1. Let $\Omega = \{(x,t) : x > 0, \ t > 0\}$. Assume $f \in C^{\infty}(\overline{\Omega})$, $f$ has bounded support and $f = 0$ on $\{t = 0\}$. Suppose $u \in C^2(\overline{\Omega})$ is a solution of
\[ u_t + u_x + u = f(x,t) \text{ on } \Omega, \]
\[ u = 0 \text{ on } \{x = 0\} \cup \{t = 0\}. \]

(a) For each $t > 0$, prove that $u(\cdot, t)$ has bounded support.
(b) For each $t > 0$, prove
\[ \int_0^\infty u_t^2 \, dx \leq \int_0^t e^{s-t} \int_0^\infty f_t^2(x,s) \, dx \, ds. \]
(c) Prove there exists $K > 0$ such that $\int_0^\infty u_t^2 \, dx \leq Ke^{-t}$ for all $t > 0$.

2. Let $a > 0$, $\Omega = (-1,1) \times (-a,a) \subset \mathbb{R}^2$. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of
\[ \Delta u = -1 \text{ on } \Omega, \quad u = 0 \text{ on } \partial \Omega. \]
Using the functions $v(x,y) = (1 - x^2)(a^2 - y^2)$, $w(x,y) = 2 - x^2 - \frac{y^2}{a^2}$ (or constant multiples of them), find positive bounds $C_1(a)$ and $C_2(a)$ such that
\[ C_1(a) \leq u(0,0) \leq C_2(a). \]

3. Suppose $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is open, bounded with $C^\infty$-smooth boundary $\partial \Omega$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of
\[ -\Delta(u^3) = u \text{ on } \Omega, \quad u = 0 \text{ on } \partial \Omega. \]

(a) Using the Green’s function show there exists a constant $C > 0$ depending only on $\Omega$, but not on the solution, such that $\int_\Omega |u(x)|^3 \, dx \leq C$, and $\sup_{\Omega} |u| \leq C$.
(b) Show that, if $u \geq 0$ on $\Omega$, then either, $u \equiv 0$ on $\Omega$ or $u > 0$ on $\Omega$.
(c) Let $v$ be the eigenfunction corresponding to the first (least) eigenvalue $\lambda$ of $-\Delta v = \lambda v$ on $\Omega$, $v = 0$ on $\partial \Omega$ (recall $v > 0$ on $\Omega$). Show that, if $u \geq v$, then $u^3 \geq \frac{1}{\lambda}v$. 

(d) Assuming also \( u^3 \in C^1(\Omega) \), prove \( \int_{\Omega} |\nabla(u^2)|^2 \, dx = C_1 \int_{\Omega} u^2 \, dx \leq C_2 \) where \( C_1, C_2 \) depend only on \( \Omega \), not on \( u \).

4. Let \( u_0 : \mathbb{R}^n \to \mathbb{R} \) be smooth and compactly supported, and
\[
m = \int_{\mathbb{R}^n} u_0(y) \, dy.
\]
Let \( u \) be a solution of the Cauchy problem
\[
\begin{align*}
  u_t - \Delta u &= 0 \quad \text{on } \mathbb{R}^n \times (0, \infty), \\
  u(x,0) &= u_0(x) \quad x \in \mathbb{R}^n,
\end{align*}
\]
with \( |u(x, t)| \leq Ae^{a|x|^2} \) for some fixed \( A, a > 0 \) and all \( (x, t) \in \mathbb{R}^n \times (0, \infty) \). Prove that there is a constant \( N \) depending only on \( n \) such that \( \sup_{x \in \mathbb{R}^n} |u(x, t) - m \Phi(x, t)| \leq \frac{N}{t^{n/2}} \int_{\mathbb{R}^n} |y| |u_0(y)| \, dy \), for all \( t > 0 \), where \( \Phi(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} \).

5. Let \( u \) be a smooth function on \( \overline{B}_1 \times [0, 1] \) that satisfies the equation
\[
\begin{align*}
  a_0 \, u_t - b_0 \, \Delta u + u &= 1 \quad \text{on } B_1 \times (0, 1), \\
  u &= 1 \quad \text{on } \partial B_1 \times (0, 1), \\
  u(x,0) &= 1 \quad x \in B_1,
\end{align*}
\]
where \( a_0, b_0 : \overline{B}_1 \times [0, 1] \to [0, \infty) \) are given continuous functions (\( B_1 \) = unit ball in \( \mathbb{R}^n \)). Prove that \( u \leq 1 \) on \( \overline{B}_1 \times [0, 1] \).

6. Assume that \( \Omega \subset \mathbb{R}^n \) is an open, bounded set with \( C^\infty \)-smooth boundary \( \partial \Omega \). Let \( T > 0 \), \( \Omega_T = \Omega \times (0, T] \). Suppose \( a \in C^1(\overline{\Omega}) \), \( a > 0 \) on \( \overline{\Omega} \), \( \phi, \psi \in C^2(\overline{\Omega}) \). Suppose \( u \in C^2(\Omega_T) \) is a solution of
\[
\begin{align*}
  u_{tt} - a(x) \Delta u &= u^3 \quad \text{on } \Omega_T, \\
  \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega \times [0, T], \\
  u &= \phi, \quad u_t = \psi \quad \text{on } \Omega \times \{ t = 0 \}.
\end{align*}
\]
Prove that \( u \) is unique.

7. Assume \( \phi \in C^2(\mathbb{R}) \) and \( h, \psi \in C^1(\mathbb{R}) \). Consider the initial-value problem with \( u \in C^2(\mathbb{R} \times [0, \infty)) \)
\[
\begin{align*}
  u_{tt} - u_{xx} &= h(x-t) \quad \text{on } \mathbb{R} \times [0, \infty),
\end{align*}
\]
(1)
\[ u = \phi(x), \quad u_t = \psi(x) \text{ at } t = 0, \quad x \in \mathbb{R}. \quad (2) \]

(a) Find a solution of the p.d.e. in (1).
(b) Find a solution of (1) and (2).
Question 1: Let \( g : \mathbb{R} \to \mathbb{R} \) be a smooth function. Find solutions of the following initial-value problem in \( \mathbb{R}^2 \)
\[
    u_x + (1 + x^2)u_y - u = 0 \quad \text{with} \quad u(x, \frac{1}{3}x^3) = g(x).
\]

Question 2: Let \( h : \mathbb{R} \to \mathbb{R} \) be a smooth function. Consider the following equation in \( \mathbb{R}^2 \)
\[
    xu_x + yu_y = 2u \quad \text{with} \quad u(x, 0) = h(x).
\]
(a) Check that the line \( \{ y = 0 \} \) is characteristic at each point and find all \( h \) satisfying the compatibility condition on \( \{ y = 0 \} \).
(b) For \( h \) as compatible in (a), solve the PDE.

Question 3: Let \( \phi \) be smooth, compactly supported function defined in the unit ball \( B_1 \subset \mathbb{R}^n \) such that \( \phi = 1 \) on \( B_1/2 \), where \( B_1/2 \subset \mathbb{R}^n \) is the ball of radius \( 1/2 \) centered at the origin. Suppose that \( u \) is harmonic in \( B_1 \).
(a) Prove that there is \( \alpha > 0 \) depending only on \( n \) and \( \sup |\Delta \phi| \) and \( \sup |\nabla \phi| \) such that
\[
    \Delta (\phi^2 |\nabla u|^2 + \alpha u^2) \geq 0 \quad \text{in} \quad B_1.
\]
(b) Use part (a) and the maximum principle to conclude that there is a constant \( C > 0 \) depending only on \( n, \phi \) such that
\[
    \sup_{B_1/2} |\nabla u| \leq C \sup_{\partial B_1} |u|.
\]

Question 4: Let \( B_1 \subset \mathbb{R}^2 \) be the unit ball with boundary \( \partial B_1 \). Let \( f, c \in C(\overline{B_1}) \) and \( g \in C(\partial B_1) \). Assume that \( c(x, y) > 0 \) for all \( (x, y) \in B_1 \). Prove that there exists at most one \( C^2 \)-solution to the following equation
\[
    \begin{aligned}
        -x^2u_{xx} - y^2u_{yy} + c(x, y)u &= f \quad \text{in} \quad B_1 \\
        u &= g \quad \text{on} \quad \partial B_1.
    \end{aligned}
\]

Question 5: Let \( a_0 \) be a smooth and compactly supported function defined on \( \mathbb{R}^n \) and \( p_0 \in (1, \infty) \). Consider the following Cauchy problem
\[
\begin{aligned}
    \left\{ \begin{array}{ll}
        u_t - \Delta u = |u|^{p_0-1}u & \text{in} \quad \mathbb{R}^n \times (0, \infty) \\
        u(x, 0) = a_0(x) & \text{in} \quad \mathbb{R}^n.
    \end{array} \right.
\end{aligned}
\]  \hfill (1)
Define the scaling
\[
    u_\lambda(x, t) = \lambda^\beta u(\lambda x, \lambda^2 t), \quad \lambda > 0.
\]
(a) Find \( \beta \) (possibly depending on \( n, p_0 \)) so that if \( u \) is a solution of (1), then \( u_\lambda \) is also a solution of (1) (with appropriate scaled initial data \( a_0^\lambda \)).
(b) Recall that the \( L^p \)-norm is defined by
\[
    \|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |u(x, t)|^pdx \right)^{\frac{1}{p}}, \quad p \in [1, \infty).
\]
For \( \beta \) found in a), find \( p \) so that if \( u \) is a solution of (1) then
\[
    \|u(\cdot, \lambda^2 t)\|_{L^p(\mathbb{R}^n)} = \|u(\cdot, t)\|_{L^p(\mathbb{R}^n)}
\]
for all \( \lambda > 0 \) and for all \( t > 0 \).
**Question 6:** Let us denote $\mathbb{R}^2_+ = \mathbb{R} \times (0, \infty)$ and $B_1^+ = B_1 \cap \mathbb{R}^2_+$, where $B_1$ is the unit ball in $\mathbb{R}^2$. Assume that $u = u(x, y, t)$ is a smooth function defined on $\overline{B_1^+} \times [0, 1]$ and satisfying

$$u_t - y^\alpha [u_{xx} + u_{yy}] + u_y + u \leq 0 \quad \text{for} \quad (x, y) \in B_1^+ \quad \text{and} \quad t \in (0, 1),$$

where $\alpha > 0$ is a given number. Assume that $u(x, y, 0) \leq 0$, and that $u \leq 0$ on $(\partial B_1 \cap \mathbb{R}^2_+) \times (0, 1)$, where $\partial B_1$ denotes the boundary of $B_1$. Prove that $u \leq 0$ on $\overline{B_1^+} \times [0, 1]$.

**Note:** We are not given any information on the boundary data on the part of the boundary where $y = 0$.

**Question 7:** Let $u_1(x)$ and $u_2(x)$ be smooth functions whose supports are in the unit ball $B_1 \subset \mathbb{R}^n$. For each $x_0 \in \mathbb{R}^n$ and each $t_0 > 0$, let $C(x_0, t_0)$ be the cone defined by

$$C(x_0, t_0) = \{(x, t) : 0 \leq t \leq t_0, \quad |x - x_0| \leq t_0 - t\}.$$

Assume that $u \in C^2$ is the solution of the equation

$$u_{tt} - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty)$$

with given initial data $u(x, 0) = u_1(x)$ and $u_t(x, 0) = u_2(x)$.

Give the proof for the finite propagation speed result for the wave equation, namely $u = 0$ on $C(x_0, t_0)$ for all $x_0 \in \mathbb{R}^n$ with $|x_0| > 1$ and $t_0 = |x_0| - 1$.

**Question 8:** Let $u$ be a smooth solution of the equation

$$u_{tt} - \Delta u = f \quad \text{on} \quad \mathbb{R}^3 \times (0, \infty)$$

with $u(\cdot, 0) = u_t(\cdot, 0) = 0$. Also, let $v$ be a smooth solution of the equation

$$v_{tt} - \Delta v = g \quad \text{on} \quad \mathbb{R}^3 \times (0, \infty)$$

with $v(\cdot, 0) = v_t(\cdot, 0) = 0$. Assume that $|f|^2 \leq g$. Prove that $2u(x, t)^2 \leq t^2 v(x, t)$ for all $x \in \mathbb{R}^3$ and $t > 0$. 

Question 1: Solve the Cauchy problem
\[
\begin{cases}
x u_x - y u_y = u - y, & x > 0, y > 0, \\
u(y^2, y) = y, & y > 0.
\end{cases}
\]

Question 2: Let \( a, R \) be positive numbers and consider the equation
\[
\begin{cases}
\partial_t u + a \partial_x u = f(x, t), & 0 < x < R, \\
u(0, t) = 0, & t > 0, \\
u(x, 0) = 0, & 0 < x < R.
\end{cases}
\]
Prove that for each solution \( u(x, t) \in C^1((0, R) \times (0, \infty)) \) we have
\[
\int_0^R u^2(x, t) dx \leq e^t \int_0^t \int_0^R f^2(x, s) dx ds, \quad \forall \ t > 0.
\]

Question 3: Let \( r > 0 \) and let \( f, g \) be continuous functions defined on \( \overline{B}_r(0) \). Let \( u \) be in \( C^2(B_r(0)) \cap C(\overline{B}_r(0)) \) be the solution of the equation
\[
\begin{cases}
-\Delta u = f, & B_r(0), \\
u = g, & \partial B_r(0).
\end{cases}
\]
Prove that
\[
u(0) = \int_{\partial B_r(0)} g(x) dS(x) + \frac{1}{n(n-2)\alpha(n)} \int_{B_r(0)} \left( \frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f(x) dx.
\]
Hint: Consider
\[
\phi(s) = \int_{\partial B_s(0)} u(y) dS, \quad 0 < s \leq r.
\]
Compute \( \phi'(s) \) and then find \( \phi(0) \).

Question 4: Let \( R > 0 \) and we denote \( B_R \) the ball of radius \( R \) centered at the origin in \( \mathbb{R}^n \).
Let \( c, f \) be continuous functions on \( \overline{B}_R \). Assume that \( c \leq 0 \) on \( \overline{B}_R \), and also assume that \( u \in C^2(B_R) \cap C(\overline{B}_R) \) satisfies
\[
\begin{cases}
\Delta u + cu = f & \text{in } B_R, \\
u = 0 & \text{on } \partial B_R.
\end{cases}
\]
Prove that
\[
\sup_{B_R} |u| \leq \frac{R^2}{2n} \sup_{B_R} |f|
\]
Hint: Let \( A = \sup_{B_R} |f| \) and
\[
v(x) = \frac{AR^2}{2n}(R^2 - |x|^2)
\]
Use the maximum principle to prove that \( |u(x)| \leq v(x) \) on \( B_R \).

