1. Suppose that $g$ is a smooth function on $\mathbb{R}$ and consider the initial value problem

$$e^x u_x + u_y = u$$
$$u(x, 0) = g(x).$$

Write a formula for the solution. Find the domain of definition of the solution.

2. Let $B_2(0) \subset \mathbb{R}^n$, a ball centered at the origin with radius 2 and define the operator

$$Lu := \Delta u + b \cdot \nabla u + (4 - |x|^2)u,$$

where $b = (b_1, b_2, \ldots, b_n)$ is a given vector of smooth functions on $\overline{B_2(0)}$. Suppose that for some $\lambda > 4$ the function $u \in C^2(\overline{B_2(0)})$ satisfies

$$Lu = \lambda u \quad \text{in } B_2(0)$$
$$\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial B_2(0).$$

(a) Show that for large $\eta > 0$ the function $v(x) = e^{-\eta|x|^2} - e^{-4\eta}$ satisfies the inequality

$$Lv \geq \lambda v \quad \text{in } B_2(0) \setminus B_1(0)$$
$$v = 0 \quad \text{on } \partial B_2(0)$$
$$v > 0 \quad \text{on } \partial B_1(0).$$

(b) Prove that the solution $u$ of (1) cannot attain its positive maximum in $B_2(0)$.

(c) Prove that the solution $u$ of (1) can have no positive maximum in $\overline{B_2(0)}$. [Hint: If $x_0 \in \partial B_2(0)$ such that $u(x_0) > 0$ is a maximum of $u$, then for appropriately chosen small $\epsilon$ work with the function $w = u + \epsilon v - u(x_0)$ on $B_2(0) \setminus B_1(0)$ where $v$ is as in part (a).]

(d) Conclude that the solution $u$ of (1) is identically 0.
3. Suppose that $u$ is harmonic on $\mathbb{R}^n$ and $B_1(0)$ represents the unit ball. For any $t > 0$ define

$$ I(t) = \int_{\partial B_1(0)} u(ty)u\left(\frac{y}{t}\right) dS_y. $$

Show that $I$ is a constant function.

4. Let $\alpha, \gamma$ be positive numbers, $\beta \in \mathbb{R}$ and $b \in \mathbb{R}^n$ be given. Consider the Cauchy problem

$$ \begin{align*}
\alpha u_t + b \cdot \nabla u + \beta u &= \gamma \Delta u \quad \text{in } \mathbb{R}^n \times (0, \infty), \\
u(x, 0) &= g(x), \quad \text{on } \mathbb{R}^n
\end{align*} $$

where $g$ is compactly supported smooth function.

(a) Find $\kappa, \mu \in \mathbb{R}$ and $a \in \mathbb{R}^n$ so that $v(x, t) = e^{\kappa t}u(\mu x + at, t)$ solves

$$ \begin{align*}
v_t &= \Delta v \quad \text{in } \mathbb{R}^n \times (0, \infty), \\
v(x, 0) &= g(\mu x) \quad \text{on } \mathbb{R}^n.
\end{align*} $$

(b) Write down an explicit formula for a solution $u(x, t)$ of (2).

5. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $c$ be continuous in $\overline{\Omega} \times [0, T]$ with $c \geq -c_0$ for a nonnegative constant $c_0$, and $u_0$ be continuous in $\Omega$ with $u_0 \geq 0$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $xf(x) \leq 0$ for all $x \in \mathbb{R}$. Suppose that $u \in C^{2,1}(\Omega \times (0, T]) \cap C(\overline{\Omega} \times [0, T])$ is a solution of

$$ \begin{align*}
u_t - \Delta u + cu &= uf(u) \quad \text{in } \Omega \times (0, T] \\
u(\cdot, 0) &= u_0 \quad \text{on } \Omega \\
u &= 0 \quad \text{on } \partial \Omega \times (0, T]
\end{align*} $$

Prove that

$$ 0 \leq u(x, t) \leq e^{c_0 T} \sup_{\Omega} u_0, \quad \text{for all } (x, t) \in \Omega \times (0, T]. $$

*Hint: For the lower bound work on $w = u e^{-Mt}$ for a suitable choice of a constant $M$.***
6. Let Ω be a bounded smooth domain. For given smooth functions \( V(x) \) and \( h(x) \) in \( \overline{\Omega} \), consider the equation

\[
\begin{align*}
    u_{tt} - \Delta u + V(x) u &= h(x) u^3, \quad x \in \Omega, t > 0 \\
    u(x, 0) &= f(x), \ u_t(x, 0) = g(x) \quad x \in \Omega \\
    |x|^2 u + \frac{\partial u}{\partial n} &= 0, \quad x \in \partial \Omega.
\end{align*}
\]

(a) Show that if \( V(x) \geq -\alpha \) for some \( \alpha > 0 \) and any \( x \in \Omega \) and there is a solution \( u \in C^2(\Omega \times [0, \infty)) \), then it is unique.

(b) In the event \( f = 0 \) and \( h \leq 0 \), if \( u \in C^2(\Omega \times [0, \infty)) \) is a solution, show that for all \( t > 0 \)

\[
\int_{\Omega} \left( u_t^2 + |\nabla u|^2 + V(x) u^2 \right) dx \leq \int_{\Omega} g^2 dx.
\]

7. Consider the equation

\[
\begin{align*}
    u_{tt} - \Delta u &= -u, \quad (x, y, t) \in \mathbb{R}^2 \times (0, \infty) \\
    u(x, y, 0) &= 0, \quad (x, y) \in \mathbb{R}^2 \\
    u_t(x, y, 0) &= h(x, y), \quad (x, y) \in \mathbb{R}^2
\end{align*}
\]

where \( h \) is a smooth function defined on \( \mathbb{R}^2 \). Find a formula for the solution \( u(x, y, t) \).

*Hint:* Introduce \( v(x, y, z, t) = \cos(z)u(x, y, t) \) defined on \( \mathbb{R}^3 \times (0, \infty) \) and notice that \( u(x, y, t) = v(x, y, 0, t) \).
1. Let $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$. Consider the first-order p.d.e.

$$u_x^2 + u_y^2 = u^2 \quad \text{on } \Omega$$

satisfying $u = 1$ on $x^2 + y^2 = 1$. Prove that there exist exactly two solutions $u \in C^1(\Omega)$. Also find $\lim_{r \to 0} u(x,y)$, $r = (x^2 + y^2)^{1/2}$.

2. Let $0 < R_1 < R_2$, $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : R_1 < |x| < R_2\}$, $|x|^2 = x_1^2 + x_2^2$. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $\Delta u \geq 0$ on $\Omega$. Denote $M(r) = \sup_{|x|=r} u$ for $R_1 \leq r \leq R_2$. Prove

$$M(r) \leq [M(R_1) \ln(R_2/r) + M(R_2) \ln(r/R_1)] (\ln(R_2/R_1))^{-1}$$

for $r \in [R_1, R_2]$.

Hint: Consider an auxiliary harmonic function $v(r)$.

3. Suppose $\Omega \subset \mathbb{R}^n$ is open and bounded. Assume $b_1, ..., b_n \in C^1(\overline{\Omega})$ and let $Lu = \Delta u + \sum_{i=1}^n b_i(x)u_{x_i}$. Suppose $u \in C^3(\overline{\Omega})$ satisfies $Lu = 0$ on $\Omega$. Define $v = u^2$, $w = |Du|^2 = \sum_{k=1}^n u_{x_k}^2$ on $\Omega$.

Prove

(a) $Lv = 2|Du|^2$ on $\Omega$.

(b) For some $M > 0$, $Lw \geq 2|H|^2 - M|Du|^2$ on $\Omega$; here the Hessian $H = [u_{x_k x_i}], |H|^2 = \sum_{i,k=1}^n u_{x_k x_i}^2$.

(c) For some $\lambda > 0$, $L(\lambda v + w) \geq 0$ on $\Omega$, and for some $C > 0$

$$||Du||_{L^\infty(\Omega)} \leq C(||Du||_{L^\infty(\partial \Omega)} + ||u||_{L^\infty(\partial \Omega)}).$$

4. Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $u_0 \in C^0(\overline{\Omega})$, $g \in C^0(\mathbb{R})$, $a(x,t) \in C^1(\overline{\Omega} \times [0,T])$, $a \geq 0$ on $\overline{\Omega} \times [0,T]$. Assume $u \in C^2(\overline{\Omega} \times [0,T])$ solves

$$u_t = \text{div}(a(x,t)\nabla u) + g(u)|\nabla u| \quad \text{on } \Omega \times [0,T]$$

with initial condition $u(x,0) = u_0(x)$ for $x \in \Omega$, and boundary condition $u(x,t) = 0$ for $(x,t) \in \partial \Omega \times [0,T]$. Prove that $|u(x,t)| \leq \max_{\overline{\Omega}} |u_0|$ for all $(x,t) \in \overline{\Omega} \times [0,T]$.

5. Let $u$ be the bounded solution to the initial value problem

$$u_t = \Delta u \quad \text{on } \mathbb{R}^n \times [0,\infty)$$

with initial condition \( u(\cdot, 0) = u_0 \) where \( u_0 \) is bounded on \( \mathbb{R}^n \) and satisfies, for some \( \alpha \in (0, 1) \) and \( C > 0, \ |u_0(x) - u_0(y)| \leq C|x-y|^{\alpha}, \ x, y \in \mathbb{R}^n \). Prove that there exists a constant \( C_1 > 0 \) such that \( |u(x, t) - u(x, s)| \leq C_1|t^{\alpha/2} - s^{\alpha/2}| \) for all \( x \in \mathbb{R}^n, s, t \geq 0 \).

6. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a function such that for every \( R > 0 \) there exists \( N = N(R) > 0 \) such that
\[
|f(s, t)| \leq N(|s| + |t|) \quad \text{for all } (s, t) \in \mathbb{R}^2, |s| + |t| \leq R.
\]
Let \( u \) be a smooth compactly supported solution of the nonlinear wave equation
\[
u_{tt} - \Delta u + f(u, u_t) = 0 \quad \text{on } \mathbb{R}^3 \times (0, \infty).
\]
Assume that there is \( x_0 \in \mathbb{R}^3 \) and \( t_0 > 0 \) such that
\[
u(x, 0) = u_t(x, 0) = 0 \quad \text{for all } x \in B(x_0, t_0)
\]
(\( B(x_0, t_0) \) is the open ball in \( \mathbb{R}^3 \) with radius \( t_0 \) and centered at \( x_0 \)). Prove that \( u = 0 \) in the cone \( K(x_0, t_0) \) defined by
\[
K(x_0, t_0) = \{(x, t) \in \mathbb{R}^4 : 0 \leq t \leq t_0, |x - x_0| \leq t_0 - t\}.
\]
Hint: One may consider the energy function \( e(t) = \frac{1}{2} \int_{B(x_0, t_0 - t)} (u_t^2 + |\nabla u|^2 + u^2) \, dx \).

7. Let \( g : \mathbb{R} \to \mathbb{R} \) be defined by \( g(x) = 1 \) if \( |x| < 1, \ g(x) = 0 \) if \( |x| \geq 1 \). Use d'Alembert's formula to find the solution \( u \) of the wave equation
\[
u_{tt} - \nu_{xx} = 0 \quad \text{on } \mathbb{R} \times (0, \infty)
\]
with \( u(x, 0) = x^2 \) and \( u_t(x, 0) = g(x), \ x \in \mathbb{R} \). Show that \( u \) is not differentiable with respect to the variable \( t \) at \( (x_0, t_0) = (0, 1) \).
1. Let $\Omega = \{(x,t) : x > 0, t > 0\}$. Assume $f \in C^\infty(\overline{\Omega})$, $f$ has bounded support and $f = 0$ on $\{t = 0\}$. Suppose $u \in C^2(\overline{\Omega})$ is a solution of

$$u_t + u_x + u = f(x,t) \text{ on } \Omega,$$

$$u = 0 \text{ on } \{x = 0\} \cup \{t = 0\}.$$

(a) For each $t > 0$, prove that $u(\cdot,t)$ has bounded support.
(b) For each $t > 0$, prove

$$\int_0^\infty u_t^2 \, dx \leq \int_0^t e^{s-t} \int_0^\infty f_t^2(x,s) \, dx \, ds.$$

(c) Prove there exists $K > 0$ such that $\int_0^\infty u_t^2 \, dx \leq Ke^{-t}$ for all $t > 0$.

2. Let $a > 0$, $\Omega = (-1,1) \times (-a,a) \subset \mathbb{R}^2$. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of

$$\Delta u = -1 \text{ on } \Omega, \quad u = 0 \text{ on } \partial \Omega.$$

Using the functions $v(x,y) = (1 - x^2)(a^2 - y^2)$, $w(x,y) = 2 - x^2 - \frac{y^2}{a^2}$ (or constant multiples of them), find positive bounds $C_1(a)$ and $C_2(a)$ such that

$$C_1(a) \leq u(0,0) \leq C_2(a).$$

3. Suppose $\Omega \subset \mathbb{R}^n \ (n \geq 3)$ is open, bounded with $C^\infty$-smooth boundary $\partial \Omega$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of

$$-\Delta(u^3) = u \text{ on } \Omega, \quad u = 0 \text{ on } \partial \Omega.$$

(a) Using the Green’s function show there exists a constant $C > 0$ depending only on $\Omega$, but not on the solution, such that $\int_\Omega |u(x)|^3 \, dx \leq C$, and $\sup_{\Omega} |u| \leq C$.
(b) Show that, if $u \geq 0$ on $\Omega$, then either, $u \equiv 0$ on $\Omega$ or $u > 0$ on $\Omega$.
(c) Let $v$ be the eigenfunction corresponding to the first (least) eigenvalue $\lambda$ of $-\Delta v = \lambda v$ on $\Omega$, $v = 0$ on $\partial \Omega$ (recall $v > 0$ on $\Omega$). Show that, if $u \geq v$, then $u^3 \geq \frac{1}{\lambda} v$. 

1
(d) Assuming also \( u^3 \in C^1(\Omega) \), prove \( \int_{\Omega} |\nabla (u^2)|^2 \, dx = C_1 \int_{\Omega} u^2 \, dx \leq C_2 \) where \( C_1, C_2 \) depend only on \( \Omega \), not on \( u \).

4. Let \( u_0 : \mathbb{R}^n \to \mathbb{R} \) be smooth and compactly supported, and

\[
m = \int_{\mathbb{R}^n} u_0(y) \, dy.
\]

Let \( u \) be a solution of the Cauchy problem

\[
u_t - \Delta u = 0 \quad \text{on} \quad \mathbb{R}^n \times (0, \infty),
\]

\[
u(x, 0) = u_0(x) \quad x \in \mathbb{R}^n,
\]

with \( |u(x, t)| \leq Ae^{a|x|^2} \) for some fixed \( A, a > 0 \) and all \( (x, t) \in \mathbb{R}^n \times (0, \infty) \). Prove that there is a constant \( N \) depending only on \( n \) such that

\[
\sup_{x \in \mathbb{R}^n} |u(x, t) - m \Phi(x, t)| \leq \frac{N}{t^{n/2}} \int_{\mathbb{R}^n} |y| |u_0(y)| \, dy, \quad \text{for all} \quad t > 0,
\]

where \( \Phi(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \).

5. Let \( u \) be a smooth function on \( \overline{B}_1 \times [0, 1] \) that satisfies the equation

\[
a_0 \, u_t - b_0 \, \Delta u + u = 1 \quad \text{on} \quad B_1 \times (0, 1),
\]

\[
u = 1 \quad \text{on} \quad \partial B_1 \times (0, 1),
\]

\[
u(x, 0) = 1 \quad x \in B_1,
\]

where \( a_0, b_0 : \overline{B}_1 \times [0, 1] \to [0, \infty) \) are given continuous functions (\( B_1 = \text{unit ball in} \ \mathbb{R}^n \)). Prove that \( u \leq 1 \) on \( \overline{B}_1 \times [0, 1] \).

