answer all questions and show all work. You will not receive credit if you do not justify your answers where it is necessary.

2. There are 10 questions worth a total of 100 points.

3. Remember to sign the Honor Pledge.

4. Before submitting your answers, staple everything together and use the front page of the exam with your name on it as the cover page of your final submission.

5. Good luck!

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the University Honor Code.

Pledge: _______________________

.
1. Suppose \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space and \(B \in \mathcal{F}\) satisfies \(\mathbb{P}(B) > 0\). Let \(Q : \mathcal{F} \to [0, 1]\) be defined by \(Q(A) = \mathbb{P}(A|B)\). Show that \((\Omega, \mathcal{F}, Q)\) is a probability space. If \(C \in \mathcal{F}\) and \(Q(C) > 0\), show that \(Q(A|C) = \mathbb{P}(A|BC)\).

2. There are 10 coins in a bag. Five of them are normal coins (one head and one tail), one coin has two heads and four coins have two tails. You pull one coin out, look at one of its sides and see that it is a tail. What is the probability that it is a normal coin?

3. Let \(X\) be a nonnegative random variable following a distribution \(F\). Show that \(\mathbb{E}X = \int_0^\infty (1 - F(x))dx\).

4. Suppose that the real-valued random variables \(\xi, \eta\) are independent, that \(\xi\) has a bounded density \(p(x)\) (for \(x \in \mathbb{R}\), with respect to Lebesgue measure), and that \(\eta\) is integer valued.
   
   i. Prove that \(\zeta = \xi + \eta\) has a density.
   
   ii. Calculate the density of \(\zeta\) in the case where \(\xi \sim \text{Uniform}[0,1]\) and \(\eta \sim \text{Poisson}(1)\).

5. Let \(X_1, X_2, \ldots\) be independent random variables with zero means and \(S_n = X_1 + \cdots + X_n\). Let \(M_n = \max_{1 \leq k \leq n} |S_k|\).
   
   i. Show that \(\mathbb{E}(S_n^2 I_{A_k}) > c^2 \mathbb{P}(A_k)\), where \(A_k = \{M_{k-1} \leq c < M_k\}\) and \(c > 0\).
   
   ii. Deduce Kolmogorov's inequality: \(\mathbb{P}(M_n > c) \leq \frac{\mathbb{E}(S_n^2)}{c^2}, c > 0\).

6. Let \(X_t, t > 0\) be random variables defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Assuming that \(E|X_t|^2 = E|X|^2 < \infty\) for all \(t\), prove that \(\mathbb{P}(\lim_{t \to 0} X_t = X) = 1\) implies \(\lim_{t \to 0} E|X_t - X|^2 = 0\), i.e. under the above assumptions, almost sure convergence implies convergence in mean square.

7. Let \(\{X_n\}\) be a sequence of independent random variables which converges in probability to the limit \(X\). Show that \(X\) is almost surely constant.

8. Let \(X_n\) be independent and identically distributed random variables with \(\mathbb{P}(X_n = 1) = \frac{1}{2}\) and \(\mathbb{P}(X_n = -1) = \frac{1}{2}\). Show that if \(S_n = \sum_{j=1}^n X_j\) then
   
   i. \(\frac{1}{n} S_n \xrightarrow{p} 0\)
   
   ii. \(\frac{1}{n^2} S_n^2 \xrightarrow{a.s.} 0\)

9. Let \(X_1, X_2, \ldots\) be independent identically distributed random variables such that \(\mathbb{E}X_n = 0\) and \(|X_n| \leq 1\) a.s. Define \(S_n = \sum_{k=1}^n X_k\). Find a number \(c\) such that \(S_n^2 - cn\) is a martingale and justify the martingale property.