Question 5: Let \( u_0 \) be the smooth and compactly supported function defined on \( \mathbb{R}^n \). Assume that \( u \) is a solution of the Cauchy problem
\[
\begin{cases}
\partial_t u - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n.
\end{cases}
\]
Let \( p, q \in (1, \infty) \) with \( p \geq q \) and consider the inequality
\[
\|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} \leq \frac{N}{t^\alpha} \|u_0\|_{L^q(\mathbb{R}^n)}, \quad t > 0
\]
with \( N = N(n, p, q) \) and \( \alpha = \alpha(n, p, q) \), where we denote
\[
\|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |u(x, t)|^p \, dx \right)^{\frac{1}{p}}
\]
and similar notation is also used for \( \|u_0\|_{L^q(\mathbb{R}^n)} \).

Use the scaling property of the heat equation to find the number \( \alpha \) (certainly, show all of the work).

**Question 6**: Assume that \( u \) is a smooth, bounded solution of the equation
\[
\begin{cases}
  u_t - \Delta u = u(1 - u) & \text{in } B_1 \times (0, 1] \\
  u = 0 & \text{on } \partial B_1 \times (0, 1] \\
  u = \frac{1}{2} & \text{on } B_1 \times \{0\}.
\end{cases}
\]
Prove that \( 0 \leq u \leq 1 \).

**Question 7**: Let \( \varphi \) be a smooth, compactly supported function on \( \mathbb{R}^2 \). Assume that \( u \) is a smooth solution of
\[
\begin{cases}
  u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\
  u(\cdot, 0) = 0 & \text{on } \mathbb{R}^2, \\
  u_t(\cdot, 0) = \varphi & \text{on } \mathbb{R}^2.
\end{cases}
\]
Prove that
\[
|u(x, t)| \leq \frac{1}{2\sqrt{t}} \left( \|\varphi\|_{L^1(\mathbb{R}^2)} + \|\nabla \varphi\|_{L^1(\mathbb{R}^2)} \right), \quad \forall t > 1.
\]

**Question 8**: Assume that \( u \in C^2(\mathbb{R}^n \times [0, \infty)) \) is a solution of the wave equation
\[
u_{tt} = \Delta u \text{ in } \mathbb{R}^n \times (0, \infty).
\]
Let
\[
E(t) = \frac{1}{2} \int_{B_{1-t}} \left[ |u_t(x, t)|^2 + |\nabla u(x, t)|^2 \right] \, dx \quad \text{for } t \in (0, 1),
\]
where \( \nabla u = (u_{x_1}, u_{x_2}, \ldots, u_{x_n}) \) and \( B_r \) denotes the ball in \( \mathbb{R}^n \) with radius \( r > 0 \) and centered at the origin.

(a) Prove that
\[
E'(t) = \int_{B_{1-t}} \left[ u_t(x, t)u_{tt}(x, t) + \sum_{i=1}^n u_{x_i} u_{x_it} \right] \, dx
+ \frac{1}{2} \int_{\partial B_{1-t}} \left[ u_t^2(x, t) + |\nabla u(x, t)|^2 \right] \, dS(x).
\]
(b) Use the note that
\[
[u_{x_i} u_{t}]_{x_i} = u_{x_i} u_{x_it} + u_{x_it} u_{tt},
\]
to prove that \( E'(t) \leq 0 \). Then, conclude also that \( u = 0 \) on \( \{(x, t) : |x| \leq 1 - t, \ 0 \leq t \leq 1\} \) if \( u(x, 0) = u_t(x, 0) = 0 \) for \( x \in B_1 \).
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Question 1: For $x > 0$, consider the equation:

\[
\begin{align*}
\begin{cases}
    uu_x + 2ux_y = 0 & \text{in } \mathbb{R}^2 \\
u(x,0) = \frac{1}{x} & \text{for } x > 0.
\end{cases}
\end{align*}
\]

For $t_0, t_1 > 0$ with $t_0 \neq t_1$, let $C_0$ be the characteristic passing through the point $(t_0,0,1/t_0)$ and let $C_1$ be the characteristic passing through $(t_1,0,1/t_1)$. Determine whether the projections of $C_0$ and $C_1$ onto the $x$-$y$ plane intersect for some $y > 0$ (i.e., whether a shock develops), and if they do, find the point $(x,y)$ of intersection.

Question 2: Given a bounded domain $\Omega$ in $\mathbb{R}^n$, let $h$ be the solution to

\[
\Delta h = -1 \text{ in } \Omega, \quad h = 0 \text{ on } \partial \Omega.
\]

Let $a > 0$ be a constant.

Prove: If there exists a function $u > 0$ that satisfies the equation

\[
\Delta u = \frac{1}{u} \text{ in } \Omega, \quad u \equiv a \text{ on } \partial \Omega,
\]

then $a \geq \sqrt{\max_{\Omega} h}$.

Hint: Prove $u \leq a$. Then prove a better upper bound for $u$.

Question 3:
(a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, bounded, and even (that is, $f(-x) = f(x)$ for all $x \in \mathbb{R}$). Suppose $u = u(x,t) \in C^2_t(\mathbb{R}^2_+) \cap C(\mathbb{R}^2_+)$ satisfies

\[
\begin{align*}
\begin{cases}
    u_x = u_{xx} & \text{for } x \in \mathbb{R}, 0 < t < \infty, \\
u(x,0) = f(x) & \text{for } x \in \mathbb{R}, \\
|u(x,t)| \leq Ke^{a|x|^2} & \text{for } x \in \mathbb{R}, 0 < t < \infty,
\end{cases}
\end{align*}
\]

for some positive constants $K$ and $a$. Prove that for each $t > 0$, $u(x,t)$ is an even function of $x$: i.e., $u(-x,t) = u(x,t)$ for all $t > 0$.

(b) Assume $f : [0,\infty) \to \mathbb{R}$ is continuous and bounded. For $x \geq 0$ and $t \geq 0$, suppose $u = u(x,t) \in C^2((0,\infty) \times [0,\infty))$ satisfies

\[
\begin{align*}
\begin{cases}
    u_t = u_{xx} & \text{for } 0 < x < \infty, 0 < t < \infty, \\
u(x,0) = f(x) & \text{for } 0 \leq x < \infty, \\
u_x(0,t) = 0 & \text{for } 0 < t < \infty \\
|u(x,t)| \leq Ke^{a|x|^2} & \text{for } x \in \mathbb{R}_+, 0 < t < \infty,
\end{cases}
\end{align*}
\]

for some positive constants $K$ and $a$. Here $u_x(0,t)$ is interpreted as the $x$-derivative of $u$ from the right at $(0,t)$. Find a function $H = H(x,y,t)$ such that

\[
u(x,t) = \int_0^\infty H(x,y,t)f(y) \, dy,
\]

and justify your answer.
Question 4: Consider the nonlinear PDE
\[ u_{tt} - \Delta u + u^3 = 0, \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}. \]

1. Assume that \( u \) is smooth and has compact support in \( x \) for each \( t \). What is the energy expression
\[ E(t) = \int_{\mathbb{R}^3} q(u, u_t, \nabla u) dx \]
which is conserved, i.e., \( E'(t) = 0 \)?

2. For any \( \alpha > 0 \), and \( x_0 \in \mathbb{R}^3 \), denote by
\[ E_\alpha(t) = \int_{B_\alpha(x_0)} q(u, u_t, \nabla u) dx \]
the energy contained in the ball of radius \( \alpha > 0 \) centered at \( x_0 \). Show that for any \( T > 0 \) and \( a > 0 \),
\[ E_\alpha(T) \leq E_{\alpha + T}(0) \]
Hint: Work with the 'energy'
\[ \tilde{E}(t) := \int_{B_{T+\alpha}(x_0)} q(u, u_t, \nabla u) dx \]

3. Given \( a > 0 \), show that if \( u(x, 0) = u_t(x, 0) = 0 \) for \( |x| > a \), then \( u(x, t) = 0 \) for all \( |x| \geq a + t, \ t \geq 0 \).

Question 5: Let \( B \) be the unit ball in \( \mathbb{R}^n \) and let \( u \in C^\infty(\bar{B} \times [0, \infty)) \) satisfy
\[ u_t - \Delta u + u^{1/2} = 0 \quad \text{on} \ B \times (0, \infty) \]
\[ 0 \leq u \quad \text{on} \ B \times (0, \infty) \]
\[ u = 0 \quad \text{on} \ \partial B \times (0, \infty). \]
(a) Show that, if \( u|_{t=t_0} \equiv 0 \), then \( u \equiv 0 \) for \( t > t_0 \) as well.
(b) Prove that there is a number \( T \) depending only on \( M := \max u|_{t=0} \) such that \( u \equiv 0 \) on \( B \times (T, \infty) \).

Hint: Let \( v \) be the solution of the IVP,
\[ \frac{dv}{dt} + v^{1/2} = 0, \quad v(0) = M, \]
and consider the function \( w = v - u \).

Question 6:
(a) Find a \( C^1 \) solution in \( \mathbb{R}^+ \times \mathbb{R} \ni (x, y) \) to:
\[ x^2 u_x - y^2 u_y = u^2 \quad \text{for} \ x > 0, y \in \mathbb{R}, \quad u(1, y) = \frac{1}{1 + y^2} \]
(b) Explain why this solution is not unique as a solution in \( C^1(\mathbb{R}^+ \times \mathbb{R}) \), but its restriction to some appropriate open set \( U \) containing the initial curve \( \{1\} \times \mathbb{R} \) is unique in \( C^1(U) \).
**Question 7:** Suppose $f, g \in C^\infty(\mathbb{R}^n)$. Suppose $u \in C^2(\mathbb{R}^n \times [0, \infty))$ satisfies

$$
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u, & (x,t) \in \mathbb{R}^n \times (0, \infty), \\
u(x,0) &= f(x), & x \in \mathbb{R}^n, \\
\frac{\partial u}{\partial t}(x,0) &= g(x), & x \in \mathbb{R}^n.
\end{align*}
$$

Prove that

$$
\int_{\mathbb{R}^n} u(x,t) \, dx = C_1 t + C_2,
$$

for all $t > 0$, where $C_1 = \int_{\mathbb{R}^n} g(x) \, dx$ and $C_2 = \int_{\mathbb{R}^n} f(x) \, dx$, under either of the two conditions:

(i) $n = 3$, $\int_{\mathbb{R}^3} |f(x)| \, dx < \infty$, $\int_{\mathbb{R}^3} |\nabla f(x)| \, dx < \infty$, and $\int_{\mathbb{R}^3} |g(x)| \, dx < \infty$; or

(ii) $n \in \mathbb{N}$, and $f$ and $g$ have compact support.

**Question 8:** Let $u \in C^2(\mathbb{R}^n)$ be a subharmonic function and consider the spherical averages

$$
v(r) := \frac{1}{\mathcal{S}(r)} \int_{\partial B_r(0)} u(x) \, dS(x).
$$

(a) Show that the function $x \mapsto v(|x|)$ is also subharmonic in $\mathbb{R}^n$, and that $r \mapsto r^{n-1}v'(r)$ is monotonic.

(b) Now let $n = 2$. Prove that, if $u$ is also bounded, then $u$ is a constant.
PDE Preliminary Exam, January 2018

Instruction:
Solve all eight problems. Begin your answer to each question on a separate sheet. Explain all your steps.

1. A smooth function \( u \) defined in the first quadrant on the \( xy \)-plane satisfies

\[
-y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} = -2u, \quad u(x, 0) = x.
\]

Determine \( u(0, y) \).

2. Suppose that \( u(x, t) \) is a smooth solution of

\[
\begin{cases}
  u_t + uu_x = 0 & \text{for } x \in \mathbb{R}, \ t > 0 \\
  u(x, 0) = f(x), & \text{for } x \in \mathbb{R}
\end{cases}
\]

Assume that \( f \) is a \( C^1 \) function such that

\[
f(x) = \begin{cases}
  0 & \text{for } x < -1 \\
  1 & \text{for } x > 1
\end{cases}
\]

and \( f'(x) > 0, \) for \( |x| < 1 \).

(a) Sketch the characteristics emanating from \((x_0, 0)\) for several values of \( x_0 < -1, x_0 \in (-1, 1), \) and \( x_0 > 1 \).

(b) Show that for \( t > 0, \)

\[
\lim_{r \to \infty} u(rx, rt) = \begin{cases}
  0 & \text{for } x < 0 \\
  x/t & \text{for } 0 < x < t \\
  1 & \text{for } x > t
\end{cases}
\]

3. Suppose that for all \( r > 2, \) there exists a function \( u_r : \mathbb{R}^3 \to \mathbb{R} \) that is continuous and satisfies

\[
\begin{cases}
  \Delta u = 0 & \text{in } B_r(0) \setminus B_1(0) \\
  u(x) = 0 & \text{for } |x| \geq r \\
  u(x) = 1, & \text{for } x \in B_1(0)
\end{cases}
\]

(a) Show that for all \( x \in \mathbb{R}^3, \) if \( 2 < r_1 \leq r_2, \) then

\[
0 \leq u_{r_1}(x) \leq u_{r_2}(x) \leq 1.
\]
(b) Show that

i. \( u(x) = \lim_{r \to \infty} u_r(x) \) is harmonic on \( \mathbb{R}^3 \setminus \overline{B_1(0)} \)
ii. \( \lim_{|x| \to \infty} u(x) = 0. \)

[Hint: noting that \( \frac{1}{|x|} \) is harmonic, study \( u_r(x) - \frac{1}{|x|} \) over an annulus.]

4. Denote by \( \mathbb{R}^n_+ = \{ x = (x', x_n) : x_n > 0 \} \), \( \Sigma = \{ x = (x', x_n) : x_n = 0 \} \).
Suppose that \( u \) is harmonic in \( \mathbb{R}^n_+ \), continuous on \( \mathbb{R}^n_+ \cup \Sigma \), and \( u = 0 \) on \( \Sigma \). Define

\[
\overline{u}(x', x_n) := \begin{cases} 
  u(x', x_n) & \text{for } x_n \geq 0, \\
  -u(x', -x_n) & \text{for } x_n < 0.
\end{cases}
\]

Then show that \( \overline{u} \) is harmonic in \( \mathbb{R}^n_+ \).