6. Assume that \( \Omega \subset \mathbb{R}^n \) is an open, bounded set with \( C^\infty \)-smooth boundary \( \partial \Omega \). Let \( T > 0 \), \( \Omega_T = \Omega \times (0, T] \). Suppose \( a \in C^1(\overline{\Omega}), a > 0 \) on \( \overline{\Omega} \), \( \phi, \psi \in C^2(\overline{\Omega}) \). Suppose \( u \in C^2(\overline{\Omega_T}) \) is a solution of

\[
u_{tt} - a(x) \Delta u = u^3 \quad \text{on} \quad \Omega_T,
\]

\[
\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial \Omega \times [0, T],
\]

\[
u = \phi, \quad \nu_t = \psi \quad \text{on} \quad \Omega \times \{ t = 0 \}.
\]

Prove that \( u \) is unique.

7. Assume \( \phi \in C^2(\mathbb{R}) \) and \( \psi, \psi \in C^1(\mathbb{R}) \). Consider the initial-value problem

\[
u_{tt} - u_{xx} = h(x - t) \quad \text{on} \quad \mathbb{R} \times [0, \infty),
\]

(1)
\[ u = \phi(x), \quad u_t = \psi(x) \quad \text{at} \quad t = 0, \quad x \in \mathbb{R}. \quad (2) \]

(a) Find a solution of the p.d.e. in (1).
(b) Find a solution of (1) and (2).
Question 1: Let $g: \mathbb{R} \to \mathbb{R}$ be a smooth function. Find solutions of the following initial-value problem in $\mathbb{R}^2$

$$u_x + (1 + x^2)u_y - u = 0 \quad \text{with} \quad u(x, \frac{1}{3}x^3) = g(x).$$

Question 2: Let $h: \mathbb{R} \to \mathbb{R}$ be a smooth function. Consider the following equation in $\mathbb{R}^2$

$$xu_x + yu_y = 2u \quad \text{with} \quad u(x, 0) = h(x).$$

(a) Check that the line $\{y = 0\}$ is characteristic at each point and find all $h$ satisfying the compatibility condition on $\{y = 0\}$.

(b) For $h$ as compatible in (a), solve the PDE.

Question 3: Let $\phi$ be smooth, compactly supported function defined in the unit ball $B_1 \subset \mathbb{R}^n$ such that $\phi = 1$ on $B_{1/2}$, where $B_{1/2} \subset \mathbb{R}^n$ is the ball of radius $1/2$ centered at the origin. Suppose that $u$ is harmonic in $B_1$.

(a) Prove that there is $\alpha > 0$ depending only on $n$ and $\sup |\Delta \phi|$ and $\sup |\nabla \phi|$ such that

$$\Delta (\phi^2 |\nabla u|^2 + \alpha u^2) \geq 0 \quad \text{in} \quad B_1.$$

(b) Use part (a) and the maximum principle to conclude that there is a constant $C > 0$ depending only on $n, \phi$ such that

$$\sup_{B_{1/2}} |\nabla u| \leq C \sup_{\partial B_1} |u|.$$

Question 4: Let $B_1 \subset \mathbb{R}^2$ be the unit ball with boundary $\partial B_1$. Let $f, c \in C(\overline{B}_1)$ and $g \in C(\partial B_1)$. Assume that $c(x, y) > 0$ for all $(x, y) \in B_1$. Prove that there exists at most one $C^2$-solution to the following equation

$$\left\{ \begin{array}{ll}
-x^2u_{xx} - y^2u_{yy} + c(x, y)u &= f \quad \text{in} \quad B_1 \\
u &= g \quad \text{on} \quad \partial B_1.
\end{array} \right.$$ 

Question 5: Let $a_0$ be a smooth and compactly supported function defined on $\mathbb{R}^n$ and $p_0 \in (1, \infty)$. Consider the following Cauchy problem

$$\left\{ \begin{array}{ll}
u_t - \Delta u &= |u|^{p_0-1}u \quad \text{in} \quad \mathbb{R}^n \times (0, \infty) \\
u(x, 0) &= a_0(x) \quad x \in \mathbb{R}^n.
\end{array} \right. \quad (1)$$

Define the scaling

$$u_\lambda(x, t) = \lambda^\beta u(\lambda x, \lambda^2 t), \quad \lambda > 0.$$

(a) Find $\beta$ (possibly depending on $n, p_0$) so that if $u$ is a solution of (1), then $u_\lambda$ is also a solution (1) (with appropriate scaled initial data $a_\lambda^0$).

(b) Recall that the $L^p$-norm is defined by

$$\|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |u(x, t)|^p dx \right)^{1/p}, \quad p \in [1, \infty).$$

For $\beta$ found in a), find $p$ so that if $u$ is a solution of (1) then

$$\|u(\cdot, \lambda^2 t)\|_{L^p(\mathbb{R}^n)} = \|u_\lambda(\cdot, t)\|_{L^p(\mathbb{R}^n)}$$

for all $\lambda > 0$ and for all $t > 0.$
Question 6: Let us denote $\mathbb{R}_+^2 = \mathbb{R} \times (0, \infty)$ and $B_1^+ = B_1 \cap \mathbb{R}_+^2$, where $B_1$ is the unit ball in $\mathbb{R}^2$. Assume that $u = u(x, y, t)$ is a smooth function defined on $B_1^+ \times [0, 1]$ and satisfying

$$u_t - y^n |u_{xx} + u_{yy}| + u_y + u \leq 0 \quad \text{for} \quad (x, y) \in B_1^+ \quad \text{and} \quad t \in (0, 1),$$

where $\alpha > 0$ is a given number. Assume that $u(x, y, 0) \leq 0$, and that $u \leq 0$ on $(\partial B_1 \cap \mathbb{R}_+^2) \times (0, 1)$, where $\partial B_1$ denotes the boundary of $B_1$. Prove that $u \leq 0$ on $B_1^+ \times [0, 1]$.

Note: We are not given any information on the boundary data on the part of the boundary where $y = 0$.

Question 7: Let $u_1(x)$ and $u_2(x)$ be smooth functions whose supports are in the unit ball $B_1 \subset \mathbb{R}^n$. For each $x_0 \in \mathbb{R}^n$ and each $t_0 > 0$, let $C(x_0, t_0)$ be the cone defined by

$$C(x_0, t_0) = \{(x, t) : 0 \leq t \leq t_0, \quad |x - x_0| \leq t_0 - t\}.$$

Assume that $u \in C^2$ is the solution of the equation

$$u_{tt} - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty)$$

with given initial data $u(x, 0) = u_1(x)$ and $u_t(x, 0) = u_2(x)$.

Give the proof for the finite propagation speed result for the wave equation, namely $u = 0$ on $C(x_0, t_0)$ for all $x_0 \in \mathbb{R}^n$ with $|x_0| > 1$ and $t_0 = |x_0| - 1$.

Question 8: Let $u$ be a smooth solution of the equation

$$u_{tt} - \Delta u = f \quad \text{on} \quad \mathbb{R}^3 \times (0, \infty)$$

with $u(\cdot, 0) = u_t(\cdot, 0) = 0$. Also, let $v$ be a smooth solution of the equation

$$v_{tt} - \Delta v = g \quad \text{on} \quad \mathbb{R}^3 \times (0, \infty)$$

with $v(\cdot, 0) = v_t(\cdot, 0) = 0$. Assume that $|f|^2 \leq g$. Prove that $2u(x, t)^2 \leq t^2 v(x, t)$ for all $x \in \mathbb{R}^3$ and $t > 0$. 
Question 1: Solve the Cauchy problem
\[
\begin{align*}
    xu_x - yu_y &= u - y, \quad x > 0, y > 0, \\
    u(y^2, y) &= y, \quad y > 0.
\end{align*}
\]

Question 2: Let \(a, R\) be positive numbers and consider the equation
\[
\begin{align*}
    u_t + au_x &= f(x, t), \quad 0 < x < R, \quad t > 0, \\
    u(0, t) &= 0, \quad t > 0, \\
    u(x, 0) &= 0, \quad 0 < x < R.
\end{align*}
\]
Prove that for each solution \(u(x, t) \in C^1((0, R) \times (0, \infty))\) we have
\[
\int_0^R u^2(x, t)dx \leq e^t \int_0^t \int_0^R f^2(x, s)dxds, \quad \forall \ t > 0.
\]

Question 3: Let \(r > 0\) and let \(f, g\) be continuous functions defined on \(B_r(0)\). Let \(u\) be in \(C^2(B_r(0)) \cap C(\overline{B_r(0)})\) be the solution of the equation
\[
\begin{align*}
    -\Delta u &= f, \quad B_r(0), \\
    u &= g, \quad \partial B_r(0).
\end{align*}
\]
Prove that
\[
u(0) = \int_{\partial B_r(0)} g(x) dS(x) + \frac{1}{n(n-2)\alpha(n)} \int_{B_r(0)} \left[ \frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right] f(x) dx.
\]
Hint: Consider
\[
\phi(s) = \int_{\partial B_s(0)} u(y) dS, \quad 0 < s \leq r.
\]
Compute \(\phi'(s)\) and then find \(\phi(0)\).

Question 4: Let \(R > 0\) and we denote \(B_R\) the ball of radius \(R\) centered at the origin in \(\mathbb{R}^n\). Let \(c, f\) be continuous functions on \(\overline{B_R}\). Assume that \(c \leq 0\) on \(\overline{B_R}\), and also assume that \(u \in C^2(B_R) \cap C(\overline{B_R})\) satisfies
\[
\begin{align*}
    \Delta u + cu &= f \quad \text{in} \ B_R, \\
    u &= 0 \quad \text{on} \ \partial B_R.
\end{align*}
\]
Prove that
\[
\sup_{B_R} |u| \leq \frac{R^2}{2n} \sup_{B_R} |f|
\]
Hint: Let \(A = \sup_{B_R} |f|\) and
\[
v(x) = \frac{AR^2}{2n} (R^2 - |x|^2)
\]
Use the maximum principle to prove that \(|u(x)| \leq v(x)\) on \(B_R\).

Question 5: Let \(u_0\) be the smooth and compactly supported function defined on \(\mathbb{R}^n\). Assume that \(u\) is a solution of the Cauchy problem
\[
\begin{align*}
    u_t - \Delta u &= 0 \quad \text{in} \ \mathbb{R}^n \times (0, \infty) \\
    u(x, 0) &= u_0(x) \quad x \in \mathbb{R}^n.
\end{align*}
\]
Let $p, q \in (1, \infty)$ with $p \geq q$ and consider the inequality
\[
\|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} \leq \frac{N}{t^\alpha} \|u_0\|_{L^q(\mathbb{R}^n)}, \quad t > 0
\]
with $N = N(n, p, q)$ and $\alpha = \alpha(n, p, q)$, where we denote
\[
\|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(x, t)|^p dx\right)^{\frac{1}{p}}
\]
and similar notation is also used for $\|u_0\|_{L^q(\mathbb{R}^n)}$.

Use the scaling property of the heat equation to find the number $\alpha$ (certainly, show all of the work).

**Question 6:** Assume that $u$ is a smooth, bounded solution of the equation
\[
\begin{aligned}
&u_t - \Delta u = u(1 - u) \quad \text{in} \quad B_1 \times (0, 1] \\
&u = 0 \quad \text{on} \quad \partial B_1 \times (0, 1] \\
&u = \frac{1}{2} \quad \text{on} \quad B_1 \times \{0\}.
\end{aligned}
\]
Prove that $0 \leq u \leq 1$.

**Question 7:** Let $\varphi$ be a smooth, compactly supported function on $\mathbb{R}^2$. Assume that $u$ is a smooth solution of
\[
\begin{aligned}
&u_{tt} - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^2 \times (0, \infty), \\
&u(\cdot, 0) = 0 \quad \text{on} \quad \mathbb{R}^2, \\
&u_t(\cdot, 0) = \varphi \quad \text{on} \quad \mathbb{R}^2.
\end{aligned}
\]
Prove that
\[
|u(x, t)| \leq \frac{1}{2\sqrt{t}} \left(\|\varphi\|_{L^1(\mathbb{R}^2)} + \|\nabla\varphi\|_{L^1(\mathbb{R}^2)}\right), \quad \forall \ t > 1.
\]

**Question 8:** Assume that $u \in C^2(\mathbb{R}^n \times [0, \infty))$ is a solution of the wave equation
\[
u_{tt} = \Delta u \quad \text{in} \quad \mathbb{R}^n \times (0, \infty).
\]

Let
\[
E(t) = \frac{1}{2} \int_{B_1 - t} \left[|u_t(x, t)|^2 + |\nabla u(x, t)|^2\right] dx \quad \text{for} \quad t \in (0, 1),
\]
where $\nabla u = (u_{x_1}, u_{x_2}, \cdots, u_{x_n})$ and $B_r$ denotes the ball in $\mathbb{R}^n$ with radius $r > 0$ and centered at the origin.

(a) Prove that
\[
E'(t) = \int_{B_1 - t} \left[|u_t(x, t)|^2 + \sum_{i=1}^n u_{x_i}u_{x_i,t}\right] dx \\
- \frac{1}{2} \int_{\partial B_1 - t} \left[u_t^2(x, t) + |\nabla u(x, t)|^2\right] dS(x).
\]
(b) Use the note that
\[
[u_{x_i}u_{x_i,t} + u_{x_i}u_{x_i,t}] = u_{x_i}u_{x_i,t} + u_{x_i}u_{x_i,t}
\]
to prove that $E'(t) \leq 0$. Then, conclude also that $u = 0$ on $\{(x, t) : |x| \leq 1 - t, \ 0 \leq t \leq 1\}$ if $u(x, 0) = u_t(x, 0) = 0$ for $x \in B_1$. 
PDE Preliminary Exam, August 2018 — UTK

Question 1: For $x > 0$, consider the equation:

$$
\begin{align*}
    \left\{
        \begin{array}{l}
            uu_x + 2xu_y = 0 \quad \text{in } \mathbb{R}^2 \\
            u(x,0) = \frac{1}{x} \quad \text{for } x > 0.
        \end{array}
    \right.
\end{align*}
$$

For $t_0, t_1 > 0$ with $t_0 \neq t_1$, let $C_0$ be the characteristic passing through the point $(t_0,0,1/t_0)$ and let $C_1$ be the characteristic passing through $(t_1,0,1/t_1)$. Determine whether the projections of $C_0$ and $C_1$ onto the $x$-$y$ plane intersect for some $y > 0$ (i.e., whether a shock develops), and if they do, find the point $(x,y)$ of intersection.

Question 2: Given a bounded domain $\Omega$ in $\mathbb{R}^n$, let $h$ be the solution to

$$
\Delta h = -1 \quad \text{in } \Omega, \quad h = 0 \quad \text{on } \partial \Omega.
$$

Let $a > 0$ be a constant.

Prove: If there exists a function $u > 0$ that satisfies the equation

$$
\Delta u = \frac{1}{u} \quad \text{in } \Omega, \quad u \equiv a \quad \text{on } \partial \Omega,
$$

then $a \geq \sqrt{\max_{\Omega} h}$.

Hint: Prove $u \leq a$. Then prove a better upper bound for $u$.

Question 3:
(a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, bounded, and even (that is, $f(-x) = f(x)$ for all $x \in \mathbb{R}$). Suppose $u = u(x,t) \in C^2_t(\mathbb{R}^2_+) \cap C(\mathbb{R}^2_+)$ satisfies

$$
\begin{align*}
    \left\{ \begin{array}{l}
        u_t = u_{xx} \quad &\text{for } x \in \mathbb{R}, 0 < t < \infty, \\
        u(x,0) = f(x) \quad &\text{for } x \in \mathbb{R}, \\
        |u(x,t)| \leq Ke^{a|x|^2} \quad &\text{for } x \in \mathbb{R}, 0 < t < \infty,
    \end{array} \right.
\end{align*}
$$

for some positive constants $K$ and $a$. Prove that for each $t > 0$, $u(x,t)$ is an even function of $x$: i.e., $u(-x,t) = u(x,t)$ for all $t > 0$.

