Algebra Preliminary Examination  
August 2021

Attempt all questions, and justify each answer.

Part I

1. Let $G$ be a group. Recall that the \textit{commutator subgroup} $[G, G]$ of $G$ is the subgroup generated by all commutators $[g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2$ $(g_1, g_2 \in G)$. Also recall that a subgroup $H$ of $G$ is \textit{characteristic in} $G$, written $H \text{ char } G$, if each automorphism of $G$ maps $H$ onto itself.

(a) Define subgroups $G^{(n)}$ $(n \in \mathbb{Z}, n \geq 0)$ inductively as follows:

$$G^{(0)} = G, \quad G^{(n+1)} = [G^{(n)}, G^{(n)}].$$

Prove that $G^{(n)} \text{ char } G$ for all $n \geq 0$.

(b) Suppose that $G$ is a non-trivial finite group, such that $G^{(n)} = 1$ for some $n > 0$. Prove that $G$ has a non-trivial characteristic subgroup of prime power order. \textit{(Hint: consider the subgroup $G^{(n-1)}$, where $n$ is the smallest integer for which $G^{(n)} = 1$.)}

2. The \textit{holomorph} of a group $G$, denoted $\text{Hol}(G)$, is defined to be the semidirect product $G \rtimes \text{Aut}(G)$, where $\phi : \text{Aut}(G) \rightarrow \text{Aut}(G)$ is the identity map. Thus we may identify $\text{Aut}(G)$ with the subgroup $K = \{(1, \sigma) : \sigma \in \text{Aut}(G)\}$ of the semidirect product $\text{Hol}(G)$. As usual we identify $G$ with the (normal) subgroup $\{ (g, 1) : g \in G \}$ of $\text{Hol}(G)$.

Let $G = \{1, z_1, z_2, z_3\}$ be the non-cyclic group of order 4 (\textit{i.e.} $G$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$).

Prove that $\text{Hol}(G)$ is isomorphic to the symmetric group $S_4$. \textit{(Hint: Consider the action by left multiplication of $\text{Hol}(G)$ on the four left cosets $K, z_1K, z_2K, z_3K$ of $K$.)}

Part II

1. Let $R$ be an integral domain with the property that every ideal generated by two elements of $R$ is principal.

(a) Prove that every finitely generated ideal of $R$ is principal.

(b) Suppose that $R$ also satisfies the ascending chain condition on principal ideals, \textit{i.e.} given any chain of principal ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$, there exists a positive integer $k$ such that $I_k = I_{k+n}$ for all positive integers $n$. Prove that $R$ is a principal ideal domain.

2. Recall that an element $e$ of a ring $R$ is \textit{idempotent} if $e^2 = e$. In this question all rings are assumed to be commutative and with $1 \neq 0$.

(a) Let $R$ be a ring containing an idempotent $e$ distinct from $0, 1$. Prove that $R$ is isomorphic to a direct product of two rings. \textit{(Hint: if $e$ is idempotent, then so is $1 - e$.)}

(b) Suppose that $R$ is a finite ring and that every element of $R$ is idempotent. Prove that $R$ is isomorphic to the direct product of finitely many copies of the field with two elements.
Part III

In this part, all $R$–modules $M$ are assumed to be unital, i.e. $1 \cdot m = m$ for all $m \in M$.

1. Recall that given left $R$–modules $D, M, N$, an $R$–module homomorphism $\phi : M \rightarrow N$ induces a homomorphism of Abelian groups $\psi : \text{Hom}_R(D, M) \rightarrow \text{Hom}_R(D, N)$ given by $\psi(\alpha) = \phi \circ \alpha$.

Let $R$ be a ring with $1 \neq 0$ and let $D, L, M, N$ be left $R$–modules. Prove that if the sequence

$$0 \rightarrow L \xrightarrow{\phi} M \xrightarrow{\psi} N \rightarrow 0$$

of module homomorphisms is exact, then the sequence of induced homomorphisms of Abelian groups

$$0 \rightarrow \text{Hom}_R(D, L) \xrightarrow{\phi'} \text{Hom}_R(D, M) \xrightarrow{\psi'} \text{Hom}_R(D, N)$$

is also exact.

2. Let $I = (2, x)$ be the ideal generated by $2$ and $x$ in the ring $R = \mathbb{Z}[x]$, $x$ being an indeterminate. The ring $R/I \cong \mathbb{Z}/2\mathbb{Z}$ inherits from $R$ a natural $R$–module structure, with annihilator $I$.