5. Let \( \Omega \subseteq \mathbb{R}^n \) be a \( C^\infty \) bounded domain. Assume that \( u_0 \in C^\infty(\overline{\Omega}) \), \( \alpha \in C([0, \infty)) \), and \( \lim_{t \to \infty} \alpha(t) \leq 0 \). Suppose also \( u \in C^2(\overline{\Omega} \times [0, \infty)) \) satisfies

\[
\begin{cases}
  u_t = \Delta u + \alpha(t)u & \text{on } \Omega \times (0, \infty), \\
  u = 0 & \text{on } \partial\Omega \times (0, \infty), \\
  u = u_0 & \Omega \times \{t = 0\}.
\end{cases}
\]

Prove that

\[
\lim_{t \to \infty} \int_{\Omega} u^2(x, t)dx = 0
\]

(Hint: Use the Energy method. You may apply Poincaré's inequality.)

6. Let \( \Omega \subseteq \mathbb{R}^n \) be a \( C^\infty \) bounded domain, \( T > 0 \), and \( \alpha \in \mathbb{R}^n \) is a given vector. Suppose \( u \in C^2(\overline{\Omega} \times [0, T]) \) satisfies

\[
\begin{cases}
  u_t = \Delta u + \alpha \cdot \nabla u + u^2 & \text{on } \Omega \times (0, T), \\
  u = 0 & \text{on } \partial\Omega \times (0, T], \\
  u = 0 & \Omega \times \{t = 0\}.
\end{cases}
\]

Prove that

(a) \( u \geq 0 \), on \( \Omega \times (0, T] \),
(b) \( u_t \geq 0 \) on \( \Omega \times (0, T] \).

(Hint: What equation does \( u_t \) solve? )
7. Let $\Omega \subseteq \mathbb{R}^n$ be a $C^\infty$ bounded domain and let $T > 0$. Suppose $V = V(x)$ and $h = h(x)$ are continuous functions on $\overline{\Omega}$, with $V(x) \geq 0$. Suppose $u = u(x, t) \in C^2(\overline{\Omega} \times [0, T])$, where $x \in \Omega$ and $t \in [0, T]$, and $u$ satisfies

\[
\begin{cases}
  u_t - \Delta u + V(x)u = h(x) & \text{on } \Omega \times (0, T); \\
  u(x, 0) = 0 & \text{on } \Omega; \\
  u_t(x, 0) = 0 & \text{on } \Omega; \\
  u = -D_n u & \text{on } \partial \Omega \times (0, T),
\end{cases}
\]

where $D_n u$ is the outward normal derivative of $u$ on $\partial \Omega$.

(a) Prove that $\int_\Omega h(x)u(x, t) \, dx \geq 0$ for all $t \geq 0$.

\textbf{Hint:} Consider

\[
E(t) = \frac{1}{2} \int_\Omega u_t^2 + |\nabla u|^2 + Vu^2 - 2hu \, dx + \frac{1}{2} \int_{\partial \Omega} u^2 \, d\sigma,
\]

where $d\sigma$ is surface measure on $\partial \Omega$.

(b) Suppose in addition that $V(x) \geq A$ and $|h(x)| \leq B$, for all $x \in \Omega$, for some constants $A > 0$ and $B > 0$. Prove that

\[
\int_\Omega |u(x, t)| \, dx \leq \frac{2B|\Omega|}{A},
\]

for all $t \geq 0$, where $|\Omega| = \int_\Omega \, dx$ is the measure of $\Omega$.

\textbf{Hint:} Start by writing $\int_\Omega |u| \, dx = \int_\Omega \frac{\sqrt{V}|u|}{\sqrt{V}} \, dx$, and apply Cauchy Schwartz.

8. Suppose $u \in C^2(\mathbb{R}^n \times [0, \infty))$ is a solution of

\[
\begin{cases}
  u_t = \Delta u & \text{on } \mathbb{R}^n \times (0, \infty); \\
  u(x, 0) = f(x) & \text{on } \mathbb{R}^n; \\
  u_t(x, 0) = g(x) & \text{on } \mathbb{R}^n,
\end{cases}
\]

where $f, g \in C^\infty(\mathbb{R}^n)$ have compact support: there exists $R > 0$ such that $f(x) = 0$ and $g(x) = 0$ if $|x| > R$. Consider the statement:

\textbf{(S):} For all such $f, g$ and $R$, and all $x_0 \in \mathbb{R}^n$, there exists $T = T(x_0, R) > 0$ such that $u(x_0, t) = 0$ for all $t > T$.

(a) Is (S) true if $n = 1$? Either prove (S) or give an example showing that $S$ fails.

(b) Is (S) true if $n = 3$? Either prove (S) or give an example showing that $S$ fails.
1. For a given continuous function $f$, solve the initial-boundary value problem

$$\begin{align*}
&\begin{cases}
  u_t + (x + 1)^2 u_x = x, & \text{for } x > 0, t > 0 \\
u(x, 0) = f(x), & \text{for } x > 0 \\
u(0, t) = -1 + t, & \text{for } t > 0.
\end{cases}
\end{align*}$$

Find a condition on $f$ so that the solution $u(x, t)$ is continuous on the first quadrant of $\mathbb{R}^2$, i.e. the region $\{ (x, t) \in \mathbb{R}^2 : x > 0, t > 0 \}$.

2. Determine an integral (weak) solution to the Burger’s equation

$$u_t + \left( \frac{1}{2} u^2 \right)_x = 0, \quad (x, t) \in \mathbb{R} \times (0, \infty)$$

with initial data

$$u(x, 0) = \begin{cases}
1 & \text{if } x < 0 \\
1 - x & \text{if } 0 < x < 1 \\
0 & \text{if } x > 1.
\end{cases}$$

3. Let $n \geq 2$, and let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with $C^\infty$-smooth boundary. Suppose $p$ and $q$ are non-negative continuous functions defined on $\Omega$, satisfying $p(x) + q(x) > 0$ (strict inequality) for all $x \in \Omega$. Find all functions $u \in C^2(\overline{\Omega})$ satisfying

$$\begin{align*}
&\begin{cases}
  \Delta u = pu^3 + qu & \text{in } \Omega, \\
  \frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega,
\end{cases}
\end{align*}$$

where $n(x)$ is the outward unit normal to $\Omega$ at $x \in \partial \Omega$.

4. Suppose $u$ is harmonic on a $C^\infty$ domain $\Omega \subseteq \mathbb{R}^n$, and let $u(x) = 0$ for $x \notin \Omega$. Suppose $\varphi$ is a $C^\infty$ function on $\mathbb{R}^n$ such that $\varphi(x) = 0$ if $|x| \geq 1$, and $\varphi$ is radial: there exists a function $\varphi_0 : [0, \infty) \to \mathbb{R}$ such that $\varphi(x) = \varphi_0(|x|)$. For $\epsilon > 0$, let

$$\varphi_\epsilon(x) = \frac{1}{\epsilon^n} \varphi \left( \frac{x}{\epsilon} \right).$$

Let

$$A = \int_{\mathbb{R}^n} \varphi(x) \, dx.$$ 

Fix $x_0 \in \Omega$ and let $R > 0$ be such that $x \in \Omega$ if $|x - x_0| < R$. For $0 < \epsilon < R$, prove that

$$\varphi_\epsilon \ast u(x_0) = Au(x_0),$$

where $\ast$ denotes convolution: by definition, $f \ast g(x) = \int_{\mathbb{R}^n} f(x - y)g(y) \, dy$. 

5. Suppose that $\mathbf{b} \in \mathbb{R}^n$, and $\beta \in \mathbb{R}$ are given. Consider the Cauchy problem

\[(*) \quad \begin{cases} u_t + \mathbf{b} \cdot \nabla u + \beta u = \Delta u, & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(x,0) = f(x), & \text{on } \mathbb{R}^n. \end{cases} \]

(a) Determine $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that if $u$ is a smooth solution to $(*)$, then $v(x,t) = e^{-(a \cdot x + \alpha t)}u(x,t)$ solves the Cauchy problem

\[\begin{cases} v_t = \Delta v, & \text{in } \mathbb{R}^n \times (0, \infty) \\ v(x,0) = e^{-\frac{b}{2} \cdot x}f(x), & \text{on } \mathbb{R}^n. \end{cases} \]

(b) Write down an explicit formula for a solution $u(x,t)$ to $(*)$.

6. Let $\Omega \subset \mathbb{R}^n$ a bounded domain with smooth boundary, and $T > 0$. Denote the cylinder $\Omega_T = \Omega \times (0, T]$ and its parabolic boundary $\partial_p \Omega_T = (\partial \Omega \times [0, T]) \cup (\Omega \times \{0\})$.

(a) Prove the following version of the maximum principle. Suppose that $u$ and $v$ are two functions in $C^2(\Omega_T)$ such that

\[u_t - \Delta u \leq v_t - \Delta v \quad \text{in } \Omega_T, \quad u \leq v \quad \text{on } \partial_p \Omega_T.\]

Then $u \leq v$ in $\Omega_T$.

(b) Suppose that $f(x,t), u_0(x)$ and $\phi(x,t)$ are continuous functions in their respective domains. Let $u \in C^2(\overline{\Omega_T})$ satisfy

\[\begin{cases} u_t - \Delta u = e^{-u} - f(x,t), & \text{in } \Omega_T \\ u|_{t=0} = u_0, & \text{in } \Omega \\ u|_{\partial \Omega \times (0,T)} = \phi. \end{cases} \]

Let $a = \|f\|_{L^\infty}$ and $b = \sup\{\|u_0\|_{L^\infty}, \|\phi\|_{L^\infty}\}$.

i. Show that $-(at + b) \leq u(x,t)$, for all $(x,t) \in \overline{\Omega_T}$.

**Hint: Introduce $v(x,t) = -(at + b)$ and use part a).**

ii. Prove $u(x,t) \leq Te^{aT+b} + at + b$, for all $(x,t) \in \overline{\Omega_T}$.
7. Suppose that \( f \in C^2(\mathbb{R}) \) is odd and 2-periodic (i.e. \( f(x + 2) = f(x) \) for all \( x \in \mathbb{R} \)). Let \( u \in C^2([0, 1] \times \mathbb{R}) \) solve

\[
\begin{cases}
    u_{tt} - u_{xx} = \sin(\pi x) & \text{in } (0, 1) \times \mathbb{R} \\
    u(x, 0) = f(x), \quad u_t(x, 0) = 0, & x \in [0, 1] \\
    u(0, t) = 0 = u(1, t), & t \in \mathbb{R}.
\end{cases}
\]

(a) Prove uniqueness of the solution \( u \in C^2([0, 1] \times \mathbb{R}) \).

(b) Find the solution \( u \), and show that it satisfies \( u(x, t + 2) = u(x, t) \), and \( u(x, -t) = u(x, t) \) for all \( (x, t) \in [0, 1] \times \mathbb{R} \).

8. Assume that \( \Omega \subset \mathbb{R}^n \) is open, bounded with \( C^\infty \)-smooth boundary \( \partial \Omega \). Let \( T > 0 \), and denote \( \Omega_T = \Omega \times (0, T] \). Suppose also that \( f \in C^1(\mathbb{R}^{n+2}) \), \( \phi, \psi \in C^2(\overline{\Omega}) \), and \( u \in C^2(\overline{\Omega_T}) \) is a solution of

\[
\begin{cases}
    u_{tt} - \Delta u = f(u, u_t, \nabla u), & \text{in } \Omega_T \\
    u = \phi, \quad u_t = \psi, & \text{on } \Omega \times \{t = 0\}, \\
    \frac{\partial u}{\partial n} = 0, & \text{on } \partial \Omega \times [0, T].
\end{cases}
\]

Prove that \( u \) is unique.

**Hint:** You may use an energy function of the form

\[
E(t) = \frac{1}{2} \int_{\Omega} (w_t^2 + |\nabla w|^2 + w^2) dx.
\]
1.) Consider the PDE, for \( x \in \mathbb{R} \) and \( y \in \mathbb{R} \):

\[
\begin{aligned}
\left( \ast \right) \quad \begin{cases}
2yu_x + u_y = u^4, \\
u(x, 0) = f(x),
\end{cases}
\end{aligned}
\]

for some \( C^2 \) function \( f \).

(a) Show that \( \left( \ast \right) \) has a solution that exists for all \( x \in \mathbb{R} \) and all \( y > 0 \) if and only if \( f(t) \leq 0 \) for all \( t \in \mathbb{R} \).

(b) Show that if \( \left( \ast \right) \) has a solution for all \( (x, y) \in \mathbb{R}^2 \), then \( f(t) = 0 \) for all \( t \) and \( u \) is identically 0.

2.) Suppose \( n \geq 2 \), \( R > 0 \), \( B(0,R) \subseteq \mathbb{R}^n \), and \( u : \overline{B(0,R)} \to \mathbb{R} \) satisfies \( u \in C(\overline{B(0,R)}) \), \( u \) is harmonic on \( B(0,R) \), and \( u \geq 0 \) on \( B(0,R) \).

(a) Prove that

\[
\frac{(R - |x|)R^{n-2}}{(R + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{(R + |x|)R^{n-2}}{(R - |x|)^{n-1}} u(0),
\]

for all \( x \in B(0,R) \).

(b) Prove that

\[
|ux_j(x)| \leq \frac{(2n + 2)R^{n-1}}{(R - |x|)^n} u(0),
\]

for \( x \in B(0,R) \) and \( j = 1, 2, \ldots, n \).

3.) Suppose \( n \geq 3 \), and \( \Omega \subseteq \mathbb{R}^n \) is a \( C^\infty \) bounded domain. Let

\[
\Gamma(x) = \frac{1}{(2 - n)\omega_n|x|^{n-2}},
\]

for \( x \in \mathbb{R}^n \setminus \{0\} \), be the fundamental solution for the Laplacian on \( \mathbb{R}^n \). Let \( G(x,y) \) be the Green's function for the Laplacian on \( \Omega \) (i.e., \( G(x,y) = h(x,y) + \Gamma(x-y) \)), where, for each \( x \in \Omega \), \( h(x,y) \) is a harmonic function of \( y \) on \( \Omega \), and \( h(x,y) = -\Gamma(x-y) \) for \( x \in \Omega \) and \( y \in \partial\Omega \). You can assume that \( G \in C^2(\overline{\Omega} \times \overline{\Omega} \setminus \{(x,y) \in \overline{\Omega} \times \overline{\Omega} : x = y\}) \). Prove that \( \Gamma(x-y) < G(x,y) < 0 \), for \( (x, y) \in \Omega \times \Omega \) with \( x \neq y \).
4.) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded $C^1$ domain and suppose $T > 0$. Let $\Omega_T = \Omega \times (0, T]$. Suppose $u \in C^2_1(\overline{\Omega_T}) \cap C(\overline{\Omega_T})$ satisfies

\[
\begin{aligned}
    &\begin{cases}
    u_t - \Delta u + |\nabla u|^2 - u(u - 1)(u - 2), & \text{for } (x, t) \in \Omega_T, \\
    u(x, t) = e^{-t}[1 + \sin(|x|^2)], & \text{for } (x, t) \in \partial \Omega \times [0, T], \\
    u(x, 0) = 1 + \sin(|x|^2), & \text{for } x \in \Omega.
    \end{cases}
\end{aligned}
\]

Prove that $0 \leq u \leq 2$ on $\overline{\Omega_T}$.