(b) Assume $f : [0,\infty) \to \mathbb{R}$ is continuous and bounded. For $x \geq 0$ and $t \geq 0$, suppose $u = u(x,t) \in C^2([0,\infty) \times [0,\infty))$ satisfies

$$
\begin{align*}
    \left\{ \begin{array}{l}
        u_t = u_{xx} \quad &\text{for } 0 < x < \infty, 0 < t < \infty, \\
        u(x,0) = f(x) \quad &\text{for } 0 \leq x < \infty, \\
        u_x(0,t) = 0 \quad &\text{for } 0 < t < \infty \\
        |u(x,t)| \leq Ke^{a|x|^2} \quad &\text{for } x \in \mathbb{R}_+, 0 < t < \infty,
    \end{array} \right.
\end{align*}
$$

for some positive constants $K$ and $a$. Here $u_x(0,t)$ is interpreted as the $x$-derivative of $u$ from the right at $(0,t)$. Find a function $H = H(x,y,t)$ such that

$$
u(x,t) = \int_0^\infty H(x,y,t)f(y) \, dy,$$

and justify your answer.
**Question 4:** Consider the nonlinear PDE

\[ u_{tt} - \Delta u + u^3 = 0, \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}. \]

1. Assume that \( u \) is smooth and has compact support in \( x \) for each \( t \). What is the energy expression

\[ E(t) = \int_{\mathbb{R}^3} q(u, u_t, \nabla u) dx \]

which is conserved, i.e., \( E'(t) = 0 \)?

2. For any \( \alpha > 0 \), and \( x_0 \in \mathbb{R}^3 \), denote by

\[ E_\alpha(t) = \int_{B_\alpha(x_0)} q(u, u_t, \nabla u) dx \]

the energy contained in the ball of radius \( \alpha > 0 \) centered at \( x_0 \). Show that for any \( T > 0 \) and \( a > 0 \),

\[ E_a(T) \leq E_{a+T}(0) \]

*Hint: Work with the 'energy'*

\[ \bar{E}(t) := \int_{B_{T+a}(x_0)} q(u, u_t, \nabla u) dx \]

3. Given \( a > 0 \), show that if \( u(x, 0) = u_t(x, 0) = 0 \) for \( |x| > a \), then \( u(x, t) = 0 \) for all \( |x| \geq a + t, \ t \geq 0 \).

**Question 5:** Let \( B \) be the unit ball in \( \mathbb{R}^n \) and let \( u \in C^\infty(\bar{B} \times [0, \infty)) \) satisfy

\[ u_t - \Delta u + u^{1/2} = 0 \quad \text{on } B \times (0, \infty) \]
\[ 0 \leq u \quad \text{on } B \times (0, \infty) \]
\[ u = 0 \quad \text{on } \partial B \times (0, \infty). \]

(a) Show that, if \( u|_{t=t_0} \equiv 0 \), then \( u \equiv 0 \) for \( t > t_0 \) as well.

(b) Prove that there is a number \( T \) depending only on \( M := \max u|_{t=0} \) such that \( u \equiv 0 \) on \( B \times (T, \infty) \).

*Hint: Let \( v \) be the solution of the IVP,*

\[ \frac{dv}{dt} + v^{1/2} = 0, \quad v(0) = M, \]

and consider the function \( w = v - u \).

**Question 6:**

(a) Find a \( C^1 \) solution in \( \mathbb{R}^+ \times \mathbb{R} \) \( \ni (x, y) \) to:

\[ x^2 u_x - y^2 u_y = u^2 \quad \text{for } x > 0, y \in \mathbb{R}, \quad u(1, y) = \frac{1}{1+y^2} \]

(b) Explain why this solution is not unique as a solution in \( C^1(\mathbb{R}^+ \times \mathbb{R}) \), but its restriction to some appropriate open set \( U \) containing the initial curve \( \{1\} \times \mathbb{R} \) is unique in \( C^1(U) \).
**Question 7:** Suppose \( f, g \in C^\infty(\mathbb{R}^n) \). Suppose \( u \in C^2(\mathbb{R}^n \times [0, \infty)) \) satisfies

\[
\begin{align*}
    u_{tt} &= \Delta u, \quad (x, t) \in \mathbb{R}^n \times (0, \infty), \\
    u(x, 0) &= f(x), \quad x \in \mathbb{R}^n, \\
    u_t(x, 0) &= g(x), \quad x \in \mathbb{R}^n. 
\end{align*}
\]

Prove that

\[
\int_{\mathbb{R}^n} u(x, t) \, dx = C_1 t + C_2,
\]

for all \( t > 0 \), where \( C_1 = \int_{\mathbb{R}^n} g(x) \, dx \) and \( C_2 = \int_{\mathbb{R}^n} f(x) \, dx \), under either of the two conditions:

(i) \( n = 3, \int_{\mathbb{R}^3} |f(x)| \, dx < \infty, \int_{\mathbb{R}^3} |\nabla f(x)| \, dx < \infty, \text{ and } \int_{\mathbb{R}^3} |g(x)| \, dx < \infty; \) or

(ii) \( n \in \mathbb{N}, \) and \( f \) and \( g \) have compact support.

**Question 8:** Let \( u \in C^2(\mathbb{R}^n) \) be a subharmonic function and consider the spherical averages

\[
v(r) := \int_{\partial B_r(0)} u(x) \, dS(x).
\]

(a) Show that the function \( x \mapsto v(|x|) \) is also subharmonic in \( \mathbb{R}^n \), and that \( r \mapsto r^{n-1}v'(r) \) is monotonic.

(b) Now let \( n = 2 \). Prove that, if \( u \) is also bounded, then \( u \) is a constant.
PDE Preliminary Exam, January 2018

Instruction:

Solve all eight problems. Begin your answer to each question on a separate sheet. Explain all your steps.

1. A smooth function $u$ defined in the first quadrant on the $xy$-plane satisfies

$$-y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} = -2u, \quad u(x, 0) = x.$$

Determine $u(0, y)$.

2. Suppose that $u(x, t)$ is a smooth solution of

$$\begin{cases} 
  u_t + uu_x = 0 & \text{for } x \in \mathbb{R}, \ t > 0 \\
  u(x, 0) = f(x), & \text{for } x \in \mathbb{R}
\end{cases}$$

Assume that $f$ is a $C^1$ function such that

$$f(x) = \begin{cases} 
  0 & \text{for } x < -1 \\
  1 & \text{for } x > 1
\end{cases} \quad \text{and } f'(x) > 0, \text{ for } |x| < 1.$$

(a) Sketch the characteristics emanating from $(x_0, 0)$ for several values of $x_0 < -1, x_0 \in (-1, 1)$, and $x_0 > 1$.

(b) Show that for $t > 0$,

$$\lim_{r \to \infty} u(rx, rt) = \begin{cases} 
  0 & \text{for } x < 0 \\
  x/t & \text{for } 0 < x < t \\
  1 & \text{for } x > t
\end{cases}$$

3. Suppose that for all $r > 2$, there exists a function $u_r : \mathbb{R}^3 \to \mathbb{R}$ that is continuous and satisfies

$$\begin{cases} 
  \Delta u = 0 & \text{in } B_r(0) \setminus \overline{B_1(0)} \\
  u(x) = 0 & \text{for } |x| \geq r \\
  u(x) = 1, & \text{for } x \in \overline{B_1(0)}.
\end{cases}$$

(a) Show that for all $x \in \mathbb{R}^3$, if $2 < r_1 \leq r_2$, then

$$0 \leq u_{r_1}(x) \leq u_{r_2}(x) \leq 1.$$
(b) Show that

i. \( u(x) = \lim_{r \to \infty} u_r(x) \) is harmonic on \( \mathbb{R}^3 \setminus \overline{B_1(0)} \)

ii. \( \lim_{|x| \to \infty} u(x) = 0. \)

[Hint: noting that \( \frac{1}{|x|} \) is harmonic, study \( u_r(x) - \frac{1}{|x|} \) over an annulus.]

4. Denote by \( \mathbb{R}^n_+ = \{x = (x', x_n): x_n > 0\}, \Sigma = \{x = (x', x_n): x_n = 0\}. \)

Suppose that \( u \) is harmonic in \( \mathbb{R}^n_+ \), continuous on \( \mathbb{R}^n_+ \cup \Sigma \), and \( u = 0 \) on \( \Sigma \). Define

\[
\overline{u}(x', x_n) := \begin{cases} 
    u(x', x_n) & \text{for } x_n \geq 0, \\
    -u(x', -x_n) & \text{for } x_n < 0.
\end{cases}
\]

Then show that \( \overline{u} \) is harmonic in \( \mathbb{R}^n \).

5. Let \( \Omega \subseteq \mathbb{R}^n \) be a \( C^\infty \) bounded domain. Assume that \( u_0 \in C^\infty(\bar{\Omega}), a \in C([0, \infty)), \) and \( \lim_{t \to \infty} a(t) \leq 0. \) Suppose also \( u \in C^2(\bar{\Omega} \times [0, \infty)) \) satisfies

\[
\begin{cases} 
    u_t = \Delta u + a(t) u & \text{on } \Omega \times (0, \infty), \\
    u = 0 & \text{on } \partial \Omega \times (0, \infty), \\
    u = u_0 & \Omega \times \{t = 0\}.
\end{cases}
\]

Prove that

\[
\lim_{t \to \infty} \int_\Omega u^2(x, t) dx = 0
\]

(Hint: Use the Energy method. You may apply Poincaré's inequality.)

6. Let \( \Omega \subseteq \mathbb{R}^n \) be a \( C^\infty \) bounded domain, \( T > 0, \) and \( a \in \mathbb{R}^n \) is a given vector. Suppose \( u \in C^2(\bar{\Omega} \times [0, T]) \) satisfies

\[
\begin{cases} 
    u_t = \Delta u + a \cdot \nabla u + u^2 & \text{on } \Omega \times (0, T), \\
    u = 0 & \text{on } \partial \Omega \times (0, T), \\
    u = 0 & \Omega \times \{t = 0\}.
\end{cases}
\]

Prove that

(a) \( u \geq 0, \) on \( \Omega \times (0, T], \)
(b) \( u_t \geq 0 \) on \( \Omega \times (0, T]. \)

(Hint: What equation does \( u_t \) solve?)
7. Let $\Omega \subseteq \mathbb{R}^n$ be a $C^\infty$ bounded domain and let $T > 0$. Suppose $V = V(x)$ and $h = h(x)$ are continuous functions on $\overline{\Omega}$, with $V(x) \geq 0$. Suppose $u = u(x,t) \in C^2(\overline{\Omega} \times [0,T])$, where $x \in \Omega$ and $t \in [0,T]$, and $u$ satisfies

$$\begin{cases}
    u_t - \Delta u + V(x)u = h(x) & \text{on } \Omega \times (0,T); \\
    u(x,0) = 0 & \text{on } \Omega; \\
    u_t(x,0) = 0 & \text{on } \Omega; \\
    u = -D_n u & \text{on } \partial \Omega \times (0,T),
\end{cases}$$

where $D_n u$ is the outward normal derivative of $u$ on $\partial \Omega$.

(a) Prove that $\int_\Omega h(x)u(x,t) \, dx \geq 0$ for all $t \geq 0$.

*Hint: Consider*

$$E(t) = \frac{1}{2} \int_\Omega u_t^2 + |\nabla u|^2 + Vu^2 - 2hu \, dx + \frac{1}{2} \int_{\partial \Omega} u^2 \, d\sigma,$$

where $d\sigma$ is surface measure on $\partial \Omega$.

(b) Suppose in addition that $V(x) \geq A$ and $|h(x)| \leq B$, for all $x \in \Omega$, for some constants $A > 0$ and $B > 0$. Prove that

$$\int_\Omega |u(x,t)| \, dx \leq \frac{2B|\Omega|}{A},$$

for all $t \geq 0$, where $|\Omega| = \int_\Omega \, dx$ is the measure of $\Omega$.

*Hint: Start by writing $\int_\Omega |u| \, dx = \int_\Omega \frac{\sqrt{V}|u|}{\sqrt{V}} \, dx$, and apply Cauchy Schwartz.*

8. Suppose $u \in C^2(\mathbb{R}^n \times [0,\infty))$ is a solution of

$$\begin{cases}
    u_t = \Delta u & \text{on } \mathbb{R}^n \times (0,\infty); \\
    u(x,0) = f(x) & \text{on } \mathbb{R}^n; \\
    u_t(x,0) = g(x) & \text{on } \mathbb{R}^n,
\end{cases}$$

where $f, g \in C^\infty(\mathbb{R}^n)$ have compact support: there exists $R > 0$ such that $f(x) = 0$ and $g(x) = 0$ if $|x| > R$. Consider the statement:

(S): For all such $f, g$ and $R$, and all $x_0 \in \mathbb{R}^n$, there exists $T = T(x_0, R) > 0$ such that $u(x_0, t) = 0$ for all $t > T$.

(a) Is (S) true if $n = 1$? Either prove (S) or give an example showing that $S$ fails.

(b) Is (S) true if $n = 3$? Either prove (S) or give an example showing that $S$ fails.
PDE Preliminary Exam, August 2017

1. For a given continuous function \( f \), solve the initial-boundary value problem

\[
\begin{cases}
    u_t + (x + 1)^2 u_x = x, & \text{for } x > 0, t > 0 \\
    u(x, 0) = f(x), & x > 0 \\
    u(0, t) = -1 + t, & t > 0.
\end{cases}
\]

Find a condition on \( f \) so that the solution \( u(x, t) \) is continuous on the first quadrant of \( \mathbb{R}^2 \), i.e. the region \( \{(x, t) \in \mathbb{R}^2 : x > 0, t > 0\} \).

2. Determine an integral (weak) solution to the Burger’s equation

\[
u_t + \left( \frac{1}{2} u^2 \right)_x = 0, \quad (x, t) \in \mathbb{R} \times (0, \infty)
\]

with initial data

\[
u(x, 0) = \begin{cases}
    1 & \text{if } x < 0 \\
    1 - x & \text{if } 0 < x < 1 \\
    0 & \text{if } x > 1.
\end{cases}
\]

3. Let \( n \geq 2 \), and let \( \Omega \subseteq \mathbb{R}^n \) be a bounded domain with \( C^\infty \)-smooth boundary. Suppose \( p \) and \( q \) are non-negative continuous functions defined on \( \Omega \), satisfying \( p(x) + q(x) > 0 \) (strict inequality) for all \( x \in \Omega \). Find all functions \( u \in C^2(\Omega) \) satisfying

\[
\begin{cases}
    \Delta u = pu^3 + qu & \text{on } \Omega, \\
    \frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( n(x) \) is the outward unit normal to \( \Omega \) at \( x \in \partial \Omega \).