10. Let \((Y, \mathcal{F})\) be a martingale and suppose that there exists a sequence \(K_1, K_2, \ldots\) of real numbers such that \(\mathbb{P}(|Y_n - Y_{n-1}| \leq K_n) = 1\) for all \(n\). Show that \(\mathbb{P}(|Y_n - Y_0| \geq x) \leq 2 \exp\left(-\frac{x^2}{\sum_{i=1}^n K_i}\right), x > 0\). (Hint: Consider a random variable \(D\) with 0 mean and \(\mathbb{P}(|D| \leq 1) = 1\). Use the inequality \(e^{\psi} \leq \frac{1}{2}(1 - d)e^{-\psi} + \frac{1}{2}(1 + d)e^\psi\), if \(|d| \leq 1, \psi > 0\) to bound \(\mathbb{E}(e^{\psi D})\). Furthermore, find a pertinent \(D\) based on the hypothesis of the problem.)

End of exam.
1. (11 points) Let $X$ be a random variable with $\mathbb{E}X = 0$ and finite variance $\sigma^2$.
   a) For any $t$, show that
      \[
      \mathbb{E}\frac{X^2}{X^2 + t^2} \leq \frac{\sigma^2}{\sigma^2 + t^2}.
      \]
   b) Use last inequality to prove that for $t > 0$,
      \[
      \mathbb{P}(X \leq t) \geq \frac{t^2 - \sigma^2}{t^2 + \sigma^2}.
      \]

2. (12 points) A biased coin is tossed repeatedly. Each time there is a probability $p$ of a head turning up. Let $p_n$ be the probability that an even number of heads has occurred after $n$ tosses. Show that $p_0 = 1$ and
   \[
p_n = p + (1 - 2p)p_{n-1} \quad \text{if } n \geq 1.
   \]

Solve this difference equation.

3. (12 points) A telephone sales company attempts repeatedly to sell new kitchens to each of the $N$ families in a village. Family $i$ agrees to buy a new kitchen after it has been solicited $K_i$ times, where the $K_i$ are i.i.d. random variables with probability mass function
   \[
f(n) = \mathbb{P}(K_i = n), \quad n = 1, 2, \ldots.
   \]

Let $X_n$ be the number of kitchens sold at the $n$th round of solicitations, so that
   \[
   X_n = \sum_{i=1}^{N} 1_{K_i = n}.
   \]

Suppose that $N$ is a Poisson random variable (independent of $K_i$'s) with parameter $\nu$. 

1
a) Show that the $X_n$ are independent random variables, $X_r$ having the Poisson distribution with parameter $\nu f(r)$. (Hint: Use characteristic function).

b) Let

$$T = \inf\{n : X_n = 0\} \text{ and } S = X_1 + X_2 + \cdots + X_T.$$  

Show that

$$\mathbb{E}(S) = \nu \mathbb{E}(T),$$  

where

$$F(k) = \sum_{j=1}^{k} f(j).$$  

4. (12 points) Let $X_n, n = 1, 2, \cdots$ be i.i.d. random variable with common probability density function $f(x) = e^{-x}, x > 0$. Show that

$$\limsup_{n \to \infty} \frac{X_n}{\log n} = 1, \quad a.s.$$  

Hint: Prove that

$$P\left(\limsup_{n} A_{n}^{1-\varepsilon}\right) = 1 \quad \text{and} \quad P\left(\limsup_{n} A_{n}^{1+\varepsilon}\right) = 0,$$  

where

$$A_{n}^{c} = \left\{ \frac{X_n}{\log n} > c \right\}.$$  

5. (10 points) Let $f$ be an increasing function on interval $[a, b]$. Use probability method to show that

$$\int_{a}^{b} 2xf(x)dx \geq (a + b) \int_{a}^{b} f(x)dx.$$  

6. (12 points) a) Let $X_\lambda$ be a Poisson random measure with parameter $\lambda$. Prove that $\frac{X_\lambda}{\sqrt{\lambda}}$ converges in distribution. Identify the limiting distribution.

b) Prove that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n-1} \frac{(nt)^k}{k!} = \begin{cases} 
1 & \text{if } 0 < t < 1, \\
\frac{1}{2} & \text{if } t = 1, \\
0 & \text{if } t > 1.
\end{cases}$$
(Hint: Use part a.)