5.) Suppose $g = g(x, t) \in C^2_1(\mathbb{R}^{n+1}_+)$, where $x \in \mathbb{R}^n$ and $t \geq 0$, and suppose $g$ has compact support. Suppose $u \in C^2_1(\mathbb{R}^{n+1}_+ \cap C(\mathbb{R}^{n+1}_+)$ satisfies, for some positive constants $K$ and $a$,

\[
\begin{aligned}
    &\begin{cases}
    u_t - \Delta u = g(x, t) & \text{for } x \in \mathbb{R}^n, t \in (0, \infty), \\
    u(x, 0) = 0 & \text{for } x \in \mathbb{R}^n, \\
    |u(x, t)| \leq Ke^{a|x|^2} & \text{for } x \in \mathbb{R}^n, t \in [0, \infty).
    \end{cases}
\end{aligned}
\]

Suppose $p > n/2$ and $M = \max_{x \geq 0} \int_{\mathbb{R}^n} |g(x, t)|^p \, dx$. Prove that there exists a constant $C$, depending only on $n$ and $p$, such that

\[
|u(x, t)| \leq CM^{1/p} t^{1 - \frac{n}{2p}},
\]

for all $(x, t) \in \mathbb{R}^{n+1}$.

6.) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is harmonic, and $g : \mathbb{R}^3 \to \mathbb{R}$ is $C^\infty$. Suppose $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ satisfies

\[
\begin{aligned}
    &\begin{cases}
    u_{tt} = \Delta u, & x \in \mathbb{R}^3, \ t > 0 \\
    u(x, 0) = f(x), & x \in \mathbb{R}^3, \\
    u_t(x, 0) = g(x), & x \in \Omega.
    \end{cases}
\end{aligned}
\]

(a) Prove that

\[
|u(x, t)| \leq |f(x)| + \sup_{y \in B(0, 1)} |g(y)|
\]

for $x \in \mathbb{R}^3$ and $0 < t < 1$.

(b) Prove that

\[
|u(x, t)| \leq \frac{3}{4\pi t^2} \int_{B(x,t)} |g(y)| \, dy + \frac{1}{4\pi t} \int_{B(x,t)} |\nabla g(y)| \, dy,
\]

for $x \in \mathbb{R}^3$ and $t \geq 1$. 
7.) Let \( n \geq 2 \), let \( \Omega \subseteq \mathbb{R}^n \) be a \( C^\infty \) bounded domain, and let \( T > 0 \). Suppose \( \vec{h} = (h_1, h_2, \ldots, h_n) \), where each component \( h_j = h_j(x, t) : \overline{\Omega} \times [0, T] \to \mathbb{R} \) satisfies \( h_j \in C(\overline{\Omega} \times [0, T]) \). Suppose \( f, g : \overline{\Omega} \to \mathbb{R} \) are continuous. Show that there is at most one function \( u = u(x, t) \in C^2(\overline{\Omega} \times [0, T]) \) satisfying

\[
\begin{align*}
  u_{tt} &= \Delta u + \nabla u \cdot \vec{h}, \quad x \in \Omega, \ 0 < t < T \\
  u &= 0, \quad x \in \partial \Omega, \ 0 \leq t \leq T, \\
  u(x, 0) &= f(x), \quad x \in \Omega, \\
  u_t(x, 0) &= g(x), \quad x \in \Omega.
\end{align*}
\]
In the following, unless otherwise stated, \( \Omega \subset \mathbb{R}^n \) is an open, bounded set with \( C^\infty \)-smooth boundary \( \partial \Omega \). Denote \( \Omega_T = \Omega \times (0,T) \).

1. Let \( \Omega = \{(x,t) : x \in \mathbb{R}, t > 0\} \) and assume \( u_0, v_0 \in C^1(\mathbb{R}) \). Suppose \( u, v \in C^1(\overline{\Omega}) \) solve the system

\[
\begin{align*}
    u_t + u_x &= u & \text{on } \overline{\Omega}, \\
    v_t + v_x &= -v + u & \text{on } \overline{\Omega}, \\
    u(x,0) &= u_0(x), & v(x,0) &= v_0(x) & x \in \mathbb{R}.
\end{align*}
\]

Find \( u(x,t) \), \( v(x,t) \) in terms of \( u_0, v_0 \).

2. Let \( R > 0 \). Assume \( u \in C^2(\overline{B_R(0)}) \) is nonnegative and satisfies \( u(0) = 0 \),

\[
0 \leq \Delta u \leq 1 \quad \text{on } \quad B_R(0).
\]

Let \( u_1, u_2 \) be the solutions of the following problems

\[
\begin{align*}
    \Delta u_1 &= \Delta u & \text{on } B_R(0), \\
    u_1 &= 0 & \text{on } \partial B_R(0).
\end{align*}
\]

\[
\begin{align*}
    \Delta u_2 &= 0 & \text{on } B_R(0), \\
    u_2 &= u & \text{on } \partial B_R(0).
\end{align*}
\]

(a) Prove that \( u = u_1 + u_2 \) on \( B_R(0) \) and \( u_1 \leq 0, u_2 \geq 0 \) on \( B_R(0) \).

(b) Prove that \( |u_1(x)| \leq \frac{R^2}{2n} \) for all \( x \in B_R(0) \). Hint: Compare \( u_1 \) with \( \phi(x) = \frac{1}{2n}(R^2 - |x|^2) \).

(c) Prove that \( u_2(x) \leq \frac{2^n-1}{n}R^2 \) for all \( x \in B_{R/2}(0) \). Conclude \( |u(x)| \leq \frac{1+2^n}{2n}R^2 \) for all \( x \in B_{R/2}(0) \).

3. Let \( n \geq 3, f \in C^\infty_0(\mathbb{R}^n) \). Assume \( u \in C^\infty(\mathbb{R}^n) \) is a solution of

\[
-\Delta u = f \quad \text{on } \mathbb{R}^n
\]

and \( u(x) \to 0 \) as \( |x| \to \infty \). Prove there exists \( C > 0 \) such that

\[
|u(x)| \leq \frac{C}{|x|^{n-2}}
\]
for all \( x \in \mathbb{R}^n, x \neq 0 \).

4. Let \( T > 0 \) and assume \( \phi, h, f, g \) are \( C^\infty \)-smooth functions. Suppose \( u, v \in C^2(\overline{\Omega}_T) \) satisfy

\[
\begin{align*}
  u_t - \Delta u &= \phi \quad \text{on } \Omega_T, \\
  u &= h \quad \text{on } \partial \Omega \times (0, T], \\
  u &= f \quad \text{on } \Omega \times \{t = 0\}, \\
  v_t - \Delta v &= \phi \quad \text{on } \Omega_T, \\
  v &= h \quad \text{on } \partial \Omega \times (0, T], \\
  v &= g \quad \text{on } \Omega \times \{t = 0\}.
\end{align*}
\]

Prove that \( \int_\Omega |u(x, t) - v(x, t)|^2 \, dx \leq \int_\Omega |f(x) - g(x)|^2 \, dx \) for all \( t \in [0, T] \).

5. Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) is continuous, bounded and \( \int_{\mathbb{R}^n} |f| \, dx < \infty \). Show there exists a unique solution \( u \in C^\infty(\mathbb{R}^n \times (0, \infty)) \cap C^0(\mathbb{R}^n \times [0, \infty)) \) of

\[
\begin{align*}
  u_t &= \Delta u - 2u, & \text{on } \mathbb{R}^n \times (0, \infty), \\
  u &= f, & \text{on } \mathbb{R}^n \times \{t = 0\}, \\
  |u(x, t)| &\leq Ce^{-2t}(1 + t)^{-\frac{n}{2}}, \quad \text{for } (x, t) \in \mathbb{R}^n \times [0, \infty),
\end{align*}
\]

for some constant \( C \) depending on \( f, n \) but not on \( x, t \).

6. Let \( f \in C^1(\mathbb{R}) \) with \( f' \) bounded on \( \mathbb{R} \) and \( f(0) = 0 \). Suppose \( \phi, \psi \in C^2(\overline{\Omega}) \) and \( u \in C^2(\overline{\Omega}_T) \) is a solution of

\[
\begin{align*}
  u_{tt} - \Delta u &= f(u) \quad \text{on } \Omega_T, \\
  u &= 0 \quad \text{on } \partial \Omega \times (0, T], \\
  u &= \phi, \quad u_t = \psi \quad \text{on } \Omega \times \{t = 0\}.
\end{align*}
\]

(a) Denoting \( E(t) = \frac{1}{2} \int_\Omega (u_t^2 + |\nabla u|^2 + u^2) \, dx \), prove \( E(t) \leq E(0) e^{Ct} \) for all \( t \in [0, T] \), and for some constant \( C > 0 \).
(b) Prove the solution \( u \) is unique.

7. Let \( p > n/2 \). Suppose \( \phi, \psi \in C_0^\infty(\mathbb{R}^n) \) and \( u \in C^2(\mathbb{R}^n \times [0, \infty)) \) is a solution of

\[
\begin{align*}
  u_{tt} - \Delta u &= 0 \quad \text{on } \mathbb{R}^n \times [0, \infty), \\
  u &= \phi, \quad u_t = \psi \quad \text{on } \mathbb{R}^n \times \{t = 0\}.
\end{align*}
\]

Prove that there exists \( C > 0 \) such that

\[
\int_{\mathbb{R}^n} \frac{|u_t| + |\nabla u|}{(1 + |x| + t)^p} \, dx \leq \frac{C}{(1 + t)^{n/2}}
\]

for all \( t \geq 0 \).
In the following, unless otherwise stated, $\Omega \subset \mathbb{R}^n$ is an open, bounded set with $C^\infty$-smooth boundary $\partial \Omega$. Denote $\Omega_T = \Omega \times (0,T]$.

1. Let $\Omega = \{(x,t) : x \in \mathbb{R}, t > 0\}$, $b \in \mathbb{R}$ and assume $a \in C^1(\overline{\Omega})$, $\phi \in C^1(\mathbb{R})$ are bounded. Suppose $u \in C^1(\overline{\Omega})$ is a solution of

$$u_t + a(x,t)u_x + bu = 0 \quad \text{on} \quad \Omega,$$

$$u(x,0) = \phi(x), \quad x \in \mathbb{R}.$$

(a) Prove $\sup_{t \geq 0} |u(x,t)| \leq e^{-bt} \sup_{x \in \mathbb{R}} |\phi|$ for all $t \geq 0$.
(b) Find the solution when $a = a(t)$.

2. Let $\Omega \subset \mathbb{R}^2$ and suppose $g \in C^0(\partial \Omega)$. Show that there exists at most one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying

$$\Delta u + u_x - u_y = u^3 \quad \text{on} \quad \Omega,$$

$$u = g \quad \text{on} \quad \partial \Omega.$$

3. Let $\Omega \subset \mathbb{R}^n$. A function $v \in C^0(\Omega)$ is subharmonic on $\Omega$ iff for every $x \in \Omega$, there exists $r(x) > 0$ such that $v$ satisfies the mean-value property:

$$v(x) \leq \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x,r)} v(\xi) dS(\xi)$$

for all $r \in (0,r(x)]$, where $\omega_n$ is the surface area of the unit sphere in $\mathbb{R}^n$.

(a) Suppose $u,v \in C^0(\Omega)$, $u$ is harmonic on $\Omega$, $v$ is subharmonic on $\Omega$, $v \leq u$ on $\partial \Omega$. Prove $v \leq u$ on $\Omega$. You can assume the maximum principle for subharmonic functions.

(b) Let $v \in C^0(\Omega)$ be subharmonic on $\Omega$ and $B(x_0, R) \subset \Omega$. For $r \in (0,R)$ define

$$g(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x_0,r)} v(\xi) dS(\xi).$$

Prove $g$ is nondecreasing on $(0,R)$. Deduce the mean-value property

$$v(x_0) \leq \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x_0,r)} v(\xi) dS(\xi).$$
holds for any \( \overline{B(x_0, r)} \subset \Omega \) (note, in the definition of subharmonic function, this is assumed only for sufficiently small \( r \)). Hint: for \( r_1 < r_2 \) use the Poisson Integral Formula on \( B(x_0, r_2) \) to get a harmonic function.

4. Let \( m > 0, \ T > 0 \) and assume \( u_0 \in C^0(\overline{\Omega}) \) is nonnegative on \( \Omega \). Suppose \( u \in C^{\alpha,1}(\Omega_T) \cap C^0(\overline{\Omega_T}) \) is a solution of

\[
\begin{align*}
    u_t &= \Delta u + |\nabla u|^2 + u(m - u) \quad \text{on} \quad \Omega_T, \\
    u &= 0 \quad \text{on} \quad \partial\Omega \times (0,T], \\
    u &= u_0 \quad \text{on} \quad \Omega \times \{t = 0\}.
\end{align*}
\]

Prove \( 0 \leq u \leq \max\{m, \sup_{\Omega} u_0\} \) on \( \overline{\Omega_T} \).

5. Let \( 1 < p < \infty, \ u_0 \in C^0(\overline{\Omega}) \). Consider

\[
\begin{align*}
    u_t &= \Delta u + |u|^{p-1}u \quad \text{on} \quad \Omega_T, \\
    u &= 0 \quad \text{on} \quad \partial\Omega \times (0,T], \\
    u &= u_0 \quad \text{on} \quad \Omega \times \{t = 0\}.
\end{align*}
\]

For each \( u_0 \), let \( T_{\max} = T_{\max}(u_0) \in (0,\infty) \) be the maximal time such that the problem above has a solution \( u \in C^{2,1}(\overline{\Omega} \times [0,T_{\max})) \). Let \( E(t) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} dx \), \( y(t) = \int_{\Omega} u^2 dx \) for \( t \in [0,T_{\max}) \).

(a) Prove \( \frac{d}{dt}E(t) = -\int_{\Omega} u_t^2 dx \), \( t \in (0,T_{\max}) \).

(b) With \( c = \frac{2(p-1)}{p+1} |\Omega|^{\frac{1}{2}} \), prove \( \frac{d}{dt}y(t) \geq -4E(0) + cy(t)^{\frac{p+1}{2}} \), \( t \in (0,T_{\max}) \).

(c) Assume \( u_0 \) is nontrivial, \( E(0) < 0 \) and prove \( T_{\max}(u_0) < \infty \).