(a) Show that there is an $R$–module homomorphism from $I \otimes_R I$ to $\mathbb{Z}/2\mathbb{Z}$ mapping $p(x) \otimes q(x)$ to $\frac{p(0)}{2} q'(0)$, where $q'$ denotes the usual polynomial derivative of $q$.

(b) Show that $2 \otimes x \neq x \otimes 2$ in $I \otimes_R I$.

Part IV

In this part, $x$ denotes an indeterminate.

1. This question concerns the polynomial $f(x) := x^{p^n} - x + 1 \in \mathbb{F}_p[x]$ ($n \geq 1$). We take some fixed algebraic closure $\mathbb{A}$ of $\mathbb{F}_p$, and denote by $\mathbb{F}_{p^n}$ the unique field of order $p^n$ contained in $\mathbb{A}$. You may assume that each extension of finite degree of $\mathbb{F}_p$ is Galois over $\mathbb{F}_p$, with cyclic Galois group generated by the Frobenius automorphism $\phi : a \mapsto a^p$.

(a) Let $E$ be the splitting field over $\mathbb{F}_p$ of $f(x) = x^{p^n} - x + 1$ in $\mathbb{A}$. Show that $E$ contains $\mathbb{F}_{p^n}$ as a subfield. (Hint: If $\alpha$ is a root of $f(x)$, then so is $\alpha + a$ for each $a \in \mathbb{F}_{p^n}$.)

(b) Determine the order of the Frobenius automorphism $\phi : E \rightarrow E$, $\phi : \beta \mapsto \beta^p$. (Hint: First compute $\phi^n(\alpha)$, where $\alpha$ is a root of $f(x)$.)

(c) Show that if $f(x)$ is irreducible over $\mathbb{F}_p$, then $pn = p^n$.

[Observation (you may omit the easy proof): from $pn = p^n$ it follows that $n = 1$ or $n = p = 2$.]

2. Determine the Galois group over $\mathbb{Q}$ of $x^4 + 9$, describing how each automorphism permutes the roots of this polynomial.
Part I

1. Let \( p \) be a prime, and let \( S_p \) denote the symmetric group of degree \( p \). Prove that if \( P \) is a subgroup of \( S_p \) of order \( p \), then the normalizer of \( P \) in \( S_p \) has order \( p(p-1) \).

2. Classify, up to isomorphism, the groups of order 63.

Part II

1. A local ring is a commutative ring with \( 1 \neq 0 \) that has a unique maximal ideal. Prove that if \( R \) is a local ring with maximal ideal \( M \), then every element of \( R \setminus M \) is a unit. Also prove that if \( R \) is a commutative ring with \( 1 \neq 0 \), in which the set of nonunits forms an ideal \( M \), then \( R \) is a local ring with maximal ideal \( M \).

2. Let \( p \in \mathbb{Z}_+ \) be prime, and let \( \mathbb{Z}[i] \) denote the usual ring of Gaussian integers \( \{ a+bi \mid a, b \in \mathbb{Z} \} \). For which \( p \) is the quotient ring \( \mathbb{Z}[i]/(p) \) (i) a field, (ii) a product of fields? Justify your answer. (You may use the following facts: (i) \( \mathbb{Z}[i] \) is a Euclidean Domain with respect to the field norm, hence is also a Unique Factorization Domain, and (ii) a prime \( p \in \mathbb{Z}_+ \) with \( p \equiv 1 \pmod{4} \) can be written as the sum of two integer squares.)

Hint: Use the Chinese Remainder Theorem where appropriate. Also note that a product of fields cannot contain a nonzero nilpotent element.

Part III

1. Let \( V \) be a finite dimensional vector space over a field \( F \), and let \( v_1, v_2 \) be nonzero elements of \( V \). Prove that \( v_1 \otimes v_2 = v_2 \otimes v_1 \) in \( V \otimes_F V \) if and only if \( v_1 = \lambda v_2 \) for some \( \lambda \in F \).

2. Let \( R \) be a ring with \( 1 \neq 0 \), let \( P, M, N \) be \( R \)-modules, and let there be an exact sequence of \( R \)-module homomorphisms \( M \to N \to 0 \).

(a) Prove that if \( P \) is a direct summand of a free \( R \)-module, then the induced sequence of Abelian group homomorphisms

\[
\text{Hom}_R(P, M) \to \text{Hom}_R(P, N) \to 0
\]

is exact. (Here \( \phi' \) is the homomorphism \( \psi \mapsto \phi \circ \psi \).)