7. (11 points) For any nonnegative random variable $X$, prove that

$$EX \leq \sum_{n=0}^{\infty} P(X > n) \leq EX + 1.$$ 

8. (10 points) Let $X$ and $Y$ be two integrable random variables. Prove that for any bounded Borel function $f$,

$$E\left( (X - f(Y))^2 \right) \geq E\left( (X - E(X|Y))^2 \right).$$

9. (10 points) Let $\{X_n : n = 0, 1, 2, \cdots\}$ be a sequence of random variables. Define

$$\mathcal{F}_n = \sigma(X_1, X_2, \cdots, X_n).$$

Suppose that for any bounded stopping times $\sigma \leq \tau$, we have

$$EX_\sigma \leq EX_\tau, \quad a.s.$$ 

Prove that $(X_n, \mathcal{F}_n)$ is a submartingale.
Probability Prelim
August 8, 2011

1. Let \( \{X_n\}_{k \geq 1} \) be an independent sequence of random variables such that

\[
P\{X_n = 0\} = 1 - n^{-p} \quad \text{and} \quad P\{X_n = n^p\} = n^{-p} \quad n = 1, 2, \ldots,
\]

where \( p > 0 \).

(a) Determine the values of \( p \) that make the limit of \( \{X_n\}_{k \geq 1} \) exist almost surely, and find the strong limit \( \lim_{n \to \infty} X_n \) when it exists.

(b) Directly exam "true or false" for the statements

\[
E\left( \lim_{n \to \infty} X_n \right) = \lim_{n \to \infty} E X_n \quad \text{and} \quad E\left( \liminf_{n \to \infty} X_n \right) < \liminf_{n \to \infty} E X_n
\]

whenever the problem is well-posted.

2. Let \( h(x) \) be a bounded, strictly increasing and continuous function on \([0, \infty)\) such that \( h(0) = 0 \). Prove that for any random variables \( X_n \) and \( X \), \( X_n \xrightarrow{p} X \) if and only if

\[
\lim_{n \to \infty} E h(|X_n - X|) = 0.
\]

3. Let \( \{X_k\}_{k \geq 1} \) be an i.i.d. sequence and let \( \{\xi_n\}_{n \geq 1} \) be a sequence of Poisson random variables with \( E \xi_n = n \) \( (n = 1, 2, \ldots) \). Assume independence between \( \{X_k\}_{k \geq 1} \) and \( \{\xi_n\}_{n \geq 1} \).

(a). Compute the characteristic function of the variable

\[
Z_n = \sum_{k=1}^{\xi_n} X_k
\]

(More precisely, represent the characteristic function of \( Z_n \) in terms of the characteristic function of \( X_1 \)).

(b). Assume that \( E X_1 = 0 \) and \( E X_1^2 = 1 \). Prove that the random sequence \( Z_n / \sqrt{n} \) converges in distribution and identify the limiting distribution.

4. Let \( \varphi(x) \) be a non-negative and convex function. Let \( X \) and \( Y \) be two independent random variables with \( E X = 1 \). Prove that

\[
E \varphi(Y) \leq E \varphi(XY).
\]

5. Let \( X \in L^2(\Omega, \mathcal{A}, P) \) and let \( \mathcal{G} \subset \mathcal{A} \) be a sub \( \sigma \)-algebra. Assume that \( X \overset{d}{=} E[X|\mathcal{G}] \).

Prove that \( X = E[X|\mathcal{G}] \) a.s.
6. Let \((X, Y)\) be a 2-dimensional Gaussian random variable.