6. Consider the initial-boundary value problem

\[
\begin{align*}
    u_{tt} - u_{xx} &= -2 + \sin x \quad \text{on} \quad (0,\pi) \times (0,\infty), \\
    u &= x^2 - \pi x, \quad u_t = 0 \quad \text{at} \quad t = 0, \\
    u &= 0 \quad \text{at} \quad x = 0,\pi.
\end{align*}
\]

(a) Find the steady state solution \( u = f(x) \) of the differential equation and boundary conditions.

(b) Find the solution of the entire problem.

7. Suppose \( a \in C^0(\mathbb{R}^n), a \geq 1 \) on \( \mathbb{R}^n \) and \( u_0, u_1 \in C_0^\infty(\mathbb{R}^n) \). Suppose \( u \in C^2(\mathbb{R}^n \times [0,\infty)) \) is a solution of the problem

\[
\begin{align*}
    u_{tt} - \Delta u + a(x)u_t &= 0 \quad \text{on} \quad \mathbb{R}^n \times (0,\infty),
\end{align*}
\]
\[ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \]

\[ u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n. \]

Let \( E(t) = \int_{\Omega} (u_t^2 + |\nabla u|^2)dx \), \( K(t) = \int_{\Omega} (uu_t + \frac{1}{2}a u^2)dx \), \( t \in [0, \infty) \).

(a) Prove \( \frac{d}{dt}E \leq 0 \), \( \frac{d}{dt}(K+E) \leq -E \), and \( K+E \geq 0 \) for all \( t \geq 0 \). You may assume finite speed of propagation of solutions (the support of \( u(\cdot, t) \) is bounded in \( \mathbb{R}^n \) for each \( t \geq 0 \)).

(b) Prove \( E(t) \leq Ct^{-1} \) for all \( t > 0 \). Hint: Integrate an inequality in (a).
1. In the region $R := \{(x, t) : x > 0, t > 0\}$, solve the PDE

$$u_t + t^2 u_x = 4u, \quad \text{with}, \quad u(0, t) = h(t), \quad u(x, 0) = 1.$$ 

Find the conditions on $h$ so that the solution is continuous on $R$.

2. Solve the following PDE (also state the domain of the solution)

$$x^2 u_x + xy u_y = u^3, \quad \text{and} \quad u = 1, \quad \text{on the curve} \quad y = x^2.$$ 

3. Let $a > 0$ and $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < a^2\}$. Consider the equation

$$\begin{cases} 
\Delta u = 0, & \text{in} \ D, \\
 u = 1 + x^2 + 3xy, & \text{on} \ D.
\end{cases}$$

without solving the equation, find $u(0, 0)$, $\max D u$, and $\min D u$.

4. Let $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$ for $n > 2$. Let $u$ be defined on $\overline{B_1} \setminus \{0\}$. Assume that $u \in C(\overline{B_1} \setminus \{0\}) \cap C^2(B_1 \setminus \{0\})$, $u$ is harmonic in $B_1 \setminus \{0\}$, and

$$\lim_{|x| \to 0} \frac{u(x)}{|x|^{2-n}} = 0.$$ 

Prove that $u$ can be extended to 0 so that $u \in C^2(B_1)$.

**Hint:** By using the maximum principle on $B_1 \setminus B_r$ for $0 < r < 1$, one proves that $u = v$ in $B_1 \setminus \{0\}$, where $v$ is the solution of the equation

$$\begin{cases} 
\Delta v = 0, & \text{in} \ B_1, \\
v = u, & \text{on} \ \partial B_1.
\end{cases}$$

5. Let $\Omega$ be a non-empty, smooth bounded domain in $\mathbb{R}^n$. Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^1$ function such that $|f'|$ is bounded. Consider the reaction-diffusion equation

$$\begin{cases} 
u_t - \Delta u + f(u) = 0, & \text{in} \ \Omega \times (0, \infty), \\
u = 0, & \text{on} \ \partial \Omega \times (0, \infty), \\
u(x, 0) = u_0(x), & x \in \Omega.
\end{cases}$$

Prove that $C^2$ solutions to the problem are unique.
6. Let \( u_0 \in C^\infty_c(\Omega) \) for some non-empty, open, smooth bounded domain \( \Omega \subset \mathbb{R}^n \) with \( n > 2 \). Assume also that \( u_0 \geq 0 \). Let \( u \in C^\infty(\Omega \times [0, \infty)) \) be a solution of the equation
\[
\begin{cases}
  u_t = \Delta u, & \text{in } \Omega \times (0, \infty), \\
  u(\cdot, t) = 0, & \text{on } \partial \Omega \times (0, \infty), \\
  u(\cdot, 0) = u_0(\cdot), & \text{on } \Omega.
\end{cases}
\]

(a) Prove that for all \( t > 0 \),
\[
\|u(\cdot, t)\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}, \quad \text{and} \quad \|u(\cdot, t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^1(\Omega)} \|u(\cdot, t)\|_{L^2(\Omega)}^{1-\alpha},
\]
where
\[
\alpha = \frac{2^* - 2}{2(2^* - 1)}, \quad \text{for} \quad 2^* = \frac{2n}{n - 2}.
\]

(b) Prove that there is \( C > 0 \) depending on \( n, \Omega \) such that
\[
\frac{d}{dt} \int_\Omega u^2(x, t) dx \leq -C\|u_0\|^{-\frac{2\alpha}{\alpha}}_{L^1(\Omega)} \left( \int_\Omega u^2(x, t) dx \right)^{-\frac{1}{\alpha}}.
\]

(c) Prove that (for some new \( C = C(n, \Omega) > 0 \))
\[
\|u(\cdot, t)\|_{L^2(\Omega)} \leq C\|u_0\|_{L^2(\Omega)}(1 + t)^{-\frac{3}{2}}, \quad t \geq 0.
\]

Remark: The following inequalities maybe useful

(i) Hölder’s inequality:
\[
\|f\|_{L^p(\Omega)} \leq \|f\|_{L^{p_1}(\Omega)}^{\theta_1} \|f\|_{L^{p_2}(\Omega)}^{\theta_2},
\]
with
\[
\frac{1}{p} = \frac{\theta_1}{p_1} + \frac{\theta_2}{p_2}, \quad \theta_1 + \theta_2 = 1, \quad p, p_1, p_2 \in (1, \infty), \quad \theta_1, \theta_2 \in (0, 1).
\]

(ii) Sobolev - Poincaré inequality:
\[
\|\varphi\|_{L^\infty(\Omega)} \leq C(n, \Omega)\|
abla \varphi\|_{L^2(\Omega)}, \quad \forall \varphi \in C^\infty(\Omega), \quad \varphi|_{\partial \Omega} = 0.
\]

7. Let \( c > 0 \) be a fixed number. Solve the following wave equation
\[
\begin{cases}
  u_{tt} = c^2 u_{xx} + \cos(ct) \cos(x), & -\infty < x < \infty, \quad t > 0, \\
  u(x, 0) = x, \quad u_t(x, 0) = \sin(x), & -\infty < x < \infty.
\end{cases}
\]

8. Let \( u(x, t) \) be a \( C^2 \), compactly supported solution to the equation
\[
\begin{aligned}
  u_{tt} - \Delta u &= 0, & u(x, 0) &= 0, & u_t(x, 0) &= g(x), & x \in \mathbb{R}^3, & t > 0,
\end{aligned}
\]
Assume that \( \int_{\mathbb{R}^3} g(x)^2 dx < \infty \). Show that
\[
\int_0^\infty u(0, t)^2 dt \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} g(x)^2 dx.
\]
1. Let $g$ be a given smooth function on $\mathbb{R}$. Solve the PDE

$$\begin{cases}
    u_x + u_y = u^2, & \text{on } \{(x,y) \in \mathbb{R}^2, \ y > 0\}, \\
    u(x,0) = g(x), & x \in \mathbb{R}.
\end{cases}$$

2. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial \Omega$ for $n \in \mathbb{N}$. Let $u$ be a harmonic function in $\Omega$ and $x_0 \in \Omega$. Prove that

$$\left| \frac{\partial u(x_0)}{\partial x_i} \right| \leq \frac{n}{d} \sup_{x \in \Omega} \left| u(x) - u(x_0) \right|, \quad \text{where} \quad d = \text{dist}(x_0, \partial \Omega), \quad \forall \ i = 1, 2, \ldots, n.$$ 

Assume in addition that $u \geq 0$ in $\Omega$, show that

$$\left| \frac{\partial u(x_0)}{\partial x_i} \right| \leq \frac{n}{d} u(x_0), \quad \forall \ i = 1, 2, \ldots, n.$$ 

3. Let $\Omega = \mathbb{R}^3 \setminus \overline{B_1(0)}$, where $B_1(0)$ is an open unit ball in $\Omega$. Let $u$ be a harmonic function in $\Omega$ such that $u(x) \to 0$ as $|x| \to \infty$. Prove that there exist $r_0 > 1$ and $M > 0$ such that

$$|u(x)| \leq \frac{M}{|x|}, \quad |u_{x_k}(x)| \leq \frac{M}{|x|^2}, \quad \forall |x| \geq r_0, \quad \forall k = 1, 2, 3.$$ 

4. Let $T \in (0, \infty)$ and $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial \Omega$ for $n \in \mathbb{N}$. Let $\Omega_T = \Omega \times (0, T]$ and $u \in C^2(\overline{\Omega_T})$ be a solution of the equation

$$\begin{cases}
    u_t - \Delta u + c(x,t)u = u^2(1 - u), & \text{in } \Omega_T, \\
    u + \frac{\partial u}{\partial \nu} = 0, & \partial \Omega \times (0, T], \\
    u(x,0) = g(x), & x \in \Omega,
\end{cases}$$

with some given function $c(x,t)$ and $g(x)$. Assume that $c > 0$ on $\overline{\Omega_T}$ and $0 \leq g \leq 1$ on $\overline{\Omega}$. Prove that $0 \leq u \leq 1$ on $\overline{\Omega_T}$.

5. Consider $\Omega = [0, a] \times [0, b] \subset \mathbb{R}^2$ for some fixed $a > 0, b > 0$.

(a) Use separation of variables to find the first (i.e. the smallest) eigenvalue $\lambda_1$ and eigenfunction $\phi_1$ of the eigenvalue problem

$$\begin{cases}
    -\Delta \phi = \lambda \phi, & \Omega, \\
    \phi = 0, & \partial \Omega.
\end{cases}$$

**Remark:** Eigenfunctions must be non-trivial.

(b) Let $g$ be a smooth function on $\overline{\Omega}$ and $g$ vanishes on $\partial \Omega$. Also, let $\kappa < \lambda_1$. Assume that $u$ is a solution of the heat equation

$$\begin{cases}
    u_t = \Delta u + \kappa u, & x \in \Omega, \ t > 0, \\
    u(x,t) = 0, & x \in \partial \Omega, \ t > 0, \\
    u(x,0) = g(x), & x \in \Omega.
\end{cases}$$

prove that $u(x,t) \to 0$ uniformly in $x$ as $t \to \infty$. 

6. Let $T \in (0, \infty)$ and $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial \Omega$ for $n \in \mathbb{N}$. Let us denote $\Omega_T = \Omega \times (0, T)$ and $\Gamma_T$ the parabolic boundary of $\Omega_T$. Suppose that $u \in C(\overline{\Omega_T}) \cap C^2(\Omega_T)$ satisfies the PDE
\[ u_t - \Delta u = c(x, t)u, \quad (x, t) \in \Omega_T \]
for some $c \in C(\overline{\Omega_T})$ and $c \leq 0$. Show that if $u \geq 0$ on $\Gamma_T$, then
\[ \max_{(x, t) \in \Omega_T} u(x, t) = \max_{(x, t) \in \Gamma_T} u(x, t). \]
Give a counter example showing that the conclusion does not hold if the condition $u \geq 0$ on $\Gamma_T$ is violated.

7. Let $T \in (0, \infty)$ and $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial \Omega$ for $n \in \mathbb{N}$. Suppose that $u \in C^2(\overline{\Omega} \times [0, T])$ is a classical solution of the equation
\[
\begin{cases}
  u_{tt} - \Delta u = f(x, t), & \Omega \times (0, T), \\
  u(x, t) = 0, & (x, t) \in \partial \Omega \times (0, T).
\end{cases}
\]
Let
\[ E(t) = \frac{1}{2} \int_{\Omega} \left[ u_t^2(x, t) + |\nabla u|^2(x, t) \right] dx \]
(a) Prove that
\[ E(t) \leq e^{\int_0^T f^2(x, s)ds}, \quad \forall t \in [0, T]. \]
(b) Use the energy estimate to prove the uniqueness of the classical solution of the initial value problem
\[
\begin{cases}
  u_{tt} - \Delta u = f(x, t), & \Omega \times (0, T), \\
  u(x, t) = 0, & (x, t) \in \partial \Omega \times (0, T) \\
  u(x, 0) = g(x), & x \in \Omega, \\
  u_t(x, 0) = h(x), & x \in \Omega.
\end{cases}
\]
8. Let $f \in C^1(\mathbb{R}^3)$ with compact support. Suppose that $u \in C^2(\mathbb{R}^3 \times (0, \infty))$ and $u$ solves the Cauchy problem
\[
\begin{cases}
  u_{tt} - \Delta u = 0, & \mathbb{R}^3 \times (0, \infty), \\
  u(x, 0) = 0, & x \in \mathbb{R}^3, \\
  u_t(x, 0) = f(x), & x \in \mathbb{R}^3.
\end{cases}
\]
Prove that there is $M > 0$ such that
\[ |u(x, t)| \leq \frac{M}{1 + t} \left[ \|f\|_{L^\infty(\mathbb{R}^3)} + \|f\|_{L^1(\mathbb{R}^3)} + \|\nabla f\|_{L^1(\mathbb{R}^3)} \right], \quad \forall t \geq 0. \]
1.) (a) Solve the following Cauchy problem on $\mathbb{R}^2$:

$$\left\{
\begin{array}{l}
u_x + u_y = x + y \\
u = x^3 \text{ on the line } y = -x.
\end{array}
\right.$$  

(b) For what $C^1$ function or functions $f(x)$ does the Cauchy problem on $\mathbb{R}^2$:

$$\left\{
\begin{array}{l}
u_x + u_y = 3u \\
u = f(x) \text{ on the line } y = x
\end{array}
\right.$$  

have a solution? Prove your answer.

2.) Consider Burger's equation

\[(*) \quad \left\{\begin{array}{l}
u u_x + u_y = 0, \text{ for } x \in \mathbb{R}, y > 0 \\
u(x, 0) = f(x), \text{ for } x \in \mathbb{R},
\end{array}\right.\]

with initial data

\[f(x) = \left\{\begin{array}{ll}
4, & \text{for } x < 0, \\
4 - \frac{x}{2}, & \text{for } 0 \leq x \leq 2, \\
3, & \text{for } x > 2.
\end{array}\right.\]

(a) Find, with proof, the smallest $y^* > 0$ such that a shock occurs at $(x, y^*)$ for some $x \in \mathbb{R}$.