(b) Show by means of an example that in general the induced sequence \( \text{Hom}_R(P, M) \to \text{Hom}_R(P, N) \to 0 \) need not be exact.

Note: For this question do not assume any result concerning projective modules.
Part IV

In this part, $x$ denotes an indeterminate.

1. This question concerns the splitting field over $\mathbb{Q}$ of the polynomial $x^4 - 2x^2 - 2 \in \mathbb{Q}[x]$.

   (a) Prove that $x^4 - 2x^2 - 2$ is irreducible over $\mathbb{Q}$, and that its roots in $\mathbb{C}$ are $\pm \alpha, \pm \beta$, where $\alpha = \sqrt{1 + \sqrt{3}}, \beta = \sqrt{1 - \sqrt{3}}$.

   (b) Prove that $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\beta)$, and that $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)] = 2$.

   (c) Prove that the splitting field of $x^4 - 2x^2 - 2$ has degree 8 over $\mathbb{Q}$, and that the Galois group of this polynomial over $\mathbb{Q}$ is dihedral of order 8.

   Hint for (c): The Galois group acts faithfully on the set of roots of the polynomial.

2. Let $\mathbb{F}_p$ denote the field of order $p$, let $f \in \mathbb{F}_p[x]$ be irreducible over $\mathbb{F}_p$, and let $K$ be a splitting field for $f$ over $\mathbb{F}_p$.

   Let $L$ be an intermediate field, i.e. $\mathbb{F}_p \subseteq L \subseteq K$. Prove that the irreducible factors of the polynomial $f$ in $L[x]$ all have the same degree.

   Hint: Here is one approach. Let $g \in L[x]$ be a factor of $f$ that is irreducible in $L[x]$, and let $\alpha$ be a root of $g$ in $K$. Consider the relationship between $[L(\alpha) : L]$ and $[K : L]$.
Algebra Preliminary Examination

August 2020

Attempt all questions, and justify each answer.

Part I

1. Let \( P \) be a Sylow \( p \)-subgroup of a finite group \( G \). If \( p \) is the smallest prime dividing \( |G| \) and \( P \) is cyclic, prove that \( N_G(P) = C_G(P) \). (Recall that \( N_G(P), C_G(P) \) denote the normalizer and centralizer of \( P \) in \( G \), respectively.)

   \((Hint: \) Consider the order of the automorphism group of \( P \) and the action of \( N_G(P) \) on \( P \) by conjugation.\)

2. (a) Prove that a group of order 105 contains a cyclic normal subgroup of order 35.
   (b) Prove that, up to isomorphism, there is just one non-Abelian group of order 105.

\(In \ parts \ II, III \ and \ IV, \ X \ denotes \ an \ indeterminate.\)

Part II

1. Let \( R \) be a commutative ring with \( 1 \neq 0 \). Recall that \( R \) is Artinian if it satisfies the descending chain condition on ideals, i.e. if \( I_1 \supseteq I_2 \supseteq \ldots \) is a descending chain of ideals of \( R \), then there exists \( k \in \mathbb{Z}_+ \) such that \( I_m = I_k \) for all \( m > k \).

Let \( S \) be an arbitrary commutative ring with \( 1 \neq 0 \), and let \( J \) denote the Jacobson radical of \( S[X] \). Prove that \( S[X]/J \) is not Artinian.

2. Let \( R \) be the subring of \( \mathbb{Q}[X] \) consisting of all polynomials whose constant term is an integer.
   (a) Prove that \( R \) is an integral domain in which every irreducible element is prime.
   (b) Prove that \( R \) is not a Unique Factorization Domain.

   \((Hint: \) Consider factorizations of the element \( X \).\)

Part III

1. Let \( k \) be a field, and let \( R = M_2(k) \) be the ring of \( 2 \times 2 \) matrices over \( k \). Let \( P \) be the set of \( 2 \times 1 \) matrices over \( k \) : then \( P \) is an Abelian group under matrix addition, and left matrix multiplication of elements of \( P \) by elements of \( R \) accords \( P \) the structure of a left \( R \)-module.

Prove that the \( R \)-module \( P \) is projective, but not free.

2. Let \( R = \mathbb{Z}[X] \), let \( I \subset R \) be the ideal generated by \( 2, X \), and let \( M = I \otimes_R I \).