4. Suppose \( u \) is harmonic on a \( C^\infty \) domain \( \Omega \subseteq \mathbb{R}^n \), and let \( u(x) = 0 \) for \( x \notin \Omega \). Suppose \( \varphi \) is a \( C^\infty \) function on \( \mathbb{R}^n \) such that \( \varphi(x) = 0 \) if \( |x| \geq 1 \), and \( \varphi \) is radial: there exists a function \( \varphi_0 : [0, \infty) \rightarrow \mathbb{R} \) such that \( \varphi(x) = \varphi_0(|x|) \). For \( \epsilon > 0 \), let

\[
\varphi_\epsilon(x) = \frac{1}{\epsilon^n} \varphi\left(\frac{x}{\epsilon}\right).
\]

Let

\[
A = \int_{\mathbb{R}^n} \varphi(x) \, dx.
\]

Fix \( x_0 \in \Omega \) and let \( R > 0 \) be such that \( x \in \Omega \) if \( |x - x_0| < R \). For \( 0 < \epsilon < R \), prove that

\[
\varphi_\epsilon \ast u(x_0) = Au(x_0),
\]

where \( \ast \) denotes convolution: by definition, \( f \ast g(x) = \int_{\mathbb{R}^n} f(x - y)g(y) \, dy \).
5. Suppose that $b \in \mathbb{R}^n$, and $\beta \in \mathbb{R}$ are given. Consider the Cauchy problem

\begin{equation}
(*) \quad \begin{cases}
  u_t + b \cdot \nabla u + \beta u = \Delta u, & \text{in } \mathbb{R}^n \times (0, \infty) \\
  u(x, 0) = f(x), & \text{on } \mathbb{R}^n.
\end{cases}
\end{equation}

(a) Determine $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that if $u$ is a smooth solution to $(*)$, then $v(x, t) = e^{-(a \cdot x + \alpha t)} u(x, t)$ solves the Cauchy problem

\begin{equation}
\begin{cases}
  v_t = \Delta v, & \text{in } \mathbb{R}^n \times (0, \infty) \\
  v(x, 0) = e^{-b \cdot x} f(x), & \text{on } \mathbb{R}^n.
\end{cases}
\end{equation}

(b) Write down an explicit formula for a solution $u(x, t)$ to $(*)$.

6. Let $\Omega \subset \mathbb{R}^n$ a bounded domain with smooth boundary, and $T > 0$. Denote the cylinder $\Omega_T = \Omega \times (0, T]$ and its parabolic boundary $\partial_p \Omega_T = (\partial \Omega \times [0, T]) \cup (\Omega \times \{0\})$.

(a) Prove the following version of the maximum principle. Suppose that $u$ and $v$ are two functions in $C^2(\Omega_T)$ such that

\begin{align*}
  u_t - \Delta u &\leq v_t - \Delta v \quad \text{in } \Omega_T, \\
  u &\leq v \quad \text{on } \partial_p \Omega_T.
\end{align*}

Then $u \leq v$ in $\Omega_T$.

(b) Suppose that $f(x, t), u_0(x)$ and $\phi(x, t)$ are continuous functions in their respective domains. Let $u \in C^2(\Omega_T)$ satisfy

\begin{equation}
\begin{cases}
  u_t - \Delta u = e^{-u} - f(x, t), & \text{in } \Omega_T \\
  u|_{t=0} = u_0, & \text{in } \Omega \\
  u|_{\partial \Omega \times (0, T)} = \phi.
\end{cases}
\end{equation}

Let $a = ||f||_{L^\infty}$ and $b = \sup\{||u_0||_{L^\infty}, ||\phi||_{L^\infty}\}$.

i. Show that $-(aT + b) \leq u(x, t)$, for all $(x, t) \in \overline{\Omega_T}$.

Hint: Introduce $v(x, t) = -(at + b)$ and use part a).

ii. Prove $u(x, t) \leq T e^{aT+b} + aT + b$, for all $(x, t) \in \overline{\Omega_T}$.
7. Suppose that \( f \in C^2(\mathbb{R}) \) is odd and 2-periodic (i.e. \( f(x + 2) = f(x) \) for all \( x \in \mathbb{R} \)). Let \( u \in C^2([0, 1] \times \mathbb{R}) \) solve

\[
\begin{cases}
  u_{tt} - u_{xx} = \sin(\pi x) & \text{in } (0, 1) \times \mathbb{R} \\
  u(x, 0) = f(x), \quad u_t(x, 0) = 0, & x \in [0, 1] \\
  u(0, t) = 0 = u(1, t), & t \in \mathbb{R}.
\end{cases}
\]

(a) Prove uniqueness of the solution \( u \in C^2([0, 1] \times \mathbb{R}) \).

(b) Find the solution \( u \), and show that it satisfies \( u(x, t + 2) = u(x, t) \), and \( u(x, -t) = u(x, t) \) for all \((x, t) \in [0, 1] \times \mathbb{R}\).

8. Assume that \( \Omega \subset \mathbb{R}^n \) is open, bounded with \( C^\infty \)-smooth boundary \( \partial \Omega \). Let \( T > 0 \), and denote \( \Omega_T = \Omega \times (0, T] \). Suppose also that \( f \in C^1(\mathbb{R}^{n+2}) \), \( \phi, \psi \in C^2(\overline{\Omega}) \), and \( u \in C^2(\overline{\Omega_T}) \) is a solution of

\[
\begin{cases}
  u_{tt} - \Delta u = f(u, u_t, \nabla u), & \text{in } \Omega_T \\
  u = \phi, \quad u_t = \psi, & \text{on } \Omega \times \{t = 0\}, \\
  \frac{\partial u}{\partial n} = 0, & \text{on } \partial \Omega \times [0, T].
\end{cases}
\]

Prove that \( u \) is unique.

Hint: You may use an energy function of the form

\[
E(t) = \frac{1}{2} \int_{\Omega} (w_t^2 + |\nabla w|^2 + w^2)dx.
\]
1.) Consider the PDE, for $x \in \mathbb{R}$ and $y \in \mathbb{R}$:

\[
\begin{align*}
(\ast) & \quad \begin{cases} 
2yu_x + u_y = u^4, \\
u(x, 0) = f(x),
\end{cases}
\end{align*}
\]

for some $C^2$ function $f$.

(a) Show that (\ast) has a solution that exists for all $x \in \mathbb{R}$ and all $y > 0$ if and only if $f(t) \leq 0$ for all $t \in \mathbb{R}$.

(b) Show that if (\ast) has a solution for all $(x, y) \in \mathbb{R}^2$, then $f(t) = 0$ for all $t$ and $u$ is identically 0.

2.) Suppose $n \geq 2$, $R > 0$, $B(0, R) \subseteq \mathbb{R}^n$, and $u : \overline{B(0, R)} \to \mathbb{R}$ satisfies $u \in C(\overline{B(0, R)})$, $u$ is harmonic on $B(0, R)$, and $u \geq 0$ on $B(0, R)$.

(a) Prove that

\[
\frac{(R - |x|)R^{n-2}}{(R + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{(R + |x|)R^{n-2}}{(R - |x|)^{n-1}} u(0),
\]

for all $x \in B(0, R)$.

(b) Prove that

\[
|u_{x_j}(x)| \leq \frac{(2n + 2)R^{n-1}}{(R - |x|)^n} u(0),
\]

for $x \in B(0, R)$ and $j = 1, 2, \ldots, n$.

3.) Suppose $n \geq 3$, and $\Omega \subseteq \mathbb{R}^n$ is a $C^\infty$ bounded domain. Let

\[
\Gamma(x) = \frac{1}{(2 - n)\omega_n |x|^{n-2}},
\]

for $x \in \mathbb{R}^n \setminus \{0\}$, be the fundamental solution for the Laplacian on $\mathbb{R}^n$. Let $G(x, y)$ be the Green’s function for the Laplacian on $\Omega$ (i.e., $G(x, y) = h(x, y) + \Gamma(x - y)$, where, for each $x \in \Omega$, $h(x, y)$ is a harmonic function of $y$ on $\Omega$, and $h(x, y) = -\Gamma(x - y)$ for $x \in \Omega$ and $y \in \partial \Omega$). You can assume that $G \in C^2(\overline{\Omega} \times \overline{\Omega} \setminus \{(x, y) \in \overline{\Omega} \times \overline{\Omega} : x = y\})$. Prove that $\Gamma(x - y) < G(x, y) < 0$, for $(x, y) \in \Omega \times \Omega$ with $x \neq y$. 

4.) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded $C^1$ domain and suppose $T > 0$. Let $\Omega_T = \Omega \times (0, T]$. Suppose $u \in C^2_t(\overline{\Omega_T}) \cap C(\overline{\Omega_T})$ satisfies
\[
\begin{cases}
    u_t = \Delta u + |\nabla u|^2 - u(u - 1)(u - 2), & \text{for } (x, t) \in \Omega_T, \\
    u(x, t) = e^{-t}[1 + \sin(|x|^2)], & \text{for } (x, t) \in \partial \Omega \times [0, T], \\
    u(x, 0) = 1 + \sin(|x|^2), & \text{for } x \in \Omega.
\end{cases}
\]
Prove that $0 \leq u \leq 2$ on $\overline{\Omega_T}$.

5.) Suppose $g = g(x, t) \in C^2(\overline{\mathbb{R}_+^{n+1}})$, where $x \in \mathbb{R}^n$ and $t \geq 0$, and suppose $g$ has compact support. Suppose $u \in C^2_t(\overline{\mathbb{R}_+^{n+1}}) \cap C(\overline{\mathbb{R}_+^{n+1}})$ satisfies, for some positive constants $K$ and $a$,
\[
\begin{cases}
    u_t - \Delta u = g(x, t) & \text{for } x \in \mathbb{R}^n, t \in (0, \infty), \\
    u(x, 0) = 0 & \text{for } x \in \mathbb{R}^n, \\
    |u(x, t)| \leq Ke^{a|x|^2} & \text{for } x \in \mathbb{R}^n, t \in [0, \infty).
\end{cases}
\]
Suppose $p > n/2$ and $M = \max_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |g(x, t)|^p \, dx$. Prove that there exists a constant $C$, depending only on $n$ and $p$, such that
\[|u(x, t)| \leq CM^{1/p} t^{1-\frac{n}{2p}},\]
for all $(x, t) \in \mathbb{R}_+^{n+1}$.

6.) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is harmonic, and $g : \mathbb{R}^3 \to \mathbb{R}$ is $C^\infty$. Suppose $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ satisfies
\[
\begin{cases}
    u_{tt} = \Delta u, & x \in \mathbb{R}^3, t > 0 \\
    u(x, 0) = f(x), & x \in \mathbb{R}^3 \\
    u_t(x, 0) = g(x), & x \in \Omega.
\end{cases}
\]
(a) Prove that
\[|u(x, t)| \leq |f(x)| + \sup_{y \in B(0, 1)} |g(y)|\]
for $x \in \mathbb{R}^3$ and $0 < t < 1$.

(b) Prove that
\[|u(x, t)| \leq |f(x)| + \frac{3}{4\pi t^2} \int_{B(x, t)} |g(y)| \, dy + \frac{1}{4\pi t} \int_{B(x, t)} |\nabla g(y)| \, dy,
\]
for $x \in \mathbb{R}^3$ and $t \geq 1$. 
7.) Let $n \geq 2$, let $\Omega \subseteq \mathbb{R}^n$ be a $C^\infty$ bounded domain, and let $T > 0$. Suppose
$
\vec{h} = (h_1, h_2, \ldots, h_n),
$ where each component $h_j = h_j(x, t) : \overline{\Omega} \times [0, T] \rightarrow \mathbb{R}$ satisfies
$h_j \in C(\overline{\Omega} \times [0, T]).$ Suppose $f, g : \overline{\Omega} \rightarrow \mathbb{R}$ are continuous. Show that there is at most
one function $u = u(x, t) \in C^2(\overline{\Omega} \times [0, T])$ satisfying
\[
\begin{cases}
  u_{tt} = \Delta u + \nabla u \cdot \vec{h}, & x \in \Omega, \ 0 < t < T \\
  u = 0, & x \in \partial \Omega, \ 0 \leq t \leq T, \\
  u(x, 0) = f(x), & x \in \Omega, \\
  u_t(x, 0) = g(x), & x \in \Omega.
\end{cases}
\]
In the following, unless otherwise stated, $\Omega \subset \mathbb{R}^n$ is an open, bounded set with $C^\infty$-smooth boundary $\partial \Omega$. Denote $\Omega_T = \Omega \times (0,T]$.

1. Let $\Omega = \{(x,t) : x \in \mathbb{R}, t > 0\}$ and assume $u_0, v_0 \in C^1(\mathbb{R})$. Suppose $u, v \in C^1(\overline{\Omega})$ solve the system

$$u_t + u_x = u \quad \text{on} \quad \overline{\Omega},$$
$$v_t + v_x = -v + u \quad \text{on} \quad \overline{\Omega},$$
$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad x \in \mathbb{R}.$$

Find $u(x,t), v(x,t)$ in terms of $u_0, v_0$.

2. Let $R > 0$. Assume $u \in C^2(\overline{B_R(0)})$ is nonnegative and satisfies $u(0) = 0$, $0 \leq \Delta u \leq 1$ on $B_R(0)$.

Let $u_1, u_2$ be the solutions of the following problems

$$\Delta u_1 = \Delta u \quad \text{on} \quad B_R(0),$$
$$u_1 = 0 \quad \text{on} \quad \partial B_R(0).$$

$$\Delta u_2 = 0 \quad \text{on} \quad B_R(0),$$
$$u_2 = u \quad \text{on} \quad \partial B_R(0).$$

(a) Prove that $u = u_1 + u_2$ on $B_R(0)$ and $u_1 \leq 0, u_2 \geq 0$ on $B_R(0)$.
(b) Prove that $|u_1(x)| \leq \frac{R^2}{2n}$ for all $x \in B_R(0)$. Hint: Compare $u_1$ with $\phi(x) = \frac{1}{2n}(R^2 - |x|^2)$.
(c) Prove that $u_2(x) \leq \frac{2^{n-1}}{n} R^2$ for all $x \in B_{R/2}(0)$. Conclude $|u(x)| \leq \frac{1+2^n}{2n} R^2$ for all $x \in B_{R/2}(0)$.

3. Let $n \geq 3, f \in C_0^\infty(\mathbb{R}^n)$. Assume $u \in C^\infty(\mathbb{R}^n)$ is a solution of

$$-\Delta u = f \quad \text{on} \quad \mathbb{R}^n$$

and $u(x) \to 0$ as $|x| \to \infty$. Prove there exists $C > 0$ such that

$$|u(x)| \leq \frac{C}{|x|^{n-2}}$$
for all $x \in \mathbb{R}^n, x \neq 0$.

4. Let $T > 0$ and assume $\phi, h, f, g$ are $C^\infty$ smooth functions. Suppose $u, v \in C^2(\overline{\Omega}_T)$ satisfy

\[
\begin{align*}
\phi &= \Delta u = \phi \quad \text{on } \Omega_T, \\
u &= h \quad \text{on } \partial \Omega \times (0, T], \\
u &= f \quad \text{on } \Omega \times \{t = 0\}, \\
u &= \Delta v = \phi \quad \text{on } \Omega_T, \\
v &= h \quad \text{on } \partial \Omega \times (0, T], \\
v &= g \quad \text{on } \Omega \times \{t = 0\}.
\end{align*}
\]

Prove that $\int_{\Omega} |u(x, t) - v(x, t)|^2 dx \leq \int_{\Omega} |f(x) - g(x)|^2 dx$ for all $t \in [0, T]$.

5. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, bounded and $\int_{\mathbb{R}^n} |f| dx < \infty$. Show there exists a unique solution $u \in C^\infty(\mathbb{R}^n \times (0, \infty)) \cap C^0(\mathbb{R}^n \times [0, \infty))$ of

\[
\begin{align*}
\phi &= \Delta u - 2u, \quad \text{on } \mathbb{R}^n \times (0, \infty), \\
u &= f, \quad \text{on } \mathbb{R}^n \times \{t = 0\}, \\
|u(x, t)| &\leq Ce^{-2t}(1 + t)^{-\frac{n}{2}}, \quad \text{for } (x, t) \in \mathbb{R}^n \times [0, \infty),
\end{align*}
\]

for some constant $C$ depending on $f, n$ but not on $x, t$.

6. Let $f \in C^1(\mathbb{R})$ with $f'$ bounded on $\mathbb{R}$ and $f(0) = 0$. Suppose $\phi, \psi \in C^2(\overline{\Omega})$ and $u \in C^2(\overline{\Omega}_T)$ is a solution of

\[
\begin{align*}
\phi &= \Delta u = f(u) \quad \text{on } \Omega_T, \\
u &= 0 \quad \text{on } \partial \Omega \times (0, T], \\
u &= \phi, \quad \text{on } \Omega \times \{t = 0\}.
\end{align*}
\]

(a) Denoting $E(t) = \frac{1}{2} \int_{\Omega} (u_t^2 + |\nabla u|^2 + u^2) dx$, prove $E(t) \leq E(0) e^{Ct}$ for all $t \in [0, T]$, and for some constant $C > 0$.