(b) Find $u(x, y)$ satisfying $(*)$ for $x \in \mathbb{R}$ and $0 \leq y < y^*$, except on two line segments where the partial derivatives of $u$ may not exist.

(c) Find the integral, or weak, solution $u(x, y)$ of $(*)$ for $y \geq 0$.

3.) (a) Suppose $f \in C^\infty(\mathbb{R}^n)$ satisfies $f(x) > 0$ for all $x \in \mathbb{R}^n$. Suppose $u \in C^2(\mathbb{R}^n)$ satisfies

$$\Delta u - f(x)u = 0$$

on $\mathbb{R}^n$, and $u(x) \to 0$ uniformly as $|x| \to \infty$. Prove that $u$ is identically 0.

(b) Find a non-trivial solution of $\Delta u + u = 0$ in $\mathbb{R}^3$ such that $u(x) \to 0$ uniformly as $|x| \to \infty$. Hint: look for a radial solution $u(x, y, z) = v(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$ and note that $rv'' + 2v' = (rv)'$. 

4.) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Suppose that $\{u_n\}_{n=1}^{\infty}$ is a sequence of harmonic functions on $\Omega$ such that
\[
\int_{\Omega} |u_n(x) - u_m(x)|^2 \, dx \longrightarrow 0
\]
as $\max\{n, m\} \to \infty$. Prove that $u_n$ converges to a harmonic function on $\Omega$.

5.) Suppose $u = u(x, t) \in C^2([0, 1] \times [0, T])$ satisfies
\[
\begin{align*}
&\left\{ \begin{array}{ll}
  u_t &= u_{xx} + tu_x, & x \in [0, 1], t \in [0, T] \\
  u_x(0, t) &= u_x(1, t) = 0, & t \in [0, T].
\end{array} \right.
\end{align*}
\]
Prove that
\[
\max_{[0,1] \times [0,T]} u(x, t) = \max_{[0,1]} u(x, 0).
\]
If you use a major theorem in PDE in your solution, provide the proof of that theorem.

6.) (a) Suppose $u = u(x, t) \in C(\mathbb{R}^n \times [0, \infty)) \cap C^2(\mathbb{R}^n \times (0, \infty))$ satisfies
\[
\begin{align*}
&\left\{ \begin{array}{ll}
  u_t &= \Delta u, & x \in \mathbb{R}^n, t > 0, \\
  u(x, 0) &= f(x), & x \in \mathbb{R}^n,
\end{array} \right.
\end{align*}
\]
where $f(x) \geq 0$ is a $C^\infty$, bounded function satisfying $\int_{\mathbb{R}^n} f(x) \, dx = 2$. Suppose $u$ satisfies
\[
|u(x, t)| \leq Ae^{\alpha|x|^2},
\]
for some positive constants $\alpha$ and $A$. Prove that $\lim_{t \to \infty} u(x, t) = 0$ and $\int_{\mathbb{R}^n} u(x, t) \, dx = 2$ for all $t > 0$.

(b) Does there exist a bounded solution $u(x, t) \in C(\mathbb{R}^n \times [0, \infty)) \cap C^2(\mathbb{R}^n \times (0, \infty))$ of the initial value problem
\[
\begin{align*}
&\left\{ \begin{array}{ll}
  u_t &= \Delta u + \frac{\cos(|x|^2+1)}{1+|x|^2}, & x \in \mathbb{R}^n, t > 0, \\
  u(x, 0) &= 0, & x \in \mathbb{R}^n?
\end{array} \right.
\end{align*}
\]
Justify your answer.
7.) Suppose \( u = u(x, t) \in C^2(\mathbb{R} \times [0, \infty)) \) satisfies

\[
\begin{aligned}
&u_{tt} - u_{xx} + u = 0, \quad \text{for } x \in \mathbb{R}, t > 0, \\
u(x, 0) = f(x), \quad \text{for } x \in \mathbb{R}, \\
u_t(x, 0) = g(x), \quad \text{for } x \in \mathbb{R},
\end{aligned}
\]

where \( f \) and \( g \) are \( C^\infty \) and have compact support.

(a) For any \((x_0, t_0) \in \mathbb{R} \times (0, \infty)\) and \(0 \leq t \leq t_0\), let \( I(t) \) be the interval

\[
I(t) = [x_0 - t_0 + t, x_0 + t_0 - t].
\]

Define

\[
e(t) = \int_{I(t)} [u^2 + u_t^2 + u_x^2] dx,
\]

for \(0 \leq t \leq t_0\). Prove that \( e \) is non-increasing on \([0, t_0]\).

(b) Suppose that \( f(x) = 0 \) and \( g(x) = 0 \) for \(|x| \geq 1\). Prove that \( u(x, t) = 0 \) for \(|x| > t + 1\), for all \( t > 0 \).

8.) Suppose \( u = u(x, t) \in C^2(\mathbb{R} \times [0, \infty)) \), is the solution of the wave equation

\[
\begin{aligned}
u_{tt} = \Delta u, \quad x \in \mathbb{R}, t > 0 \\
u(x, 0) = f(x), \quad x \in \mathbb{R}, \\
u_t(x, 0) = g(x), \quad x \in \mathbb{R}.
\end{aligned}
\]

Suppose \( g \) and \( h \) are \( C^\infty \) with \( f(x) = g(x) = 0 \) for all \( x \) such that \(|x| \geq R\), for some \( R > 0 \). The kinetic energy is

\[
k(t) = \frac{1}{2} \int_{\mathbb{R}} u_t^2(x, t) dx
\]

and the potential energy is

\[
p(t) = \frac{1}{2} \int_{\mathbb{R}} u_x^2(x, t) dx.
\]

(a) Prove that \( k(t) + p(t) \) is constant.

(b) Prove that \( k(t) = p(t) \) for all \( t > R \).
1.) Consider the equation 
\[ (*) \quad u_x + 2u_y = u, \]
for \((x, y) \in \mathbb{R}^2\).

(a) Solve (*) with the Cauchy data \(u(x, x) = e^{3x}\) for all \(x \in \mathbb{R}\).
(b) Suppose \(u\) satisfies (*) with Cauchy data \(u(x, 2x) = f(x)\). Prove that \(f(x) = Ce^{x}\) for some constant \(C\).
(c) For each constant \(C \neq 0\), show that (*) with Cauchy data \(u(x, 2x) = Ce^{x}\) has infinitely many solutions.

2.) Reduce the following equation on \(\mathbb{R}^2\):
\[ u_{xx} + 6x^2u_{xy} + 9x^4u_{yy} + 6xu_y + y - x^3 = 0 \]
to canonical form and find the general solution.

3.) Let \(\Omega \subseteq \mathbb{R}^n\) be a smooth \((C^\infty)\), bounded open set. Consider the problem
\[ (***) \quad \begin{cases} \Delta u(x) = f(x), & \text{for } x \in \Omega \\ u(x) + \frac{\partial u}{\partial n} = g(x), & \text{for } x \in \partial \Omega. \end{cases} \]
where \(f \in C(\Omega), g \in C(\partial \Omega),\) and \(\frac{\partial}{\partial n}\) is the outward normal derivative on \(\partial \Omega\).

(a) Prove that there is at most one \(u \in C^2(\overline{\Omega})\) satisfying (**).
(b) Suppose \(u \in C^2(\overline{\Omega})\) satisfies (**), with \(f \geq 0\) on \(\Omega\) and \(g \leq 0\) on \(\partial \Omega\). Prove that \(u \leq 0\) on \(\Omega\).

4.) Suppose \(u = u(x, t) \in C([0, 1] \times [0, \infty)) \cap C^2((0, 1) \times (0, \infty))\), and \(u\) satisfies
\[ \begin{cases} u_t = u_{xx}, & \text{for } 0 < x < 1, t > 0, \\ u(0, t) = u(1, t) = 0, & \text{for } t \geq 0, \\ u(x, 0) = 4x(1 - x), & \text{for } 0 \leq x \leq 1. \end{cases} \]
Prove that
(a) \(0 < u(x, t) < 1\) for \(0 < x < 1, t > 0\);
(b) \(u(1 - x, t) = u(x, t)\) for \(0 \leq x \leq 1, t > 0\);
(c) \(-8 < u_{xx}(x, t) < 0\) for \(0 < x < 1, t > 0\);
(d) \(\int_0^1 u^2(x, t) \, dx\) is a strictly decreasing function of \(t\).
5.) Suppose $u = u(x, t) \in C^2([0, 1] \times [0, \infty))$ satisfies

$$
\begin{align*}
\begin{cases}
u_{tt} - \nu_{xx} = -\frac{u}{1 + u^2}, & \text{for } 0 < x < 1, t > 0 \\
u(0, t) = u(1, t) = 0, & \text{for } t \geq 0, \\
u(x, 0) = g(x), & \text{for } 0 \leq x \leq 1,
\end{cases}
\end{align*}
$$

where $g$ is a given function satisfying $g(0) = g(1) = 0$.

(a) Define

$$E(t) = \frac{1}{2} \int_0^1 u_t^2 + u_x^2 + \log(1 + u^2) \, dx,$$

for $t \geq 0$. Prove that $E$ is constant.

(b) Show that there exists $C > 0$ such that $|u(x, t)| \leq C$ for all $x \in [0, 1]$ and $t \geq 0$.

6.) Let $\Omega \subseteq \mathbb{R}^n$ be an open set.

(a) Suppose $u \in C^1(\bar{\Omega})$ and

$$\int_{\partial B(x, r)} \frac{\partial u}{\partial n} \, dS \geq 0$$

for every $x \in \mathbb{R}^n$ and $r > 0$ such that $B(x, r) \subseteq \Omega$, where $\frac{\partial}{\partial n}$ is the outward normal derivative on $\partial \Omega$ and $dS$ is surface measure on $\partial \Omega$. Prove that $u$ is subharmonic on $\Omega$. Warning: a subharmonic function is not necessarily $C^2$.

(b) Prove the converse of part (a) under the additional assumption that $u \in C^2(\bar{\Omega})$.

7.) Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded open set. Let $h \leq 0$ be a continuous function on $\Omega \times [0, \infty)$. Prove that there exists at most one function $u = u(x, t) \in C^2(\bar{\Omega} \times [0, \infty))$ satisfying

$$
\begin{align*}
\begin{cases}
u_t = \Delta u + h(x, t)u, & \text{for } x \in \Omega, t \geq 0 \\
u(x, 0) = f(x), & \text{for } x \in \Omega, \\
u(x, t) = g(x, t), & \text{for } x \in \partial \Omega, t \geq 0.
\end{cases}
\end{align*}
$$

8.) Suppose $u = u(x, t) \in C^2(\mathbb{R}^3 \times [0, \infty))$, is the solution of the wave equation

$$
\begin{align*}
\begin{cases}
u_{tt} = \Delta u, & \text{for } x \in \mathbb{R}^3, t > 0 \\
u(x, 0) = 0, & \text{for } x \in \mathbb{R}^3, \\
u_t(x, 0) = g(x), & \text{for } x \in \mathbb{R}^3.
\end{cases}
\end{align*}
$$

Suppose $g(x) = 1$ for $|x| > 1$. Prove that

$$u(x, t) = t$$

if (i) $|x| > t + 1$ or (ii) $|x| < t - 1$. 
Problem 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded $C^2$ function that satisfies
\[ \nabla f = G, \]
where $G : \mathbb{R}^n \to \mathbb{R}^n$ satisfies
\[ \int_{\partial B_r(x_0)} G(x) \cdot (x - x_0) dA(x) = 0, \]
for all $x_0 \in \mathbb{R}^n$, $r > 0$. Prove that $f$ is constant.

Problem 2. Let $\Omega = \{(x, t) : 0 < x < 1, 0 < t < \infty\}$. Assume that $u \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$ satisfies the initial boundary value problem given by the equation
\[ \frac{\partial u}{\partial t}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) \]
in the interior of the region $\Omega$, together with the boundary conditions
\[ u(x, 0) = f(x), \quad u(0, t) = \alpha(t), \quad u(1, t) = \beta(t), \]
where $f(0) = \alpha(0), \quad f(1) = \beta(0)$.

(a) Show that $u(x, t)$ cannot have a maximum where $\partial^2 u / \partial x^2 < 0$ in the interior of the region in $(x, t)$ space with $t > 0$ and $0 < x < 1$.
(b) State the strong maximum/minimum principle for the previous IVBP.
(c) Using a maximum/minimum principle show that if $f(x) \geq 0$, $\alpha(t) \geq 0$, and $\beta(t) \geq 0$, then $u(x, t) \geq 0$.

Problem 3. Suppose $u : \mathbb{R}^2 \to \mathbb{R}$ is $C^1$ and bounded and satisfies the PDE
\[ u(x, y) = a(x, y)u_x(x, y) + b(x, y)u_y(x, y). \]

(a) Show that if $a$ and $b$ are constant functions, then $u$ is identically 0.
(b) Prove that if $a = 1 + x^2$ and $b = 1 + y^2$, the above PDE has non-vanishing bounded solutions.

Problem 4. Consider the cube $\Omega = (1, 2) \times (1, 2) \times (1, 2)$. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies
\[ yu_{xx} + xu_{yy} + xu_{zz} = 1 \]
in $\Omega$, with $u = 0$ on the boundary $\partial \Omega$. Prove that $u \geq -\frac{1}{8}$.

Hint. Compare with a function of the type $v(x) = a + b|x - x_0|^2$, where $a, b \in \mathbb{R}$, $x_0 \in \mathbb{R}^3$. 

1
Problem 5. Consider the unbounded domain $\Omega = \{(x, y) : y > x^2\} \subset \mathbb{R}^2$. Suppose $u$ is bounded and harmonic on $\Omega$, and vanishes on $\partial \Omega$. Show $u \equiv 0$.

*Hint.* Test with $u \chi$, where $\chi(y)$ is a cutoff function in the second variable $y$, and is nonconstant only on $y \in [\ell, 2\ell]$.

Problem 6. Suppose $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ is a solution of

$$
\begin{align*}
&u_{tt} - \Delta u = 0 \quad \text{on} \quad \mathbb{R}^3 \times [0, \infty), \\
&u(x, 0) = 0 \quad x \in \mathbb{R}^3, \\
&u_t(x, 0) = \psi(x) \quad x \in \mathbb{R}^3,
\end{align*}
$$

where $\psi \in C^\infty(\mathbb{R}^3)$ has compact support. Let $p \in [2, \infty)$. Prove that there exists $C > 0$ such that:

(a) $|\nabla u(x, t)| \leq C(1 + t)^{-1}$ for all $(x, t) \in \mathbb{R}^3 \times [0, \infty)$,

(b) $\int_{\mathbb{R}^3} |\nabla u(x, t)|^p dx \leq C(1 + t)^{2-p}$ for all $t \geq 0$.