Prove that the element \( 2 \otimes 2 + X \otimes X \in M \) cannot be written as a simple tensor \( a \otimes b \ (a, b \in I) \).

   \((Hint: \) Use a suitable \( R \)-module homomorphism defined on \( M \).\)
Part IV

1. Prove that \( \mathbb{Q}(\sqrt{5} + 2\sqrt{5}) \) is a Galois extension of \( \mathbb{Q} \), and determine its Galois group.

2. Let \( F \) be a field (possibly infinite) of finite characteristic \( p \), and let \( a \in F \) be an element not of form \( b^p - b \) for any \( b \in F \). Let \( f = X^p - X - a \in F[X] \).
   
   (a) Prove that the polynomial \( f \) is separable and irreducible over \( F \).
   
   (b) Prove that if \( \alpha \) is a root of \( f \) in an extension field of \( F \), then \( F(\alpha) \) is a splitting field for \( f \) over \( F \).

   (Hint: Consider the effect of substituting \( X + 1 \) for \( X \) in the polynomial \( f \).)
ALGEBRA PRELIMINARY EXAM

JANUARY 2020

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have \( 1 \neq 0 \) [and their subrings contain 1] and all modules are unitary.

Part I

1. Let \( G \) be a finite group and \( \phi : G \rightarrow H \) a surjective homomorphism. Prove that if \( y \in H \) is such that \( |y| = p^s \), for some prime \( p \) and \( r \in \mathbb{Z}_{>0} \), then there is \( x \in G \) such that \( \phi(x) = y \) and \( |x| = p^s \), for some \( s \in \mathbb{Z}_{>0} \).

[Hint: Let \( g \in G \) such that \( \phi(g) = y \), and write \( |g| = n \cdot p^k \), where \( p \nmid n \).]

2. Let \( G \) be a group of order 60 and assume that 4 divides \( |Z(G)| \) [where \( Z(G) \) denotes the center of \( G \)]. Prove that \( G \) must be Abelian.

Part II

1. Let \( I \) be the ideal of \( \mathbb{Z}[x] \) generated by 7 and \( x^2 + 1 \). Prove that \( I \) is a maximal ideal.

2. Let \( R \) be an integral domain such that for any descending chain of ideals

\[
I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots
\]

there is a positive integer \( N \) such that \( I_i = I_N \) for all \( i \geq N \). Prove that \( R \) is a field.

Part III

1. Let \( R \) be a subring of \( S \). Prove that \( S \otimes_R S \neq 0 \).

2. Let \( R \) be a ring containing \( \mathbb{Z} \) such that \( R \) is a free \( \mathbb{Z} \)-module of finite rank \( n > 0 \) and every non-zero ideal of \( R \) has a non-zero element of \( \mathbb{Z} \). Prove that for every non-zero ideal \( I \) we have that \( R/I \) is finite.

Part IV

1. Given a prime \( p \) and a positive integer \( n \), show that there is an Abelian extension [i.e., Galois with Abelian Galois group] \( K \) of \( \mathbb{Q} \) with \( [K : \mathbb{Q}] = p^n \).

2. Let \( F \) be a field of characteristic \( p \) with exactly \( p^n \) elements. If \( K \) is a finite extension of \( F \) with \( K = F[\alpha] \), for some \( \alpha \in K \), and \( f \) is the minimal polynomial of \( \alpha \) over \( F \), then show that if \( \beta \) is another root of \( f \), then \( \beta \in K \) and \( \beta = \alpha^{p^k} \) for some \( k \in \mathbb{Z} \).
ALGEBRA PRELIMINARY EXAM

AUGUST 2019

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have \( 1 \neq 0 \) [and their subrings contain 1] and all modules are unitary.

Part I

1. Let \( G_1, G_2 \) be groups, \( N \trianglelefteq G_1 \), and \( \phi : G_1 \to G_2 \) be an onto homomorphism such that \( N \cap \ker(\phi) = \{1\} \). Prove that for \( x \in N \) we have that \( C_{G_2}(\phi(x)) = \phi(C_{G_1}(x)) \). [Remember: \( C_G(x) \overset{\text{def}}{=} \{ g \in G : gx = xg \} \) is the centralizer of \( x \) in \( G \).]

2. Let \( G \) be a group of order 992 = \( 2^5 \cdot 31 \). Prove that either \( G \) has a normal subgroup of order 32 = \( 2^5 \) or it has a normal subgroup of order 62.