(b) Prove the solution $u$ is unique.

7. Let $p > n/2$. Suppose $\phi, \psi \in C_0^\infty(\mathbb{R}^n)$ and $u \in C^2(\mathbb{R}^n \times [0, \infty))$ is a solution of

\[
\begin{align*}
\phi &= \Delta u = 0 \quad \text{on } \mathbb{R}^n \times [0, \infty), \\
u &= \phi, \quad \text{on } \mathbb{R}^n \times \{t = 0\}.
\end{align*}
\]

Prove that there exists $C > 0$ such that

\[
\int_{\mathbb{R}^n} \frac{|u_t| + |\nabla u|}{(1 + |x| + t)^p} dx \leq \frac{C}{(1 + t)^{p-n/2}}
\]

for all $t \geq 0$. 2
In the following, unless otherwise stated, \( \Omega \subset \mathbb{R}^n \) is an open, bounded set with \( C^\infty \)-smooth boundary \( \partial \Omega \). Denote \( \Omega_T = \Omega \times (0,T] \).

1. Let \( \Omega = \{ (x,t) : x \in \mathbb{R}, t > 0 \}, \ b \in \mathbb{R} \) and assume \( a \in C^1(\overline{\Omega}), \phi \in C^1(\mathbb{R}) \) are bounded. Suppose \( u \in C^1(\overline{\Omega}) \) is a solution of

\[
\begin{align*}
  &u_t + a(x,t)u_x + bu = 0 \quad \text{on} \quad \Omega, \\
  &u(x,0) = \phi(x), \quad x \in \mathbb{R}.
\end{align*}
\]

(a) Prove \( \sup_{\Omega} \{ u(x,t) \} \leq e^{-bt} \sup_{\mathbb{R}} \{ \phi \} \) for all \( t \geq 0 \).

(b) Find the solution when \( a = a(t) \).

2. Let \( \Omega \subset \mathbb{R}^2 \) and suppose \( g \in C^0(\partial \Omega) \). Show that there exists at most one solution \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) satisfying

\[
\begin{align*}
  &\Delta u + u_x - u_y = u^3 \quad \text{on} \quad \Omega, \\
  &u = g \quad \text{on} \quad \partial \Omega.
\end{align*}
\]

3. Let \( \Omega \subset \mathbb{R}^n \). A function \( v \in C^0(\Omega) \) is subharmonic on \( \Omega \) iff for every \( x \in \Omega \), there exists \( r(x) > 0 \) such that \( v \) satisfies the mean-value property:

\[
v(x) \leq \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x,r)} v(\xi) dS(\xi)
\]

for all \( r \in (0,r(x)] \), where \( \omega_n \) is the surface area of the unit sphere in \( \mathbb{R}^n \).

(a) Suppose \( u,v \in C^0(\Omega) \), \( u \) is harmonic on \( \Omega \), \( v \) is subharmonic on \( \Omega \), \( v \leq u \) on \( \partial \Omega \). Prove \( v \leq u \) on \( \Omega \). You can assume the maximum principle for subharmonic functions.

(b) Let \( v \in C^0(\Omega) \) be subharmonic on \( \Omega \) and \( B(x_0,R) \subset \Omega \). For \( r \in (0,R) \) define

\[
g(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x_0,r)} v(\xi) dS(\xi).
\]

Prove \( g \) is nondecreasing on \((0,R)\). Deduce the mean-value property

\[
v(x_0) \leq \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x_0,r)} v(\xi) dS(\xi)
\]
holds for any $B(x_0, r) \subset \Omega$ (note, in the definition of subharmonic function, this is assumed only for sufficiently small $r$). Hint: for $r_1 < r_2$ use the Poisson Integral Formula on $B(x_0, r_2)$ to get a harmonic function.

4. Let $m > 0$, $T > 0$ and assume $u_0 \in C^0(\bar{\Omega})$ is nonnegative on $\Omega$. Suppose $u \in C^{0,1}(\Omega_T) \cap C^0(\bar{\Omega}_T)$ is a solution of

$$u_t = \Delta u + |\nabla u|^2 + u(m - u) \text{ on } \Omega_T,$$

$$u = 0 \text{ on } \partial \Omega \times (0,T],$$

$$u = u_0 \text{ on } \Omega \times \{t = 0\}.$$

Prove $0 \leq u \leq \max\{m, \sup_{\bar{\Omega}} u_0\}$ on $\bar{\Omega}_T$.

5. Let $1 < p < \infty$, $u_0 \in C^0(\bar{\Omega})$. Consider

$$u_t = \Delta u + |u|^{p-1}u \text{ on } \Omega_T,$$

$$u = 0 \text{ on } \partial \Omega \times (0,T],$$

$$u = u_0 \text{ on } \Omega \times \{t = 0\}.$$

For each $u_0$, let $T_{\max} = T_{\max}(u_0) \in (0, \infty]$ be the maximal time such that the problem above has a solution $u \in C^{2,1}(\bar{\Omega} \times [0, T_{\max}))$. Let $E(t) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} dx, \quad y(t) = \int_{\Omega} u^2 dx$ for $t \in [0, T_{\max})$.

(a) Prove $\frac{d}{dt} E(t) = - \int_{\Omega} u^2 dx$, $t \in (0, T_{\max})$.

(b) With $c = \frac{2(p-1)}{p+1} |\Omega|^{\frac{1}{p+2}}$ prove $\frac{d}{dt} y(t) \geq -4E(0) + cy(t)^{\frac{p+1}{2}}, \quad t \in (0, T_{\max})$.

(c) Assume $u_0$ is nontrivial, $E(0) < 0$ and prove $T_{\max}(u_0) < \infty$.

6. Consider the initial-boundary value problem

$$u_{tt} - u_{xx} = -2 + \sin x \text{ on } (0, \pi) \times (0, \infty),$$

$$u = x^2 - \pi x, \quad u_t = 0 \text{ at } t = 0,$$

$$u = 0 \text{ at } x = 0, \pi.$$

(a) Find the steady state solution $u = f(x)$ of the differential equation and boundary conditions.

(b) Find the solution of the entire problem.

7. Suppose $a \in C^0(\mathbb{R}^n), a \geq 1$ on $\mathbb{R}^n$ and $u_0, u_1 \in C_0^\infty(\mathbb{R}^n)$. Suppose $u \in C^2(\mathbb{R}^n \times [0, \infty))$ is a solution of the problem

$$u_{tt} - \Delta u + a(x)u_t = 0 \text{ on } \mathbb{R}^n \times (0, \infty),$$

$$u = 0 \text{ on } \partial \mathbb{R}^n \times (0, \infty).$$
\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n, \]
\[ u_t(x, 0) = u_1(x), \quad x \in \mathbb{R}^n. \]

Let \( E(t) = \int_{\Omega} (u_t^2 + |\nabla u|^2) \, dx, \quad K(t) = \int_{\Omega} (uu_t + \frac{1}{2} au^2) \, dx, \quad t \in [0, \infty). \)

(a) Prove \( \frac{d}{dt} E \leq 0, \quad \frac{d}{dt} (K+E) \leq -E, \quad \text{and} \quad K+E \geq 0 \quad \text{for all} \quad t \geq 0. \) You may assume finite speed of propagation of solutions (the support of \( u(\cdot, t) \) is bounded in \( \mathbb{R}^n \) for each \( t \geq 0 \)).

(b) Prove \( E(t) \leq Ct^{-1} \quad \text{for all} \quad t > 0. \) Hint: Integrate an inequality in (a).
1. In the region $R := \{(x, t) : x > 0, t > 0\}$, solve the PDE

$$u_t + t^2u_x = 4u, \quad \text{with,} \quad u(0, t) = h(t), \quad u(x, 0) = 1.$$ 

Find the conditions on $h$ so that the solution is continuous on $R$.

2. Solve the following PDE (also state the domain of the solution)

$$x^2u_x + xyu_y = u^3, \quad \text{and} \quad u = 1, \quad \text{on the curve} \quad y = x^2.$$ 

3. Let $a > 0$ and $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < a^2\}$. Consider the equation

$$\begin{cases} 
\Delta u &= 0, \quad \text{in} \ D, \\
 u &= 1 + x^2 + 3xy, \quad \text{on} \ \partial D.
\end{cases}$$

without solving the equation, find $u(0, 0)$, $\max_D u$, and $\min_D u$.

4. Let $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$ for $n > 2$. Let $u$ be defined on $\overline{B_1} \setminus \{0\}$. Assume that $u \in C(\overline{B_1} \setminus \{0\}) \cap C^2(B_1 \setminus \{0\})$, $u$ is harmonic in $B_1 \setminus \{0\}$, and

$$\lim_{|x| \to 0} \frac{u(x)}{|x|^{2-n}} = 0.$$ 

Prove that $u$ can be extended to 0 so that $u \in C^2(B_1)$.

**Hint:** By using the maximum principle on $B_1 \setminus B_r$ for $0 < r < 1$, one proves that $u = v$ in $B_1 \setminus \{0\}$, where $v$ is the solution of the equation

$$\begin{cases} 
\Delta v &= 0, \quad \text{in} \ B_1, \\
v &= u, \quad \text{on} \ \partial B_1.
\end{cases}$$

5. Let $\Omega$ be a non-empty, smooth bounded domain in $\mathbb{R}^n$. Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^1$ function such that $|f'|$ is bounded. Consider the reaction-diffusion equation

$$\begin{cases} 
u_t - \Delta u + f(u) &= 0, \quad \text{in} \ \Omega \times (0, \infty), \\
u &= 0, \quad \text{on} \ \partial\Omega \times (0, \infty), \\
u(x, 0) &= u_0(x), \quad x \in \Omega.
\end{cases}$$

Prove that $C^2$ solutions to the problem are unique.
6. Let $u_0 \in C^\infty_c(\Omega)$ for some non-empty, open, smooth bounded domain $\Omega \subset \mathbb{R}^n$ with $n > 2$. Assume also that $u_0 \geq 0$. Let $u \in C^\infty(\Omega \times [0, \infty))$ be a solution of the equation

$$
\begin{cases}
  u_t = \Delta u, & \text{in } \Omega \times (0, \infty), \\
  u(\cdot, t) = 0, & \text{on } \partial\Omega \times (0, \infty), \\
  u(\cdot, 0) = u_0(\cdot), & \text{on } \Omega.
\end{cases}
$$

(a) Prove that for all $t > 0$,

$$
\|u(\cdot, t)\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}, \quad \text{and} \quad \|u(\cdot, t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^1(\Omega)}^{2^* - 2} \|u(\cdot, t)\|_{L^2(\Omega)}^{1 - \frac{2}{2^*}},
$$

where

$$
\alpha = \frac{2^* - 2}{2(2^* - 1)}, \quad \text{for} \quad 2^* = \frac{2n}{n - 2}.
$$

(b) Prove that there is $C > 0$ depending on $n, \Omega$ such that

$$
\frac{d}{dt} \int_\Omega u^2(x, t)dx \leq -C\|u_0\|_{L^1(\Omega)}^{-\frac{2\alpha}{1 + \alpha}} \left( \int_\Omega u^2(x, t)dx \right)^{\frac{1}{1 + \alpha}}.
$$

(c) Prove that (for some new $C = C(n, \Omega) > 0$)

$$
\|u(\cdot, t)\|_{L^2(\Omega)} \leq C\|u_0\|_{L^2(\Omega)}(1 + t)^{-\frac{2}{n}}, \quad t \geq 0.
$$

**Remark:** The following inequalities maybe useful

(i) Hölder's inequality:

$$
\|f\|_{L^p(\Omega)} \leq \|f\|_{L^{p_1}(\Omega)}^{\frac{\theta_1}{p}} \|f\|_{L^{p_2}(\Omega)}^{\frac{\theta_2}{p}},
$$

with

$$
\frac{1}{p} = \frac{\theta_1}{p_1} + \frac{\theta_2}{p_2}, \quad \theta_1 + \theta_2 = 1, \quad p, p_1, p_2 \in (1, \infty), \quad \theta_1, \theta_2 \in (0, 1).
$$

(ii) Sobolev - Poincaré inequality:

$$
\|\varphi\|_{L^\infty(\Omega)} \leq C(n, \Omega)\|\nabla \varphi\|_{L^2(\Omega)}, \quad \forall \varphi \in C^\infty_c(\Omega), \quad \varphi|_{\partial\Omega} = 0.
$$

7. Let $c > 0$ be a fixed number. Solve the following wave equation

$$
\begin{cases}
  u_{tt} = c^2 u_{xx} + \cos(c t) \cos(x), & -\infty < x < \infty, \quad t > 0, \\
  u(x, 0) = x, \quad u_t(x, 0) = \sin(x), & -\infty < x < \infty.
\end{cases}
$$

8. Let $u(x, t)$ be a $C^2$, compactly supported solution to the equation

$$
u_{tt} - \Delta u = 0, \quad u(x, 0) = 0, \quad u_t(x, 0) = g(x), \quad x \in \mathbb{R}^3, \quad t > 0.
$$

Assume that $\int_{\mathbb{R}^3} g(x)^2dx < \infty$. Show that

$$
\int_0^\infty u(0, t)^2dt \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} g(x)^2dx.
$$
1. Let $g$ be a given smooth function on $\mathbb{R}$. Solve the PDE

$$\begin{cases}
    u_x + u_y = u^2, & \text{on } \{(x, y) \in \mathbb{R}^2, \ y > 0\}, \\
    u(x, 0) = g(x), & x \in \mathbb{R}.
\end{cases}$$

2. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial \Omega$ for $n \in \mathbb{N}$. Let $u$ be a harmonic function in $\Omega$ and $x_0 \in \Omega$. Prove that

$$\left| \frac{\partial u(x_0)}{\partial x_i} \right| \leq \frac{n}{d} \sup_{x \in \Omega} \left| u(x) - u(x_0) \right|, \quad \text{where } d = \text{dist}(x_0, \partial \Omega), \ \forall \ i = 1, 2, \ldots, n.$$

Assume in addition that $u \geq 0$ in $\Omega$, show that

$$\left| \frac{\partial u(x_0)}{\partial x_i} \right| \leq \frac{n}{d} u(x_0), \quad \forall \ i = 1, 2, \ldots, n.$$

3. Let $\Omega = \mathbb{R}^3 \setminus \overline{B_1(0)}$, where $B_1(0)$ is an open unit ball in $\Omega$. Let $u$ be a harmonic function in $\Omega$ such that $u(x) \to 0$ as $|x| \to \infty$. Prove that there exist $r_0 > 1$ and $M > 0$ such that

$$|u(x)| \leq \frac{M}{|x|}, \quad |u_{x_k}(x)| \leq \frac{M}{|x|^2}, \quad \forall |x| \geq r_0, \ \forall k = 1, 2, 3.$$

4. Let $T \in (0, \infty)$ and $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial \Omega$ for $n \in \mathbb{N}$. Let $\Omega_T = \Omega \times (0, T]$ and $u \in C^2(\overline{\Omega_T})$ be a solution of the equation

$$\begin{cases}
    u_t - \Delta u + c(x, t)u = u^2(1 - u), & \text{in } \Omega_T, \\
    u + \frac{\partial u}{\partial \nu} = 0, & \partial \Omega \times (0, T], \\
    u(x, 0) = g(x), & x \in \Omega,
\end{cases}$$

with some given function $c(x, t)$ and $g(x)$. Assume that $c > 0$ on $\overline{\Omega_T}$ and $0 \leq g \leq 1$ on $\overline{\Omega}$. Prove that $0 \leq u \leq 1$ on $\Omega_T$.