(a). Prove that there are constants \(a\) and \(b\) such that
\[
E[Y|X] = aX + b
\]
and determine \(a\) and \(b\) as much as you can. Hint: You may start by solving and justifying the equation
\[
\begin{align*}
\text{Cov}(Y - (aX + b), X) &= 0 \\
EY &= aEX + b.
\end{align*}
\]

(b). Prove that the conditional variance defined as
\[
\text{Var}(Y|X) = E\left\{ (Y - E[Y|X])^2 | X \right\}
\]
is equal to a constant almost surely.

7. Let \(\{X_k\}_{k \geq 1}\) be an independent sequence such that \(EX_k = 0\) and
\[
\sum_{k=1}^{\infty} EX_k^2 < \infty.
\]
Prove that the random series
\[
\sum_{k=1}^{\infty} X_k
\]
converges almost surely.

8. Let \(\{X_k\}_{k \geq 1}\) be an i.i.d. sequence with the common distribution
\[
P(X_1 = -1) = P(X_1 = 1) = \frac{1}{2}.
\]
The sequence
\[
S_n = \sum_{k=1}^{n} X_k \quad n = 1, 2, \cdots
\]
is called simple random walk in literature.

(a). For each integer \(a \geq 1\), define the stopping time \(T_a\) as
\[
T_a = \inf\{k \geq 1; |S_k| = a\}
\]
Prove that \(T_a\) and \(S_{T_a}\) are independent.

(b). Prove that for any real number \(\theta\), the sequence \(\{M_n\}_{n \geq 0}\) defined as
\[
M_0 = 1 \quad \text{and} \quad M_n = \left( \cosh \theta \right)^{-n} \exp \{ \theta S_n \} \quad n = 1, 2, \cdots
\]
is a martingale.

(c). Prove that
\[
E\left( \cosh \theta \right)^{-T_a} = \left( \cosh a\theta \right)^{-1}.
\]
1. (10 points) Let $X, Y, Z$ be real random variables such that $X \leq Y \leq Z$ and $X, Z \in L^1$. Prove that $Y \in L^1$.

2. (10 points) Show that if random variables $X$ and $-X$ have the same distributions, then the characteristic function of $X$ is real valued.

3. (15 points) Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. random variables such that $X_n$ has density

$$f(x) = \begin{cases} xe^{-x^2/2} & x > 0, \\ 0 & x \leq 0. \end{cases}$$

Prove that

$$\limsup_{n \to \infty} \frac{X_n}{\sqrt{2 \log n}} = 1 \quad \text{a.s.}$$

4. (14 points) Let $X_n$ be a Poisson random variable with parameter $\lambda_n > 0$, $n = 1, 2, \ldots$. Prove that

(a) the sequence $(X_n)_{n \geq 1}$ converges in distribution if and only if $(\lambda_n)_{n \geq 1}$ converges.

(b) Deduce from (a) that if $X_n \xrightarrow{\mathcal{D}} X$ then $X$ is either a Poisson random variable or $X = 0$ a.s.

5. (12 points) Let $S_n = Z_1^2 + \cdots + Z_n^2$, where $(Z_n)_{n \geq 1}$ is a sequence of i.i.d. standard normal random variables.

(a) Show that $\sqrt{n} \frac{S_n}{n} \xrightarrow{\mathcal{D}} Y$, where $Y$ is $\text{NORMAL}(0, 1/2)$.

(b) Deduce from (a) that $\mathbb{P}(S_n \leq x) \approx \Phi(\sqrt{2x} - \sqrt{2n})$ for every $x \geq 0$ when $n$ is large.

6. (12 points) Let $S_n = X_1 + \cdots + X_n$ be the partial sum of i.i.d. random variables $(X_k)$ with $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 < \infty$. Prove that

(a) $S_n^2$ is a submartingale with respect to $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, $n = 1, 2, \ldots$.

(b) for any $\alpha > 1/2$,

$$\frac{1}{n^\alpha} \max_{1 \leq k \leq n} |S_k| \overset{\mathbb{P}}{\to} 0.$$