Part II

1. Let \( R \) be a UFD with exactly two non-associate prime elements \( p \) and \( q \) [i.e., \( p \) and \( q \) are non-associate primes and every prime is an associate of either \( p \) or \( q \)]. Prove that \( R \) is a PID.

2. Let \( R \) be a PID and \( P \) a prime ideal of \( R[x] \) such that \( P \cap R \neq \{0\} \). Prove that there is \( p \in R \) prime [in \( R \)] such that either \( P = (p) \) or \( P = (p, f) \) for some \( f \) prime in \( R[x] \).

Part III

1. Let \( R \) be a commutative ring and \( M \) an \( R \)-module. Prove that \( R \otimes_R \text{Hom}_R(R \oplus R, M) \) is projective if and only if \( M \) is projective.

2. Let \( R \) be a commutative ring, \( M \) and \( N \) be \( R \)-modules and \( M' \) and \( N' \) be submodules of \( M \) and \( N \) respectively. Define \( L \) as the submodule of \( M \otimes_R N \) generated by the set \( \{ x \otimes y \in M \otimes_R N : x \in M' \text{ or } y \in N' \} \).

Show that \( M/M' \otimes_R N/N' \cong (M \otimes_R N)/L \).

Part IV

1. Let \( F = \mathbb{Q}(\sqrt[3]{2} \cdot \zeta) \), where \( \zeta = -1/2 + \sqrt{3}i/2 \) [a primitive third root of unity]. Prove that \(-1\) is not a sum of squares in \( F \), i.e., there is no positive integer \( n \) and \( \alpha_1, \ldots, \alpha_n \in F \) such that \(-1 = \alpha_1^2 + \cdots + \alpha_n^2 \).

2. Let \( F \) be a field of characteristic 0 and \( K/F \) be a field extension of degree \( n \) such that there is a root of unity \( \zeta \) in the algebraic closure of \( K \) such that \( K \subseteq F[\zeta] \). Prove that if \( d \mid n \), there is \( \alpha \in K \) such that the minimal polynomial of \( \alpha \) over \( F \) has degree \( d \).
ALGEBRA PRELIMINARY EXAM

AUGUST 2018

**Instructions:** Attempt all problems in all four parts. Justify your answers.

**General assumptions:** All rings have $1 \neq 0$, their subrings contain 1, and all modules are unitary.

**Part I**

1. Let $G$ be a (possibly infinite) group, and suppose that $G$ contains a subgroup $H \neq G$ whose index $[G : H]$ is finite. Prove that $G$ contains a normal subgroup $N \neq G$ of finite index.
2. Prove that every group of order 70 contains a cyclic, normal subgroup of order 35.

**Part II**

1. Let $R$ be a commutative ring in which every element is either a unit or nilpotent. Prove that $R$ has exactly one prime ideal.
2. If $R$ is an integral domain, prove that there are infinitely many ideals in $R[x]$ that are both prime and principal.

**Part III**

1. Let $R$ be a ring, possibly non-commutative, and suppose that
   \[ 0 \to M' \to M \to M'' \to 0 \]
   is a short exact sequence of left $R$-modules, with $M'$ and $M''$ finitely generated. Prove that $M$ is finitely generated.
2. Let $M$ be a finitely-generated $\mathbb{Z}$-module, and let $T \subseteq M$ be its torsion submodule. Show that the torsion submodule of $M \otimes_{\mathbb{Z}} M$ has at least $|T|$ elements.

**Part IV**

1. Let $p$ be a prime and suppose that $f \in \mathbb{F}_p[x]$ is an irreducible polynomial. Let $K$ be a degree 2 extension of $\mathbb{F}_p$ and suppose that there exist non-constant polynomials $g, h \in K[x]$ such that $f = gh$. If $g$ is an irreducible polynomial of degree 5, what is the degree of $f$?
2. Suppose that $f \in \mathbb{Q}[x]$ is an irreducible degree 4 polynomial, and $K/\mathbb{Q}$ is an extension such that $f$ has exactly one root in $K$. Let $G$ be the Galois group of $f$, and show that $|G|$ is divisible by 12.
ALGEBRA PRELIMINARY EXAM

AUGUST 2017

Instructions: Attempt all problems in all four parts. Justify your answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Suppose that $H$ is a subgroup of a finite group $G$ of index $p$, where $p$ is the smallest prime dividing the order of $G$. Prove that $H$ is normal in $G$.
2. Show that every group of order 222 is solvable.
   