Problem 7. Suppose $u \in C^2(\mathbb{R}^n \times [0, \infty))$ is a solution of

$$
\begin{align*}
&u_{tt} - \Delta u = 0 \quad \text{on} \quad \mathbb{R}^n \times [0, \infty), \\
&u(x, 0) = \phi(x) \quad x \in \mathbb{R}^n, \\
&u_t(x, 0) = \psi(x) \quad x \in \mathbb{R}^n,
\end{align*}
$$

where $\phi, \psi \in C^\infty(\mathbb{R}^n)$ have compact support. Prove that there exists $C, T > 0$ such that

$$
\int_{\mathbb{R}^n} \frac{(|u_t| + |\nabla u|)^4}{1 + |x| + t} \, dx \geq C t^{-n-1}
$$

for all $t \geq T$. 
SOLUTIONS

Q1. \( G \) is \( C^1 \) since \( f \) is \( C^2 \). Using the integral condition and the divergence theorem we obtain that \( \int_B G \cdot n dA = \int_B \text{div} \ G = 0 \) on any ball \( B \). Since \( G \) is \( C^1 \) it follows that \( \text{div} \ G = 0 \) everywhere. Taking the divergence of the first equation we obtain \( \text{div} \ \nabla f = \Delta f = \text{div} \ G = 0 \), i.e. \( f \) is harmonic. Since \( f \) is also bounded, it must be constant.

Q2. Will type it soon.

Q3. Along the characteristic curves \( \dot{x} = a, \dot{y} = b \), the solution \( u \) satisfies the equation \( \dot{z} = z \), hence \( z(t) = z(0)e^t \). For \( t \in \mathbb{R} \), this is bounded exactly if \( z(0) = 0 \). The reasoning with \( t \in \mathbb{R} \) applies for \( a, b \) constant functions, because then the characteristic curves do exist for all \( t \), namely \( x(t) = x_0 + at, y(t) = y_0 + bt \). [The same reasoning would apply for any locally Lipschitz functions \( a(\cdot, \cdot), b(\cdot, \cdot) \) that satisfy (e.g.) linear bounds \( |a(x, y)| + |b(x, y)| \leq C_0(|x| + |y|) \), by some ODE theory that we may not assume known in this generality, and which would guarantee global existence in time for the characteristic curves.]

In contrast, for \( \dot{x} = 1 + x^2, \dot{y} = 1 + y^2 \), we cover the plane with characteristic curves \( x(t) = \tan(t + c_0) = \tan(t + \arctan x_0), y(t) = \tan(t + c_1) = \tan(t + \arctan y_0) \) that exist for an interval of finite length \( \leq \pi \) only. We do not need \( z(0) = 0 \) for \( z(t) = z(0)e^t \) to be bounded on this interval. Specifically, we can choose initial data \( x(0) = s, y(0) = -s, z(0) = f(s) \) for any bounded function \( f \). Then

\[
u(t + \arctan s), \tan(t - \arctan s) = f(s)e^t\]

i.e.,

\[
u(x, y) = \exp \left[ \frac{1}{2}(\arctan x + \arctan y) \right] f \left[ \frac{1}{2}(\arctan x - \arctan y) \right]
\]

Q4. We consider \( v(x, y, z) := M + \frac{1}{6} \left( (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 + (z - \frac{1}{2})^2 \right) \) where \( M \) is yet to be determined. (It will turn out that we want \( M = -\frac{1}{6} \).) We want to show, by maximum principle, that \( w := u - v \geq 0 \).

First we note that on \( \Omega \), it holds \( yu_{xx} + xu_{yy} + xu_{zz} = \frac{1}{2}(x + y + z) > 1 \). Therefore \( yu_{xx} + xu_{yy} + xu_{zz} < 0 \) in \( \Omega \). Now \( w \) does have a minimum on the compact \( \Omega \). If the minimum were in the interior, we'd have \( w_{xx} \geq 0, w_{yy} \geq 0, w_{zz} \geq 0 \) there, and thus \( yu_{xx} + xu_{yy} + xu_{zz} \geq 0 \) in violation of the DE. So \( \min w \) is taken on at the boundary, where it equals \(-\max v = -M - \frac{1}{6} \left( (\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 \right) = -M - \frac{1}{6} \), which equals 0 for our choice \( M = -\frac{1}{6} \).

So we have \( w \geq 0 \), i.e., \( u \geq v \geq M = -\frac{1}{6} \) on \( \Omega \).

Q5. We can design \( \chi \) in such a way that \( \chi(y) = 1 \) for \( y \leq \ell, \chi(y) = 0 \) for \( y \geq 2\ell, |x'| \leq c/\ell, |x''| \leq c/\ell^2 \).

Then

\[
0 = \int_\Omega \Delta u (ux) = -\int_\Omega \nabla u \cdot (\nabla (ux)) = -\int_\Omega |
abla u|^2 \chi - \frac{1}{2} \int_\Omega \nabla (u^2) \cdot \nabla \chi
= -\int_\Omega |
abla u|^2 \chi + \frac{1}{2} \int_\Omega u^2 \Delta \chi - \frac{1}{2} \int_{\partial \Omega} u^2 \partial_n \chi \ dS.
\]
The boundary term vanishes; the second term, with $u$ bounded by $M$, can be estimated by $M^2 (c/\ell^2)(c\ell^{3/2})$, hence it goes to 0 as $\ell \to \infty$. Hence we find, in this limit, that $0 = -\int_\Omega |\nabla u|^2$, and $u \equiv \text{const.}$ By DBC, $u \equiv 0$.

Q6 & Q7. See Henry's sheet.
In the following, unless otherwise stated, \( \Omega \subset \mathbb{R}^n \) is an open, bounded set with \( C^\infty \)-smooth boundary \( \partial \Omega \). Denote \( \Omega_T = \Omega \times (0,T] \), \( \Gamma_T = \text{parabolic boundary of } \Omega_T \setminus \Omega_T \).

**Problem 1.** Let \( Q = \{(x,y) \in \mathbb{R}^2 : x > 0, y \geq 0\} \). Find the solution \( u \in C^1(\Omega) \) of the initial-value problem

\[
-2xu_x + (x+y)u_y = 0, \quad (x,y) \in Q,
\]

\[
u(x,0) = x, \quad x > 0.
\]

**Problem 2.** Let \( \Omega = \{x \in \mathbb{R}^3 : 0 < |x| < 1\} \), \( S = \{x \in \mathbb{R}^3 : |x| = 1\} \). Suppose \( u \in C^2(\Omega) \cap C^0(\Omega \cup S) \) satisfies \( \Delta u \geq 0 \) on \( \Omega \), \( u = 0 \) on \( S \) and \( u \) is bounded on \( \Omega \). Prove \( u \leq 0 \) on \( \Omega \).

**Hint:** Consider \( v(x) = u(x) - \epsilon(1/|x| - 1) \) on an appropriate subdomain of \( \Omega \).

**Problem 3.** Suppose \( \alpha \in \mathbb{R}, T > 0 \) and \( f \in C^0(\overline{\Omega}) \) with \( f > 0 \) on \( \Omega \). Let \( u \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega_T}) \) be a solution of

\[
u_t = \Delta u + f(x) + \alpha u \quad \text{on } \Omega_T,
\]

\[
u = 0 \quad \text{on } \partial \Omega.
\]

Prove \( u \geq 0 \) and \( u_t \geq 0 \) on \( \Omega \times [0,T] \).

**Problem 4.** Let \( a, b \in \mathbb{R}, T > 0 \). Suppose \( \phi, \psi \in C^\infty(\overline{\Omega}) \) and \( u \in C^2(\Omega_T) \cap C^0(\overline{\Omega_T}) \) is a solution of

\[
u_{tt} - \Delta u + au_{x_1} + bu = 0 \quad \text{on } \Omega_T,
\]

\[
u = 0 \quad \text{on } \partial \Omega \times (0,T],
\]

\[
u = \phi \quad \text{on } \Omega \times \{t = 0\},
\]

\[
u_t = \psi \quad \text{on } \Omega \times \{t = 0\}.
\]

Denoting the energy \( E(t) = \frac{1}{2} \int_{\Omega} (u_t^2 + |\nabla u|^2) dx \), prove \( E(t) \leq E(0)e^{kt} \) for all \( t \in [0,T] \), for some constant \( k > 0 \). Here \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \).
Problem 5. Let $Q = \{(x,t) : x > 0, t > 0\}$. Find the solution $u \in C^2(Q) \cap C^1(\overline{Q})$ of
\[
\begin{align*}
    u_{tt} - u_{xx} &= 0, \quad (x,t) \in Q, \\
    u(x,0) &= x, \quad x > 0, \\
    u_t(x,0) &= -1, \quad x > 0, \\
    u_x(0,t) + tu(0,t) &= 1, \quad t > 0.
\end{align*}
\]

Problem 6. Consider the heat equation
\[
u_t = \Delta u \quad \text{on } \Omega_T
\]
and define $E(t) = \int_\Omega u(x,t)^2 \, dx, t \in [0,T]$. With Dirichlet boundary conditions $u = 0$ on $\partial\Omega \times (0,T]$, in order to prove backward uniqueness of solutions, it is sufficient to establish $E^2 \leq EE''$ on $[0,T]$. Prove the same inequality for Robin boundary conditions $\partial u/\partial n = g(x)u$ on $\partial\Omega \times (0,T]$, $g \in C^0(\partial\Omega)$.

Problem 7. Let $G(x,y)$ be the Green's function for $-\Delta$ on $\Omega$ with Dirichlet boundary conditions. Define $g(x) = \int_\Omega G(x,y) \, dy, x \in \overline{\Omega}$. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of
\[
-\Delta u = e^{-u} \quad \text{on } \Omega, \\
u = 0 \quad \text{on } \partial\Omega.
\]

(a) Find $-\Delta g$.
(b) Prove there exists a constant $m > 0$ such that $mg \leq u \leq g$ on $\Omega$. Express $m$ in some explicit form involving $g$. 

2
In the following $\Omega \subset \mathbb{R}^n$ is an open, bounded set with $C^\infty$-smooth boundary $\partial \Omega$. Denote $\Omega_T = \Omega \times (0, T]$, $\Gamma_T = \text{parabolic boundary of } \Omega_T = \overline{\Omega_T} \setminus \Omega_T$.

**Problem 1.** Find all positive solutions $u$ defined on all of $\mathbb{R}^2$ to the equation $xu_x + yu_y = (x^2 + y^2)/u$.

**Problem 2.** Suppose $f \in C^0(\partial \Omega), f \geq 0$ on $\partial \Omega$. Show that if a solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to the boundary-value problem

$$-\Delta u = \frac{1}{1+u^2} \quad \text{on } \Omega,$$

$$u = f \quad \text{on } \partial \Omega,$$

exists, then it is unique.

**Problem 3.** Suppose $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ is a solution of

$$u_{tt} - \Delta u = 0 \quad \text{on } \mathbb{R}^3 \times [0, \infty),$$

$$u(x, 0) = 0, \quad x \in \mathbb{R}^3,$$

$$u_t(x, 0) = g(x), \quad x \in \mathbb{R}^3,$$

where $g \in C^2(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. Prove that there exists $C > 0$ such that

$$\sup_{x \in \mathbb{R}^3} \int_0^\infty u(x, t)^2 \, dt \leq C\|g\|_{L^2(\mathbb{R}^3)}^2.$$

**Problem 4.** Let $T > 0$ and suppose $f \in C^1(\mathbb{R}), f(0) = 0$. Consider the problem

$$u_t = \Delta u + f(u) \quad \text{on } \Omega_T,$$

$$u = 0 \quad \text{on } \Gamma_T.$$

Prove this has a solution $u \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega_T})$ and that the solution is unique.
Problem 5. Let \( \Omega = (0, \pi), Q = \Omega \times (0, \infty), f \in C^0([0, \pi]), f(0) = f(\pi) = 0. \)
Prove the problem
\[
  u_t = u_{xx} + u^2 \quad \text{on} \quad Q,
\]
\[
u = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty),
\]
\[
u = f \quad \text{on} \quad \Omega \times \{t = 0\},
\]
has no solution \( u \in C^{2,1}(Q) \cap C^0(\bar{Q}) \) if \( I = \int_0^\pi f(x) \sin x \, dx \) is sufficiently large and positive.
Hint: Derive a differential inequality for \( F(t) = \int_0^\pi u(x, t) \sin x \, dx \) and obtain a contradiction.

Problem 6. Suppose \( u \in C^2(\Omega) \cap C^0(\bar{\Omega}) \) is a solution of
\[
  \Delta u = u^3 - u \quad \text{on} \quad \Omega,
\]
\[
u = 0 \quad \text{on} \quad \partial \Omega.
\]
Prove
(a) \(-1 \leq u \leq 1 \) on \( \Omega, \)
(b) \(|u(x)| \neq 1 \) for all \( x \in \Omega. \)

Problem 7. Let \( T > 0, 1 < p \leq m. \) Suppose \( \phi, \psi \in C^\infty(\bar{\Omega}) \) and \( u \in C^2(\Omega_T) \cap C^0(\bar{\Omega}_T) \) is a solution of
\[
u_{tt} - \Delta u + u_t|u_t|^{m-1} = u|u|^{p-1} \quad \text{on} \quad \Omega_T,
\]
\[
u = 0 \quad \text{on} \quad \partial \Omega \times (0, T],
\]
\[
u = \phi \quad \text{on} \quad \Omega \times \{t = 0\},
\]
\[
u_t = \psi \quad \text{on} \quad \Omega \times \{t = 0\}.
\]
Denote \( H(t) = \frac{1}{2}\|u_t(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|
abla u(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{p+1}\|u(\cdot, t)\|_{L^{p+1}(\Omega)}^{p+1}, \)
\( t \in [0, T] \) (\( H \) is not the energy for the p.d.e.). Prove that for some constant \( c > 0, H(t) \leq H(0)e^{ct} \) for all \( t \in [0, T]. \)
Hint: Calculate \( H(t). \)
Prelim Aug 2011 Partial Differential Equations

Problem 1:
Prove that every positive harmonic function in all of $\mathbb{R}^n$ is a constant. Conclude that every semi-bounded harmonic function in all of $\mathbb{R}^n$ is a constant.