5. Consider $\Omega = [0, a] \times [0, b] \subset \mathbb{R}^2$ for some fixed $a > 0, b > 0$.

(a) Use separation of variables to find the first (i.e. the smallest) eigenvalue $\lambda_1$ and eigenfunction $\phi_1$ of the eigenvalue problem

$$\begin{cases}
    -\Delta \phi = \lambda \phi, & \Omega, \\
    \phi = 0, & \partial \Omega.
\end{cases}$$

**Remark:** Eigenfunctions must be non-trivial.

(b) Let $g$ be a smooth function on $\overline{\Omega}$ and $g$ vanishes on $\partial \Omega$. Also, let $\kappa < \lambda_1$. Assume that $u$ is a solution of the heat equation

$$\begin{cases}
    u_t = \Delta u + \kappa u, & x \in \Omega, \ t > 0, \\
    u(x, t) = 0, & x \in \partial \Omega, \ t > 0, \\
    u(x, 0) = g(x), & x \in \Omega.
\end{cases}$$

prove that $u(x, t) \to 0$ uniformly in $x$ as $t \to \infty$. 
6. Let $T \in (0, \infty)$ and $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial \Omega$ for $n \in \mathbb{N}$. Let us denote $\Omega_T = \Omega \times (0, T)$ and $\Gamma_T$ the parabolic boundary of $\Omega_T$. Suppose that $u \in C(\overline{\Omega_T}) \cap C^2(\Omega_T)$ satisfies the PDE

$$u_t - \Delta u = c(x, t)u, \quad (x, t) \in \Omega_T$$

for some $c \in C(\overline{\Omega_T})$ and $c \leq 0$. Show that if $u \geq 0$ on $\Gamma_T$, then

$$\max_{(x,t) \in \Omega_T} u(x, t) = \max_{(x,t) \in \Gamma_T} u(x, t).$$

Give a counterexample showing that the conclusion does not hold if the condition $u \geq 0$ on $\Gamma_T$ is violated.

7. Let $T \in (0, \infty)$ and $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial \Omega$ for $n \in \mathbb{N}$. Suppose that $u \in C^2(\overline{\Omega} \times [0, T])$ is a classical solution of the equation

$$\begin{cases} u_{tt} - \Delta u = f(x,t), & \Omega \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial \Omega \times (0, T). \end{cases}$$

Let

$$E(t) = \frac{1}{2} \int_{\Omega} \left[ u_t^2(x, t) + |\nabla u|^2(x, t) \right] dx$$

(a) Prove that

$$E(t) \leq e^{T} \left[ E(0) + \frac{1}{2} \int_0^T \int_{\Omega} f^2(x, s) dx ds \right], \quad \forall t \in [0, T].$$

(b) Use the energy estimate to prove the uniqueness of the classical solution of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = f(x, t), & \Omega \times (0, T), \\ u(x, 0) = 0, & (x, t) \in \partial \Omega \times (0, T) \\ u_t(x, 0) = g(x), & x \in \Omega, \\ u(x, 0) = h(x), & x \in \Omega. \end{cases}$$

8. Let $f \in C^1(\mathbb{R}^3)$ with compact support. Suppose that $u \in C^2(\mathbb{R}^3 \times (0, \infty))$ and $u$ solves the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = 0, & \mathbb{R}^3 \times (0, \infty), \\ u(x, 0) = 0, & x \in \mathbb{R}^3, \\ u_t(x, 0) = f(x), & x \in \mathbb{R}^3. \end{cases}$$

Prove that there is $M > 0$ such that

$$|u(x, t)| \leq \frac{M}{1 + t} \left[ \|f\|_{L^\infty(\mathbb{R}^3)} + \|f\|_{L^1(\mathbb{R}^3)} + \|\nabla f\|_{L^1(\mathbb{R}^3)} \right], \quad \forall t \geq 0.$$
1.) (a) Solve the following Cauchy problem on $\mathbb{R}^2$:

$$\begin{cases}
    u_x + u_y = x + y \\
    u = x^3 \text{ on the line } y = -x.
\end{cases}$$

(b) For what $C^1$ function or functions $f(x)$ does the Cauchy problem on $\mathbb{R}^2$:

$$\begin{cases}
    u_x + u_y = 3u \\
    u = f(x) \text{ on the line } y = x
\end{cases}$$

have a solution? Prove your answer.

2.) Consider Burger's equation

$$\begin{cases}
    uu_x + u_y = 0, \text{ for } x \in \mathbb{R}, y > 0 \\
    u(x, 0) = f(x), \text{ for } x \in \mathbb{R},
\end{cases}$$

with initial data

$$f(x) = \begin{cases}
    4, & \text{for } x < 0, \\
    4 - \frac{x}{2}, & \text{for } 0 \leq x \leq 2, \\
    3, & \text{for } x > 2.
\end{cases}$$

(a) Find, with proof, the smallest $y^* > 0$ such that a shock occurs at $(x, y^*)$ for some $x \in \mathbb{R}$.

(b) Find $u(x, y)$ satisfying (*) for $x \in \mathbb{R}$ and $0 \leq y < y^*$, except on two line segments where the partial derivatives of $u$ may not exist.

(c) Find the integral, or weak, solution $u(x, y)$ of (*) for $y \geq 0$.

3.) (a) Suppose $f \in C^\infty(\mathbb{R}^n)$ satisfies $f(x) > 0$ for all $x \in \mathbb{R}^n$. Suppose $u \in C^2(\mathbb{R}^n)$ satisfies

$$\Delta u - f(x)u = 0$$

on $\mathbb{R}^n$, and $u(x) \to 0$ uniformly as $|x| \to \infty$. Prove that $u$ is identically 0.

(b) Find a non-trivial solution of $\Delta u + u = 0$ in $\mathbb{R}^3$ such that $u(x) \to 0$ uniformly as $|x| \to \infty$. Hint: look for a radial solution $u(x, y, z) = v(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$ and note that $rv'' + 2v' = (rv)'$. 
4.) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Suppose that $\{u_n\}_{n=1}^\infty$ is a sequence of harmonic functions on $\Omega$ such that

$$\int_\Omega |u_n(x) - u_m(x)|^2 \, dx \rightarrow 0$$

as $\max\{n, m\} \rightarrow \infty$. Prove that $u_n$ converges to a harmonic function on $\Omega$.

5.) Suppose $u = u(x, t) \in C^2([0, 1] \times [0, T])$ satisfies

$$\begin{align*}
  u_t &= u_{xx} + tu_x, & x \in [0, 1], t \in [0, T] \\
  u_x(0, t) &= u_x(1, t) = 0, & t \in [0, T].
\end{align*}$$

Prove that

$$\max_{[0,1] \times [0,T]} u(x, t) = \max_{[0,1]} u(x, 0).$$

If you use a major theorem in PDE in your solution, provide the proof of that theorem.

6.) (a) Suppose $u = u(x, t) \in C(\mathbb{R}^n \times [0, \infty)) \cap C^2(\mathbb{R}^n \times (0, \infty))$ satisfies

$$\begin{align*}
  u_t &= \Delta u, & \text{for } x \in \mathbb{R}^n, t > 0, \\
  u(x, 0) &= f(x), & \text{for } x \in \mathbb{R}^n,
\end{align*}$$

where $f(x) \geq 0$ is a $C^\infty$, bounded function satisfying $\int_{\mathbb{R}^n} f(x) \, dx = 2$. Suppose $u$ satisfies

$$|u(x, t)| \leq Ae^{\alpha|x|^2},$$

for some positive constants $\alpha$ and $A$. Prove that $\lim_{t \rightarrow \infty} u(x, t) = 0$ and $\int_{\mathbb{R}^n} u(x, t) \, dx = 2$ for all $t > 0$.

(b) Does there exist a bounded solution $u(x, t) \in C(\mathbb{R}^n \times [0, \infty)) \cap C^2(\mathbb{R}^n \times (0, \infty))$ of the initial value problem

$$\begin{align*}
  u_t &= \Delta u + \frac{\cos(|x|^2+1)}{1+|x|^2}, & \text{for } x \in \mathbb{R}^n, t > 0, \\
  u(x, 0) &= 0, & \text{for } x \in \mathbb{R}^n?
\end{align*}$$

Justify your answer.
7.) Suppose $u = u(x, t) \in C^2(\mathbb{R} \times [0, \infty))$ satisfies

$$
\begin{align*}
&\begin{aligned}
& u_{tt} - u_{xx} + u = 0, \quad \text{for } x \in \mathbb{R}, t > 0, \\
& u(x, 0) = f(x), \quad \text{for } x \in \mathbb{R}, \\
& u_t(x, 0) = g(x), \quad \text{for } x \in \mathbb{R},
\end{aligned}
\end{align*}
$$

where $f$ and $g$ are $C^\infty$ and have compact support.

(a) For any $(x_0, t_0) \in \mathbb{R} \times (0, \infty)$ and $0 \leq t \leq t_0$, let $I(t)$ be the interval

$$
I(t) = [x_0 - t + t, x_0 + t_0 - t].
$$

Define

$$
e(t) = \int_{I(t)} [u^2 + u_t^2 + u_x^2](x, t) \, dx,
$$

for $0 \leq t \leq t_0$. Prove that $e$ is non-increasing on $[0, t_0]$.

(b) Suppose that $f(x) = 0$ and $g(x) = 0$ for $|x| \geq 1$. Prove that $u(x, t) = 0$ for $|x| > t + 1$, for all $t > 0$.

8.) Suppose $u = u(x, t) \in C^2(\mathbb{R} \times [0, \infty))$, is the solution of the wave equation

$$
\begin{align*}
&\begin{aligned}
& u_{tt} = \Delta u, \quad x \in \mathbb{R}, t > 0 \\
& u(x, 0) = f(x), \quad x \in \mathbb{R}, \\
& u_t(x, 0) = g(x), \quad x \in \mathbb{R}.
\end{aligned}
\end{align*}
$$

Suppose $g$ and $h$ are $C^\infty$ with $f(x) = g(x) = 0$ for all $x$ such that $|x| \geq R$, for some $R > 0$. The kinetic energy is

$$
k(t) = \frac{1}{2} \int_{\mathbb{R}} u_t^2(x, t) \, dx
$$

and the potential energy is

$$
p(t) = \frac{1}{2} \int_{\mathbb{R}} u_x^2(x, t) \, dx.
$$

(a) Prove that $k(t) + p(t)$ is constant.

(b) Prove that $k(t) = p(t)$ for all $t > R$. 
1.) Consider the equation

\[(*) \quad u_x + 2u_y = u,\]

for \((x, y) \in \mathbb{R}^2\).

(a) Solve \((*)\) with the Cauchy data \(u(x, x) = e^{3x}\) for all \(x \in \mathbb{R}\).

(b) Suppose \(u\) satisfies \((*)\) with Cauchy data \(u(x, 2x) = f(x)\). Prove that \(f(x) = Ce^{x}\) for some constant \(C\).

(c) For each constant \(C \neq 0\), show that \((*)\) with Cauchy data \(u(x, 2x) = Ce^{x}\) has infinitely many solutions.

2.) Reduce the following equation on \(\mathbb{R}^2\):

\[u_{xx} + 6x^2 u_{xy} + 9x^4 u_{yy} + 6xu_y + y - x^3 = 0\]

to canonical form and find the general solution.

3.) Let \(\Omega \subseteq \mathbb{R}^n\) be a smooth \((C^\infty)\), bounded open set. Consider the problem

\[(**) \quad \begin{cases} 
\Delta u(x) = f(x), & \text{for } x \in \Omega \\
u(x) + \frac{\partial u}{\partial n} = g(x), & \text{for } x \in \partial \Omega.
\end{cases}\]

where \(f \in C(\Omega)\), \(g \in C(\partial \Omega)\), and \(\frac{\partial}{\partial n}\) is the outward normal derivative on \(\partial \Omega\).

(a) Prove that there is at most one \(u \in C^2(\overline{\Omega})\) satisfying \((**)\).

(b) Suppose \(u \in C^2(\overline{\Omega})\) satisfies \((**)\), with \(f \geq 0\) on \(\Omega\) and \(g \leq 0\) on \(\partial \Omega\). Prove that \(u \leq 0\) on \(\Omega\).

4.) Suppose \(u = u(x, t) \in C([0, 1] \times [0, \infty)) \cap C^2((0, 1) \times (0, \infty))\), and \(u\) satisfies

\[
\begin{cases}
    u_t = u_{xx}, & \text{for } 0 < x < 1, t > 0, \\
    u(0, t) = u(1, t) = 0, & \text{for } t \geq 0, \\
    u(x, 0) = 4x(1 - x), & \text{for } 0 \leq x \leq 1.
\end{cases}
\]

Prove that

(a) \(0 < u(x, t) < 1\) for \(0 < x < 1, t > 0\);

(b) \(u(1 - x, t) = u(x, t)\) for \(0 \leq x \leq 1, t > 0\);

(c) \(-8 < -u_{xx}(x, t) < 0\) for \(0 < x < 1, t > 0\);

(d) \(\int_0^1 u^2(x, t) \, dx\) is a strictly decreasing function of \(t\).
5.) Suppose \( u = u(x, t) \in C^2([0, 1] \times [0, \infty)) \) satisfies
\[
\begin{cases}
  u_{tt} - u_{xx} = -\frac{u}{1+u^2}, & \text{for } 0 < x < 1, t > 0 \\
  u(0, t) = u(1, t) = 0, & \text{for } t \geq 0, \\
  u(x, 0) = g(x), & \text{for } 0 \leq x \leq 1,
\end{cases}
\]
where \( g \) is a given function satisfying \( g(0) = g(1) = 0 \).

(a) Define
\[
E(t) = \frac{1}{2} \int_0^1 u_t^2 + u_x^2 + \log(1 + u^2) \, dx,
\]
for \( t \geq 0 \). Prove that \( E \) is constant.

(b) Show that there exists \( C > 0 \) such that \( |u(x, t)| \leq C \) for all \( x \in [0, 1] \) and \( t \geq 0 \).

6.) Let \( \Omega \subseteq \mathbb{R}^n \) be an open set.

(a) Suppose \( u \in C^1(\overline{\Omega}) \) and
\[
\int_{\partial B(x,r)} \frac{\partial u}{\partial n} \, dS \geq 0
\]
for every \( x \in \mathbb{R}^n \) and \( r > 0 \) such that \( B(x, r) \subseteq \Omega \), where \( \frac{\partial}{\partial n} \) is the outward normal derivative on \( \partial \Omega \) and \( dS \) is surface measure on \( \partial \Omega \). Prove that \( u \) is subharmonic on \( \Omega \). Warning: a subharmonic function is not necessarily \( C^2 \).

(b) Prove the converse of part (a) under the additional assumption that \( u \in C^2(\overline{\Omega}) \).

7.) Let \( \Omega \subseteq \mathbb{R}^n \) be a smooth bounded open set. Let \( h \leq 0 \) be a continuous function on \( \overline{\Omega} \times [0, \infty) \). Prove that there exists at most one function \( u = u(x, t) \in C^2(\overline{\Omega} \times [0, \infty)) \) satisfying
\[
\begin{cases}
  u_t = \Delta u + h(x, t) u, & \text{for } x \in \Omega, t \geq 0 \\
  u(x, 0) = f(x), & \text{for } x \in \Omega, \\
  u(x, t) = g(x, t), & \text{for } x \in \partial \Omega, t \geq 0.
\end{cases}
\]

8.) Suppose \( u = u(x, t) \in C^2(\mathbb{R}^3 \times [0, \infty)) \), is the solution of the wave equation
\[
\begin{cases}
  u_{tt} = \Delta u, & \text{for } x \in \mathbb{R}^3, t > 0 \\
  u(x, 0) = 0, & \text{for } x \in \mathbb{R}^3, \\
  u_t(x, 0) = g(x), & \text{for } x \in \mathbb{R}^3.
\end{cases}
\]
Suppose \( g(x) = 1 \) for \( |x| > 1 \). Prove that
\[
u(x, t) = t
\]
if (i) \( |x| > t + 1 \) or (ii) \( |x| < t - 1 \).
Problem 1. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a bounded \( C^2 \) function that satisfies
\[
\nabla f = G,
\]
where \( G : \mathbb{R}^n \to \mathbb{R}^n \) satisfies
\[
\int_{\partial B_r(x_0)} G(x) \cdot (x - x_0) dA(x) = 0,
\]
for all \( x_0 \in \mathbb{R}^n, \ r > 0 \). Prove that \( f \) is constant.