Problem 2:
Show that the damped Burger’s equation $u_t + uu_x = -u$, for $x \in \mathbb{R}, t \geq 0$, with initial data $u(x, 0) = \phi(x)$ (for a positive $C^1$ function $\phi$) has a global solution for $t \geq 0$, provided $\phi'(x) > -1$.

Problem 3:
Let $Q = \mathbb{R}^n \times (0, \infty)$, $f \in L^1(\mathbb{R}^n)$, and let $u \in C^{2,1}(Q) \cap C^0(\bar{Q})$ be the solution of the problem

$$u_t - \Delta u + u = 0 \quad \text{for} \ t > 0, x \in \mathbb{R}^n$$
$$u(x, 0) = f(x) \quad \text{for} \ x \in \mathbb{R}^n.$$

subject to the growth condition $|u(x, t)| \leq Ae^{\alpha x^2}$ for $x \in \mathbb{R}^n$ and $t \geq 0$, with certain positive constants $A, \alpha$. Show that

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-n/2} e^{-t\|f\|_{L^1(\mathbb{R}^n)}}$$

for all $t > 0$.

Problem 4:
Let $Q = \mathbb{R}^n \times (0, \infty)$, $f \in L^1(\mathbb{R}^n)$, and $g \in C^0([0, \infty)) \cap L^1(0, \infty)$. Assume that $\lim_{t \to \infty} g(t)$ exists. Suppose $u \in C^{2,1}(Q) \cap C^0(\bar{Q})$ satisfies

$$u_t - \Delta u = g(t) \quad \text{on} \ Q$$
$$u = f \quad \text{on} \ \mathbb{R}^n \times \{t = 0\}$$

and that the usual growth condition that implies uniqueness is satisfied. Show

$$\lim_{t \to \infty} u(x, t) = \int_0^\infty g(t) \, dt \quad \text{and} \quad \lim_{t \to \infty} u_t(x, t) = 0$$

for each $x \in \mathbb{R}^n$. 

1
Problem 5:
Assume in a bounded domain $\Omega \subset \mathbb{R}^n$, we have a solution $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ to $\Delta u = u^3 - 1$ and a solution $v$ to $\Delta v = v - 1$, each vanishing at the boundary. Show that $0 < v \leq u \leq 1$ in $\Omega$.

Problem 6:
Let $g \in C^2(\mathbb{R}^3)$ satisfy the conditions
\[ |g(x)| < C \quad \text{and} \quad \int_{\mathbb{R}^3} |\nabla g(x)| \, dx < 4\pi C \quad \text{and} \quad \lim_{|x| \to \infty} g(x) = 0 \]
and consider a classical solution $u$ to the wave equation
\[
\begin{align*}
    u_{tt} - \Delta u &= 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty) \\
    u(x, 0) &= C \quad \text{for } x \in \mathbb{R}^3 \\
    u_t(x, 0) &= g(x) \quad \text{for } x \in \mathbb{R}^3.
\end{align*}
\]
where $C$ is a given positive constant. Prove that $u(x, t) > 0$ for all $(x, t) \in \mathbb{R}^3 \times [0, \infty)$.

Problem 7:
Suppose $\phi \in C^\infty(\mathbb{R}^n)$ and $\psi \in C^\infty(\mathbb{R}^n)$ have support contained in the ball $B(0, \tau)$, and that $u \in C^2(\mathbb{R}^n \times [0, \infty))$ is a solution to
\[
\begin{align*}
    u_{tt} - \Delta u + \frac{1}{1+|u|} u_t &= 0 \quad \text{on } \mathbb{R}^n \times (0, \infty) \\
    u(x, 0) &= \phi(x) \quad \text{for } x \in \mathbb{R}^n \\
    u_t(x, 0) &= \psi(x) \quad \text{for } x \in \mathbb{R}^n.
\end{align*}
\]
Define $E(t) := \frac{1}{2} \int_{\mathbb{R}^n} (u_t^2 + |\nabla u|^2) \, dx$ and $I(t) := \int_t^\infty \int_{\mathbb{R}^n} \frac{1}{1+|u|} \left( u_t^2 + |\nabla u|^2 \right) \, dx \, ds$.

(a) Prove that $\int_t^\infty \int_{\mathbb{R}^n} \frac{1}{1+|u|} u_t^2(x, s) \, dx \, ds \leq E(t)$.

For your information: it can be proved that $I(t) \leq CE(t)$. You do not need to do this; only be assured of the corollary that $I(t)$ is finite.

(b) Prove that there exists a positive constant $C$ such that $I(t) \geq CE(2t)$ for all $t \geq \tau$ (with the $\tau$ from the support of the data). Hints: $I(t) \geq \int_t^{2t} \ldots$. You may assume that the support of $u$ has the same properties as solutions to the wave equation whose initial data have support in $B(0, \tau)$. And you may assume that $E(t)$ is non-increasing in $t$. 

2
In the following $\Omega \subset \mathbb{R}^n$ is an open, bounded set with $C^\infty$-smooth boundary $\partial \Omega$. Denote $\Omega_T = \Omega \times (0, T]$.

**Problem 1.** Prove the pde $u_x + 2xu_y = (y^2 - x^2)u^2 + 1$ cannot have a solution $u \in C^1(\mathbb{R}^2)$ in the entire plane $\mathbb{R}^2$.

**Problem 2.** Let $a \in \mathbb{R}$. Show the problem

$$\Delta u = u^5 + a \quad \text{on} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial \Omega,$$

has at most one solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

**Problem 3.** Let $Q = \mathbb{R}^n \times (0, \infty)$ and suppose $u \in C^{2,1}(Q) \cap C^0(\overline{Q})$ is a solution of

$$u_t - \Delta u = 0 \quad \text{on} \quad Q,$$

$$u = g(x) \quad \text{on} \quad \mathbb{R}^n \times \{t = 0\},$$

satisfying the growth condition

$$|u(x, t)| \leq Ae^{\alpha|x|^2}, \quad (x, t) \in Q,$$

where $A, \alpha$ are positive constants.

(a) Assume that $g \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ does not depend on a variable $x_j$ for some fixed $j$. Prove that the same is true for $u$.

(b) Prove that if $g \in C^\infty(\mathbb{R}^n)$ is a harmonic function on $\mathbb{R}^n$, the solution $u$ is time independent.

**Problem 4.** Let $\alpha, T > 0, \gamma \in \mathbb{R}$. Suppose $\phi \in C^0(\overline{\Omega})$ and $c \in C^0(\overline{\Omega}_T)$ with $c \geq \gamma$ on $\overline{\Omega}_T$. Suppose $u \in C^{2,1}(\Omega_T) \cap C^1(\overline{\Omega}_T)$ is a solution of

$$u_t - \Delta u + c(x, t)u = 0 \quad \text{on} \quad \Omega_T,$$

$$u = \phi \quad \text{on} \quad \Omega \times \{t = 0\},$$

$$\partial u / \partial n + \alpha u = 0 \quad \text{on} \quad \partial \Omega \times (0, T].$$

Prove $|u| \leq \sup_{\overline{\Omega}} |\phi| e^{-\gamma t}$ on $\Omega_T$ and prove $u$ is unique.
Problem 5. Solve explicitly the initial-boundary value problem
\[ u_{tt} - 4u_{xx} = 0, \quad x > 0, \quad t > 0, \]
with initial data
\[ u(x, 0) = x, \quad x > 0, \]
\[ u_t(x, 0) = -2, \quad x > 0, \]
and boundary condition
\[ u_x(0, t) + tu(0, t) = 1, \quad t > 0. \]

Problem 6. Suppose \( \Omega \subset \mathbb{R}^2 \) and \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) is a solution of
\[ (1 + u_y^2)u_{xx} + (1 + u_x^2)u_{yy} - 2u_xu_yu_{xy} = 0 \quad \text{on} \quad \Omega. \]
Show \( \inf_{\overline{\Omega}} u = \inf_{\partial \Omega} u. \)

Problem 7. Let \( T > 0, a \in \mathbb{R} \). Suppose \( \phi, \psi \in C^\infty(\overline{\Omega}) \) and \( u \in C^2(\Omega_T) \cap C^1(\overline{\Omega}_T) \) is a solution of
\[ u_{tt} - \Delta u + au_t = 0 \quad \text{on} \quad \Omega_T, \]
\[ u = \phi \quad \text{on} \quad \Omega \times \{t = 0\}, \]
\[ u_t = \psi \quad \text{on} \quad \Omega \times \{t = 0\}, \]
\[ \partial u / \partial n = 0 \quad \text{on} \quad \partial \Omega \times (0, T]. \]
Prove that for \( t \in [0, T] \) the following inequality holds \( E(t) \leq E(0)e^{at} \),
where \( E(t) = \frac{1}{2} \int_{\Omega}(u_t^2 + |\nabla u|^2)dx \) and \( a_0 = \max \{0, -2a\}. \)
In the following \( \Omega \subset \mathbb{R}^n \) is an open, bounded set with \( C^\infty \)-smooth boundary \( \partial \Omega \). Denote \( \Omega_T = \Omega \times (0,T) \).

**Problem 1.** Suppose \( u \in C^1(\mathbb{R}^2) \) is a solution of \( yu_x - xu_y = u \) on the entire plane \( \mathbb{R}^2 \). Prove \( u = 0 \) on \( \mathbb{R}^2 \).

**Problem 2.** Suppose \( f, g \in C^1(\mathbb{R}) \) with \( f(0) = g(0) = 0, f' > 0 \) and \( g' > 0 \) on \( \mathbb{R} \backslash \{0\} \). Suppose \( u \in C^2(\Omega) \cap C^1(\bar{\Omega}) \) is a solution of

\[
\Delta u = f(u) \quad \text{on} \quad \Omega,
\]

\[
\partial u / \partial n + g(u) = 0 \quad \text{on} \quad \partial \Omega.
\]

(a) Show \( u = 0 \) on \( \Omega \) using the maximum principle.
(b) Show \( u = 0 \) on \( \Omega \) using the energy method.

**Problem 3.** Let \( T > 0, c \in C^0(\bar{\Omega}_T) \). Suppose \( u \in C^{2,1}(\Omega_T) \cap C^0(\bar{\Omega}_T) \) satisfies

\[
u_t - \Delta u + c(x,t)u \leq 0 \quad \text{on} \quad \Omega_T,
\]

\[
u \leq 0 \quad \text{on} \quad \Gamma_T \quad (= \bar{\Omega}_T \backslash \Omega_T = \text{parabolic boundary of} \quad \Omega_T).
\]

Prove \( u \leq 0 \) on \( \Omega_T \).

Hint: Consider \( v = ue^{-Mt} \) for a suitable constant \( M \).

**Problem 4.** Suppose \( u \in C^2(\mathbb{R}^3 \times [0,\infty)) \) is a solution of

\[
u_{tt} - \Delta u = 0 \quad \text{on} \quad \mathbb{R}^3 \times [0,\infty),
\]

\[
u(x,0) = 0, \quad x \in \mathbb{R}^3,
\]

\[
u_t(x,0) = g(x), \quad x \in \mathbb{R}^3,
\]

where \( g \in C^2(\mathbb{R}^3) \) has compact support. Prove that there exists \( C > 0 \) such that

(a) \( |u_t(x,t)| \leq C(1 + t)^{-1} \) for all \((x,t) \in \mathbb{R}^3 \times [0,\infty), \) and
(b) \((\int_{\mathbb{R}^n} |u_t|^6dx)^{1/6} \leq C(1 + t)^{-2/3}\) for all \(t \geq 0\).

**Problem 5.** Suppose \(u \in C^2(\mathbb{R}^n)\) satisfies \(\Delta u + u^2 + 2u \leq 0\) on \(\mathbb{R}^n\). Show that the inequality \(u \geq 1\) cannot hold on all of \(\mathbb{R}^n\).

Hint: Consider the auxiliary function \(v(x) = \frac{2}{R^2} (R^2 - |x|^2)\) on \(B(0, R)\).

**Problem 6.** Suppose \(n \leq 3, \quad \phi \in C^3(\mathbb{R}^n), \psi \in C^2(\mathbb{R}^n)\) and \(\phi, \psi\) have compact support. Suppose \(u \in C^2(\mathbb{R}^n \times [0, \infty))\) is a solution of

\[
\begin{align*}
    u_{tt} - \Delta u &= u^3 \quad \text{on} \quad \mathbb{R}^n \times (0, \infty), \\
    u(x, 0) &= \phi(x), \quad x \in \mathbb{R}^n, \\
    u_t(x, 0) &= \psi(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]

where \(\int_{\mathbb{R}^n} \phi(x)^2dx > 0\). Define the energy \(E(t) = \int_{\mathbb{R}^n} \left(\frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2 - \frac{1}{4}u^4\right)dx\) and \(F(t) = \int_{\mathbb{R}^n} u^2dx\) for \(t \geq 0\). Assume \(E(0) < 0\).

(a) Prove \(E(t)\) is constant in \(t\).

(b) Find a lower bound for \(||u(\cdot, t)||_{L^4(\mathbb{R}^n)}\) and prove \(F''(t) \geq 6||u_t||_{L^2(\mathbb{R}^n)}^2\) for each \(t\).

(c) Prove \((F(t)^{-\frac{1}{2}})'' \leq 0\) for all \(t > 0\) (note \((F(t)^{-\frac{1}{2}})'' = -\frac{1}{2}(FF'' - \frac{3}{2}F'^2F^{-\frac{3}{2}})\)).

(d) Provided that \(F'(t) > 0\) for some \(t > 0\), show \(F(t) \to \infty\) as \(t \to t_0^-\) for some finite \(t_0 > 0\).

**Problem 7.** Let \(Q = \mathbb{R}^n \times (0, \infty), n = 2, 3\) and \(f \in C^0(\overline{Q})\). Suppose \(u \in C^{2,1}(Q) \cap C^0(\overline{Q})\) is a solution of

\[
\begin{align*}
    u_t - \Delta u &= f(x, t) \quad \text{on} \quad Q, \\
    u &= 0 \quad \text{on} \quad \mathbb{R}^n \times \{0\}.
\end{align*}
\]

Assume \(\int_{\mathbb{R}^n} f(x, t)^2dx \leq k\) for all \(t \geq 0\); and that for each \(\varepsilon > 0\) there exists \(C_\varepsilon > 0\) such that \(|f| \leq C_\varepsilon e^{\varepsilon|x|^2}\) on \(Q\). Assume \(|u| \leq Ae^{a|x|^2}\) holds on \(Q\) for some constants \(a, A > 0\). Show, for some \(C, \alpha > 0, \ |u| \leq Ct^\alpha\) holds on \(Q\).

Give \(\alpha\) explicitly and explain if your reasoning depends on \(n\). Explain the purpose of \(e^{a|x|^2}\).