Problem 2. Let \( \Omega = \{(x,t) : 0 < x < 1, \ 0 < t < \infty \} \). Assume that \( u \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega}) \) satisfies the initial boundary value problem given by the equation
\[
\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)
\]
in the interior of the region \( \Omega \), together with the boundary conditions
\[
u(x,0) = f(x), \ u(0,t) = a(t), \ u(1,t) = \beta(t),
\]
where \( f(0) = \alpha(0), \ f(1) = \beta(0) \).

(a) Show that \( u(x,t) \) cannot have a maximum where \( \frac{\partial^2 u}{\partial x^2} < 0 \) in the interior of the region in \( (x,t) \) space with \( t > 0 \) and \( 0 < x < 1 \).

(b) State the strong maximum/minimum principle for the previous IVBP.

(c) Using a maximum/minimum principle show that if \( f(x) \geq 0, \ a(t) \geq 0, \) and \( \beta(t) \geq 0, \) then \( u(x,t) \geq 0 \).

Problem 3. Suppose \( u : \mathbb{R}^2 \to \mathbb{R} \) is \( C^1 \) and bounded and satisfies the PDE
\[
u(x,y) = a(x,y)u_x(x,y) + b(x,y)u_y(x,y).
\]

(a) Show that if \( a \) and \( b \) are constant functions, then \( u \) is identically 0.

(b) Prove that if \( a = 1 + x^2 \) and \( b = 1 + y^2 \), the above PDE has non-vanishing bounded solutions.

Problem 4. Consider the cube \( \Omega = (1,2)^3 \). Suppose \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) satisfies
\[
yu_{xx} + xu_{yy} + xu_{zz} = 1
\]
in \( \Omega \), with \( u = 0 \) on the boundary \( \partial \Omega \). Prove that \( u \geq -\frac{1}{8} \).

Hint. Compare with a function of the type \( v(\overline{x}) = a + b |\overline{x} - \overline{x}_0|^2 \), where \( a, b \in \mathbb{R}, \ \overline{x}_0 \in \mathbb{R}^3 \).
Problem 5. Consider the unbounded domain $\Omega = \{(x,y) : y > x^2\} \subset \mathbb{R}^2$. Suppose $u$ is bounded and harmonic on $\Omega$, and vanishes on $\partial \Omega$. Show $u \equiv 0$.

*Hint.* Test with $u \chi$, where $\chi(y)$ is a cutoff function in the second variable $y$, and is nonconstant only on $y \in [\ell, 2\ell]$.

Problem 6. Suppose $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ is a solution of

$$
\begin{cases}
    u_{tt} - \Delta u &= 0 \quad \text{on } \mathbb{R}^3 \times [0, \infty), \\
    u(x,0) &= 0 \quad x \in \mathbb{R}^3, \\
    u_t(x,0) &= \psi(x) \quad x \in \mathbb{R}^3,
\end{cases}
$$

where $\psi \in C^\infty(\mathbb{R}^3)$ has compact support. Let $p \in [2, \infty)$. Prove that there exists $C > 0$ such that:

(a) $|\nabla u(x,t)| \leq C(1 + t)^{-1}$ for all $(x,t) \in \mathbb{R}^3 \times [0, \infty),$

(b) $\int_{\mathbb{R}^3} |\nabla u(x,t)|^p dx \leq C(1 + t)^{2-p}$ for all $t \geq 0$.

Problem 7. Suppose $u \in C^2(\mathbb{R}^n \times [0, \infty))$ is a solution of

$$
\begin{cases}
    u_{tt} - \Delta u &= 0 \quad \text{on } \mathbb{R}^n \times [0, \infty), \\
    u(x,0) &= \phi(x) \quad x \in \mathbb{R}^n, \\
    u_t(x,0) &= \psi(x) \quad x \in \mathbb{R}^n,
\end{cases}
$$

where $\phi, \psi \in C^\infty(\mathbb{R}^n)$ have compact support. Prove that there exists $C, T > 0$ such that

$$
\int_{\mathbb{R}^n} \frac{(|u_t| + |\nabla u|)^4}{1 + |x| + t} \, dx \geq C t^{-n-1}
$$

for all $t \geq T$. 
SOLUTIONS

Q1. $G$ is $C^1$ since $f$ is $C^2$. Using the integral condition and the divergence theorem we obtain that $\int_{\partial B} G \cdot ndA = \int_B \text{div} \; G = 0$ on any ball $B$. Since $G$ is $C^1$ it follows that $\text{div} \; G = 0$ everywhere. Taking the divergence of the first equation we obtain $\text{div} \; \nabla f = \Delta f = \text{div} \; G = 0$, i.e. $f$ is harmonic. Since $f$ is also bounded, it must be constant.

Q2. Will type it soon.

Q3. Along the characteristic curves $\dot{x} = a$, $\dot{y} = b$, the solution $u$ satisfies the equation $\dot{z} = a$, hence $z(t) = x(0)e^t$. For $t \in \mathbb{R}$, this is bounded exactly if $z(0) = 0$. The reasoning with $t \in \mathbb{R}$ applies for $a, b$ constant functions, because then the characteristic curves do exist for all $t$, namely $x(t) = x_0 + at$, $y(t) = y_0 + bt$. The same reasoning would apply for any locally Lipschitz functions $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ that satisfy (eg) linear bounds $|a(x, y)| + |b(x, y)| \leq C_0(|x| + |y|)$, by some ODE theory that we may not assume known in this generality, and which would guarantee global existence in time for the characteristic curves.

In contrast, for $\dot{x} = 1 + x^2$, $\dot{y} = 1 + y^2$, we cover the plane with characteristic curves $x(t) = \tan(t + c_0) = \tan(t + \arctan x_0)$, $y(t) = \tan(t + c_1) = \tan(t + \arctan y_0)$ that exist for an interval of finite length $\leq \pi$ only. We do not need $z(0) = 0$ for $z(t) = x(0)e^t$ to be bounded on this interval. Specifically, we can choose initial data $x(0) = s$, $y(0) = -s$, $z(0) = f(s)$ for any bounded function $f$. Then

$$u(x, y) = \exp \left[ \frac{1}{2} (\arctan x + \arctan y) \right] f \left[ \frac{1}{2} (\arctan x - \arctan y) \right]$$

Q4. We consider $v(x, y, z) := M + \frac{1}{6} \left( (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 + (z - \frac{1}{2})^2 \right)$ where $M$ is yet to be determined. (It will turn out that we want $M = -\frac{1}{6}$. We want to show, by maximum principle, that $w := u - v \geq 0$.

First we note that on $\Omega$, it holds $yw_{xx} + zw_{yy} + zw_{zz} = \frac{2}{3} (x + y + z) > 1$. Therefore $yw_{xx} + zw_{yy} + zw_{zz} < 0$ in $\Omega$. Now $w$ does have a minimum on the compact $\Omega$. If the minimum were in the interior, we'd have $w_{xx} \geq 0, w_{yy} \geq 0, w_{zz} \geq 0$ there, and thus $yw_{xx} + zw_{yy} + zw_{zz} \geq 0$ in violation of the DE. So min $w$ is taken on at the boundary, where it equals $-\max v = -M - \frac{1}{6} \left( (\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 \right) = -M - \frac{1}{6}$, which equals 0 for our choice $M = -\frac{1}{6}$.

So we have $w \geq 0$, i.e., $u \geq v \geq M = -\frac{1}{6}$ on $\Omega$.

Q5. We can design $\chi$ in such a way that $\chi(y) = 1$ for $y \leq \ell$, $\chi(y) = 0$ for $y \geq 2\ell$, $|\chi| \leq c/\ell$, $|\chi''| \leq c/\ell^2$.

Then

$$0 = \int_\Omega \Delta u(x) = \int_\Omega \nabla u \cdot (\nabla u(x)) = -\int_\Omega |\nabla u|^2 \chi + \frac{1}{2} \int_\Omega (\nabla u^2) \cdot \nabla \chi$$

$$= -\int_\Omega |\nabla u|^2 \chi + \frac{1}{2} \int_\Omega u^2 \Delta \chi - \frac{1}{2} \int_{\partial \Omega} u^2 \partial_\nu \chi \; dS.$$
The boundary term vanishes; the second term, with \( u \) bounded by \( M \), can be estimated by \( M^2 (c/\ell^2) (c\ell^{3/2}) \), hence it goes to 0 as \( \ell \to \infty \). Hence we find, in this limit, that \( 0 = -\int_{\Omega} |\nabla u|^2 \), and \( u \equiv \text{const} \). By DBC, \( u \equiv 0 \).

Q6 & Q7. See Henry's sheet.
In the following, unless otherwise stated, $\Omega \subset \mathbb{R}^n$ is an open, bounded set with $C^\infty$-smooth boundary $\partial \Omega$. Denote $\Omega_T = \Omega \times (0,T]$, $\Gamma_T$ = parabolic boundary of $\Omega_T = \overline{\Omega_T} \setminus \Omega_T$.

**Problem 1.** Let $Q = \{(x,y) \in \mathbb{R}^2 : x > 0, y \geq 0\}$. Find the solution $u \in C^1(\Omega)$ of the initial-value problem

$$-2xu_x + (x+y)u_y = 0, \quad (x,y) \in Q,$$

$$u(x,0) = x, \quad x > 0.$$

**Problem 2.** Let $\Omega = \{x \in \mathbb{R}^3 : 0 < |x| < 1\}$, $S = \{x \in \mathbb{R}^3 : |x| = 1\}$. Suppose $u \in C^2(\Omega) \cap C^0(\Omega \cup S)$ satisfies $\Delta u \geq 0$ on $\Omega$, $u = 0$ on $S$ and $u$ is bounded on $\Omega$. Prove $u \leq 0$ on $\Omega$.

Hint: Consider $v(x) = u(x) - \epsilon(1/|x| - 1)$ on an appropriate subdomain of $\Omega$.

**Problem 3.** Suppose $\alpha \in \mathbb{R}, T > 0$ and $f \in C^0(\overline{\Omega})$ with $f > 0$ on $\Omega$. Let $u \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega_T})$ be a solution of

$$u_t = \Delta u + f(x) + \alpha u \quad \text{on} \quad \Omega_T,$$

$$u = 0 \quad \text{on} \quad \Gamma_T.$$

Prove $u \geq 0$ and $u_t \geq 0$ on $\Omega \times [0,T]$.

**Problem 4.** Let $a, b \in \mathbb{R}, T > 0$. Suppose $\phi, \psi \in C^\infty(\overline{\Omega})$ and $u \in C^2(\Omega_T) \cap C^0(\overline{\Omega_T})$ is a solution of

$$u_{tt} - \Delta u + au_{x_1} + bu = 0 \quad \text{on} \quad \Omega_T,$$

$$u = 0 \quad \text{on} \quad \partial \Omega \times (0,T],$$

$$u = \phi \quad \text{on} \quad \Omega \times \{t = 0\},$$

$$u_t = \psi \quad \text{on} \quad \Omega \times \{t = 0\}.$$

Denoting the energy $E(t) = \frac{1}{2} \int_{\Omega} (u_t^2 + |\nabla u|^2)dx$, prove $E(t) \leq E(0)e^{kt}$ for all $t \in [0,T]$, for some constant $k > 0$. Here $x = (x_1, ..., x_n) \in \mathbb{R}^n$. 

1
Problem 5. Let $Q = \{(x,t) : x > 0, t > 0\}$. Find the solution $u \in C^2(Q) \cap C^1(\overline{Q})$ of
\[ \begin{align*}
    u_{tt} - u_{xx} &= 0, \quad (x,t) \in Q, \\
    u(x,0) &= x, \quad x > 0, \\
    u_t(x,0) &= -1, \quad x > 0, \\
    u_x(0,t) + fu(0,t) &= 1, \quad t > 0.
\end{align*} \]

Problem 6. Consider the heat equation
\[ u_t = \Delta u \quad \text{on} \quad \Omega_T \]
and define $E(t) = \int_{\Omega} u(x,t)^2 dx, t \in [0,T]$. With Dirichlet boundary conditions $u = 0$ on $\partial \Omega \times (0,T]$, in order to prove backward uniqueness of solutions, it is sufficient to establish $E^2 \leq EE''$ on $[0,T]$. Prove the same inequality for Robin boundary conditions $\partial u/\partial n = g(x)u$ on $\partial \Omega \times (0,T], g \in C^0(\partial \Omega)$.

Problem 7. Let $G(x,y)$ be the Green's function for $-\Delta$ on $\Omega$ with Dirichlet boundary conditions. Define $g(x) = \int_{\Omega} G(x,y)dy, x \in \Omega$. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of
\[ \begin{align*}
    -\Delta u &= e^{-u} \quad \text{on} \quad \Omega, \\
    u &= 0 \quad \text{on} \quad \partial \Omega.
\end{align*} \]

(a) Find $-\Delta g$.
(b) Prove there exists a constant $m > 0$ such that $mg \leq u \leq g$ on $\Omega$. Express $m$ in some explicit form involving $g$. 

2
In the following $\Omega \subset \mathbb{R}^n$ is an open, bounded set with $C^\infty$-smooth boundary $\partial \Omega$. Denote $\Omega_T = \Omega \times (0, T]$, $\Gamma_T$ = parabolic boundary of $\Omega_T = \overline{\Omega_T} \setminus \Omega_T$.

**Problem 1.** Find all positive solutions $u$ defined on all of $\mathbb{R}^2$ to the equation $xu_x + yu_y = (x^2 + y^2)/u$.

**Problem 2.** Suppose $f \in C^0(\partial \Omega), f \geq 0$ on $\partial \Omega$. Show that if a solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to the boundary-value problem

$$ -\Delta u = \frac{1}{1 + u^2} \quad \text{on} \quad \Omega, $$

$$ u = f \quad \text{on} \quad \partial \Omega, $$

exists, then it is unique.

**Problem 3.** Suppose $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ is a solution of

$$ u_{tt} - \Delta u = 0 \quad \text{on} \quad \mathbb{R}^3 \times [0, \infty), $$

$$ u(x, 0) = 0, \quad x \in \mathbb{R}^3, $$

$$ u_t(x, 0) = g(x), \quad x \in \mathbb{R}^3, $$

where $g \in C^2(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. Prove that there exists $C > 0$ such that

$$ \sup_{x \in \mathbb{R}^3} \int_0^\infty u(x, t)^2 \, dt \leq C \|g\|^2_{L^2(\mathbb{R}^3)}. $$

**Problem 4.** Let $T > 0$ and suppose $f \in C^1(\mathbb{R}), f(0) = 0$. Consider the problem

$$ u_t = \Delta u + f(u) \quad \text{on} \quad \Omega_T, $$

$$ u = 0 \quad \text{on} \quad \Gamma_T. $$

Prove this has a solution $u \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega_T})$ and that the solution is unique.
Problem 5. Let $\Omega = (0, \pi), Q = \Omega \times (0, \infty), f \in C^0([0, \pi]), f(0) = f(\pi) = 0$. Prove the problem
\[
\begin{align*}
    u_t &= u_{xx} + u^2 \quad \text{on } Q, \\
    u &= 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
    u &= f \quad \text{on } \Omega \times \{t = 0\},
\end{align*}
\]
has no solution $u \in C^{2,1}(Q) \cap C^0(\overline{Q})$ if $I = \int_0^\pi f(x) \sin x \, dx$ is sufficiently large and positive.

Hint: Derive a differential inequality for $F(t) = \int_0^\pi u(x, t) \sin x \, dx$ and obtain a contradiction.

Problem 6. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of
\[
\begin{align*}
    \Delta u &= u^3 - u \quad \text{on } \Omega, \\
    u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Prove
(a) $-1 \leq u \leq 1$ on $\Omega$,  
(b) $|u(x)| \neq 1$ for all $x \in \Omega$.

Problem 7. Let $T > 0, 1 < p \leq m$. Suppose $\phi, \psi \in C^\infty(\overline{\Omega})$ and $u \in C^2(\Omega_T) \cap C^0(\overline{\Omega_T})$ is a solution of
\[
\begin{align*}
    u_{tt} - \Delta u + u_t |u_t|^{m-1} &= u|u|^{p-1} \quad \text{on } \Omega_T, \\
    u &= 0 \quad \text{on } \partial \Omega \times (0, T], \\
    u &= \phi \quad \text{on } \Omega \times \{t = 0\}, \\
    u_t &= \psi \quad \text{on } \Omega \times \{t = 0\}.
\end{align*}
\]

Denote $H(t) = \frac{1}{2} \|u_t(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla u(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{p+1} \|u(\cdot, t)\|_{L^{p+1}(\Omega)}^{p+1}$, $t \in [0, T]$ ($H$ is not the energy for the p.d.e.). Prove that for some constant $c > 0, H(t) \leq H(0)e^{ct}$ for all $t \in [0, T]$.

Hint: Calculate $\dot{H}(t)$.
Prelim Aug 2011 Partial Differential Equations

Problem 1:
Prove that every positive harmonic function in all of $\mathbb{R}^n$ is a constant. Conclude that every semi-bounded harmonic function in all of $\mathbb{R}^n$ is a constant.

Problem 2:
Show that the damped Burger’s equation $u_t + uu_x = -u$, for $x \in \mathbb{R}$, $t \geq 0$, with initial data $u(x, 0) = \phi(x)$ (for a positive $C^1$ function $\phi$) has a global solution for $t \geq 0$, provided $\phi'(x) > -1$.

Problem 3:
Let $Q = \mathbb{R}^n \times (0, \infty)$, $f \in L^1(\mathbb{R}^n)$, and let $u \in C^{2,1}(Q) \cap C^0(\overline{Q})$ be the solution of the problem

$$\begin{align*}
u_t - \Delta u + u &= 0 &\text{for } t > 0, x \in \mathbb{R}^n \\
u(x, 0) &= f(x) &\text{for } x \in \mathbb{R}^n.
\end{align*}$$
subject to the growth condition $|u(x, t)| \leq Ae^{\alpha x^2}$ for $x \in \mathbb{R}^n$ and $t \geq 0$, with certain positive constants $A, \alpha$. Show that

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-n/2} e^{-t/\|f\|_{L^1(\mathbb{R}^n)}}$$
for all $t > 0$.

Problem 4:
Let $Q = \mathbb{R}^n \times (0, \infty)$, $f \in L^1(\mathbb{R}^n)$, and $g \in C^0[0, \infty) \cap L^1(0, \infty)$. Assume that $\lim_{t \to \infty} g(t)$ exists. Suppose $u \in C^{2,1}(Q) \cap C^0(\overline{Q})$ satisfies

$$\begin{align*}
u_t - \Delta u &= g(t) &\text{on } Q \\
u &= f &\text{on } \mathbb{R}^n \times \{ t = 0 \}
\end{align*}$$
and that the usual growth condition that implies uniqueness is satisfied. Show

$$\lim_{t \to \infty} u(x, t) = \int_0^\infty g(t) \, dt \text{ and } \lim_{t \to \infty} u_t(x, t) = 0$$
for each $x \in \mathbb{R}^n$. 

1
Problem 5:
Assume in a bounded domain $\Omega \subset \mathbb{R}^n$, we have a solution $u \in C^0(\Omega) \cap C^2(\Omega)$ to $\Delta u = u^3 - 1$ and a solution $v$ to $\Delta v = v - 1$, each vanishing at the boundary. Show that $0 < v \leq u \leq 1$ in $\Omega$.

Problem 6:
Let $g \in C^2(\mathbb{R}^3)$ satisfy the conditions

$$|g(x)| < C \quad \text{and} \quad \int_{\mathbb{R}^3} |\nabla g(x)| \, dx < 4\pi C \quad \text{and} \quad \lim_{|x| \to \infty} g(x) = 0$$

and consider a classical solution $u$ to the wave equation

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty)$$
$$u(x, 0) = C \quad \text{for } x \in \mathbb{R}^3$$
$$u_t(x, 0) = g(x) \quad \text{for } x \in \mathbb{R}^3$$

where $C$ is a given positive constant. Prove that $u(x, t) > 0$ for all $(x, t) \in \mathbb{R}^3 \times [0, \infty)$.

Problem 7:
Suppose $\phi \in C^\infty(\mathbb{R}^n)$ and $\psi \in C^\infty(\mathbb{R}^n)$ have support contained in the ball $B(0, r)$, and that $u \in C^2(\mathbb{R}^n \times [0, \infty))$ is a solution to

$$u_{tt} - \Delta u + \frac{1}{1+|x|} u_t = 0 \quad \text{on } \mathbb{R}^n \times (0, \infty)$$
$$u(x, 0) = \phi(x) \quad \text{for } x \in \mathbb{R}^n$$
$$u_t(x, 0) = \psi(x) \quad \text{for } x \in \mathbb{R}^n$$

Define $E(t) := \frac{1}{2} \int_{\mathbb{R}^n} (u_t^2 + |\nabla u|^2) \, dx$ and $I(t) := \int_t^\infty \int_{\mathbb{R}^n} \frac{1}{1+|x|} (u_t^2 + |\nabla u|^2) \, dx \, ds$.

(a) Prove that $\int_t^\infty \int_{\mathbb{R}^n} \frac{1}{1+|x|} u_t^2 \, dx \, ds \leq E(t)$.

For your information: it can be proved that $I(t) \leq CE(t)$. You do not need to do this; only be assured of the corollary that $I(t)$ is finite.

(b) Prove that there exists a positive constant $C$ such that $I(t) \geq CE(2t)$ for all $t \geq r$ (with the $r$ from the support of the data). Hints: $I(t) \geq \int_t^\infty \ldots$. You may assume that the support of $u$ has the same properties as solutions to the wave equation whose initial data have support in $B(0, r)$. And you may assume that $E(t)$ is non-increasing in $t$. 

2
In the following $\Omega \subset \mathbb{R}^n$ is an open, bounded set with $C^{\infty}$-smooth boundary $\partial \Omega$. Denote $\Omega_T = \Omega \times (0,T]$.

**Problem 1.** Prove the pde $u_x + 2xu_y = (y^2 - x^2)u^2 + 1$ cannot have a solution $u \in C^1(\mathbb{R}^2)$ in the entire plane $\mathbb{R}^2$.

**Problem 2.** Let $a \in \mathbb{R}$. Show the problem

$$\Delta u = u^5 + a \text{ on } \Omega,$$

$$u = 0 \text{ on } \partial \Omega,$$

has at most one solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

**Problem 3.** Let $Q = \mathbb{R}^n \times (0,\infty)$ and suppose $u \in C^{2,1}(Q) \cap C^0(\overline{Q})$ is a solution of

$$u_t - \Delta u = 0 \text{ on } Q,$$

$$u = g(x) \text{ on } \mathbb{R}^n \times \{t = 0\},$$

satisfying the growth condition

$$|u(x,t)| \leq Ae^{\alpha|x|^2}, \quad (x,t) \in Q,$$

where $A, \alpha$ are positive constants.

(a) Assume that $g \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ does not depend on a variable $x_j$ for some fixed $j$. Prove that the same is true for $u$.

(b) Prove that if $g \in C^{\infty}(\mathbb{R}^n)$ is a harmonic function on $\mathbb{R}^n$, the solution $u$ is time independent.

**Problem 4.** Let $\alpha, T > 0, \gamma \in \mathbb{R}$. Suppose $\phi \in C^0(\overline{\Omega})$ and $c \in C^0(\overline{\Omega_T})$ with $c \geq \gamma$ on $\overline{\Omega_T}$. Suppose $u \in C^{2,1}(\Omega_T) \cap C^1(\overline{\Omega_T})$ is a solution of

$$u_t - \Delta u + c(x,t)u = 0 \text{ on } \Omega_T,$$

$$u = \phi \text{ on } \Omega \times \{t = 0\},$$

$$\partial u/\partial n + \alpha u = 0 \text{ on } \partial \Omega \times (0,T].$$

Prove $|u| \leq \sup_{\overline{\Omega}} |\phi| \ e^{-\gamma t}$ on $\Omega_T$ and prove u is unique.
Problem 5. Solve explicitly the initial-boundary value problem

\[ u_{tt} - 4u_{xx} = 0, \quad x > 0, \quad t > 0, \]

with initial data

\[ u(x, 0) = x, \quad x > 0, \]
\[ u_t(x, 0) = -2, \quad x > 0, \]

and boundary condition

\[ u_x(0, t) + tu(0, t) = 1, \quad t > 0. \]

Problem 6. Suppose \( \Omega \subset \mathbb{R}^2 \) and \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) is a solution of

\[ (1 + u_x^2)u_{xx} + (1 + u_y^2)u_{yy} - 2u_xu_yu_{xy} = 0 \quad \text{on} \quad \Omega. \]

Show \( \inf_{\overline{\Omega}} u = \inf_{\partial^\Omega} u. \)

Problem 7. Let \( T > 0, \alpha \in \mathbb{R} \). Suppose \( \phi, \psi \in C^\infty(\overline{\Omega}) \) and \( u \in C^2(\Omega_T) \cap C^1(\overline{\Omega}_T) \) is a solution of

\[ u_{tt} - \Delta u + \alpha u_t = 0 \quad \text{on} \quad \Omega_T, \]
\[ u = \phi \quad \text{on} \quad \Omega \times \{ t = 0 \}, \]
\[ u_t = \psi \quad \text{on} \quad \Omega \times \{ t = 0 \}, \]
\[ \partial u / \partial n = 0 \quad \text{on} \quad \partial \Omega \times (0, T]. \]

Prove that for \( t \in [0, T] \) the following inequality holds \( E(t) \leq E(0)e^{a_0 t}, \)
where \( E(t) = \frac{1}{2} \int_{\Omega} (u_t^2 + |\nabla u|^2) dx \) and \( a_0 = \max\{0, -2\alpha\}. \)
In the following $\Omega \subset \mathbb{R}^n$ is an open, bounded set with $C^\infty$- smooth boundary $\partial \Omega$. Denote $\Omega_T = \Omega \times (0, T]$.

**Problem 1.** Suppose $u \in C^1(\mathbb{R}^2)$ is a solution of $yu_x - xu_y = u$ on the entire plane $\mathbb{R}^2$. Prove $u = 0$ on $\mathbb{R}^2$.

**Problem 2.** Suppose $f, g \in C^1(\mathbb{R})$ with $f(0) = g(0) = 0, f' > 0$ and $g' > 0$ on $\mathbb{R} \setminus \{0\}$. Suppose $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution of

$$\Delta u = f(u) \text{ on } \Omega,$$

$$\partial u/\partial n + g(u) = 0 \text{ on } \partial \Omega.$$

(a) Show $u = 0$ on $\Omega$ using the maximum principle.
(b) Show $u = 0$ on $\Omega$ using the energy method.

**Problem 3.** Let $T > 0, c \in C^0(\overline{\Omega}_T)$. Suppose $u \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega}_T)$ satisfies

$$u_t - \Delta u + c(x,t)u \leq 0 \text{ on } \Omega_T,$$

$$u \leq 0 \text{ on } \Gamma_T \ (= \overline{\Omega}_T \setminus \Omega_T = \text{parabolic boundary of } \Omega_T).$$

Prove $u \leq 0$ on $\Omega_T$.

Hint: Consider $v = ue^{-Mt}$ for a suitable constant $M$.

**Problem 4.** Suppose $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ is a solution of

$$u_{tt} - \Delta u = 0 \text{ on } \mathbb{R}^3 \times [0, \infty),$$

$$u(x,0) = 0, \ x \in \mathbb{R}^3,$$

$$u_t(x,0) = g(x), \ x \in \mathbb{R}^3,$$

where $g \in C^2(\mathbb{R}^3)$ has compact support. Prove that there exists $C > 0$ such that

(a) $|u_t(x, t)| \leq C(1 + t)^{-1}$ for all $(x, t) \in \mathbb{R}^3 \times [0, \infty)$, and
(b) \( (\int_{\mathbb{R}^3} |u_t|^6 dx)^{1/6} \leq C(1 + t)^{-2/3} \) for all \( t \geq 0 \).

**Problem 5.** Suppose \( u \in C^2(\mathbb{R}^n) \) satisfies \( \Delta u + u^2 + 2u \leq 0 \) on \( \mathbb{R}^n \). Show that the inequality \( u \geq 1 \) cannot hold on all of \( \mathbb{R}^n \).

Hint: Consider the auxiliary function \( v(x) = \frac{2}{n} (R^2 - |x|^2) \) on \( B(0, R) \).

**Problem 6.** Suppose \( n \leq 3, \phi \in C^3(\mathbb{R}^n), \psi \in C^2(\mathbb{R}^n) \) and \( \phi, \psi \) have compact support. Suppose \( u \in C^2(\mathbb{R}^n \times [0, \infty)) \) is a solution of

\[
\begin{align*}
 u_{tt} - \Delta u &= u^3 \quad \text{on } \mathbb{R}^n \times (0, \infty), \\
 u(x, 0) &= \phi(x), \quad x \in \mathbb{R}^n, \\
 u_t(x, 0) &= \psi(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]

where \( \int_{\mathbb{R}^n} \phi(x)^2 dx > 0 \). Define the energy

\[
E(t) = \int_{\mathbb{R}^n} \left( \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 - \frac{1}{4} u^4 \right) dx
\]

and \( F(t) = \int_{\mathbb{R}^n} u^2 dx \) for \( t \geq 0 \). Assume \( E(0) < 0 \).

(a) Prove \( E(t) \) is constant in \( t \).

(b) Find a lower bound for \( ||u(\cdot, t)||_{L^4(\mathbb{R}^n)} \) and prove \( F'(t) \geq 6||u_t||_{L^2(\mathbb{R}^n)}^2 \) for each \( t \).

(c) Prove \( (F(t)^{-\frac{1}{2}})'' \leq 0 \) for all \( t > 0 \) (note \( (F(t)^{-\frac{1}{2}})' = -\frac{1}{2} (FF'' - \frac{3}{2} F'F) F^{-\frac{3}{2}} \)).

(d) Provided that \( F'(t) > 0 \) for some \( t > 0 \), show \( F(t) \to \infty \) as \( t \to t_0^- \) for some finite \( t_0 > 0 \).

**Problem 7.** Let \( Q = \mathbb{R}^n \times (0, \infty), n = 2, 3 \) and \( f \in C^0(\overline{Q}) \). Suppose \( u \in C^{2,1}(Q) \cap C^0(\overline{Q}) \) is a solution of

\[
\begin{align*}
 u_t - \Delta u &= f(x, t) \quad \text{on } Q, \\
 u &= 0 \quad \text{on } \mathbb{R}^n \times \{0\}.
\end{align*}
\]

Assume \( \int_{\mathbb{R}^n} f(x, t)^2 dx \leq k \) for all \( t \geq 0 \); and that for each \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that \( |f| \leq C_\varepsilon e^{\varepsilon|x|^2} \) on \( Q \). Assume \( |u| \leq A e^{a|x|^2} \) holds on \( Q \) for some constants \( a, A > 0 \). Show, for some \( C, \alpha > 0, |u| \leq C t^\alpha \) holds on \( Q \). Give \( \alpha \) explicitly and explain if your reasoning depends on \( n \). Explain the purpose of \( e^{a|x|^2} \).