   Fun fact: The University of Tennessee was established 222 years ago.

Part II

1. Let $I$ and $J$ be ideals of a ring $R$ and assume that $P$ is a prime ideal of $R$ that contains $I \cap J$. Prove that either $I$ or $J$ is contained in $P$.
2. Let $R$ be an integral domain and suppose that every prime ideal in $R$ is principal. Prove that $R$ is a PID.

Part III

1. Let $V$ be a Noetherian right $R$-module, and $\theta : V \to V$ a homomorphism.
   (a) Show that $\ker(\theta^{n+1}) = \ker(\theta^n)$ for some $n \geq 1$.
   (b) If $\theta$ is onto, show that it is one-to-one.
2. An $R$-projection is defined to be an $R$-module homomorphism $\varphi : R^n \to R^n$ such that $\varphi^2 = \varphi$. Prove that a finitely generated $R$-module $M$ is projective if and only if it is isomorphic to the image of some $R$-projection.

Part IV

1. Let $F \subseteq E$ be fields and suppose $0 \neq \alpha \in E$ with $E = F(\alpha)$. Assume that some power of $\alpha$ lies in $F$ and let $n$ be the smallest positive integer such that $\alpha^n \in F$.
   (a) If $\alpha^m \in F$ with $m > 0$, show that $m$ is a multiple of $n$.
   (b) If $E$ is a separable extension of $F$, prove that the characteristic of $F$ does not divide $n$.
   (c) If every root of unity of $E$ lies in $F$, show that $[E : F] = n$.
2. Let $F$ be a field of characteristic 0 and let $E$ be a finite Galois extension of $F$.
   (a) If $0 \neq \alpha \in E$ with $E = F(\alpha)$, show that $F(\alpha^2) \neq E$ if and only if there exists $\sigma \in \text{Gal}(E/F)$ with $\sigma(\alpha) = -\alpha$.
   (b) Prove that there exists an element $\alpha \in E$ with $E = F(\alpha^2)$.
• Attempt all four parts. Justify your answers.

Part I.

1. Show that a group of order 255 is not a simple group.

2. A group $G$ has a cyclic normal subgroup of order 2016. If $G$ also has a subgroup of order 2017, then show that $G$ has a cyclic subgroup of order $(2016) \times (2017)$.

Part II.

Note: Rings are assumed to be commutative and with $1 \neq 0$.

1. Let $A$ and $B$ be rings. Show that each ideal of $A \times B$ is of the form $I \times J$, where $I$ is an ideal of $A$ and $J$ is an ideal of $B$.

2. Let $R$ be a ring, let $X$ be an indeterminate and let $S := \{X^n \mid 0 \leq n \in \mathbb{Z}\}$. If $S^{-1}R[[X]]$ is a field, then show that $R$ is a field.

Part III.

Note: Rings are assumed to be commutative with $1 \neq 0$ and modules are assumed to be unitary.

1. Let $A$ be a ring and let $M$, $N$ be finitely generated projective (left) $A$-modules. Show that $\text{Hom}_A(M, N)$ is a finitely generated projective $A$-module.

2. Let $R$ be a PID and let $I$, $J$ be ideals of $R$. If $I \neq R \neq J$, then show that $(R/I) \oplus (R/J)$ and $(R/I) \otimes_R (R/J)$ are not isomorphic as (left) $R$-modules.

Part IV.

Note: In what follows, $X$ is an indeterminate.

1. Let $K$ be an extension-field of $\mathbb{Q}$ such that $K/\mathbb{Q}$ is Galois with Galois group $\mathbb{Z}_{30}$. Suppose each of $f, g \in \mathbb{Q}[X]$ is an irreducible polynomial of degree 6 and $f$ has a root $a \in K$. If $g$ has a root in $K$, then show that $g$ has all its roots in $\mathbb{Q}[a]$.

2. Let $F \subset K$ be finite fields of characteristic 5 and suppose $g \in F[x]$ is irreducible in $F[x]$. If $g$ has degree 11, then show that either $g$ is irreducible in $K[x]$ or all its roots are in $K$. 
• Attempt all four parts. Justify your answers.

Part I.

1. Let $p$ be a prime number and $G$ be a non-Abelian group of order $p^3$. Show that $G$ has at least 3 (distinct) subgroups of index $p$.

2. Let $G$ be a group of order $p^3q$, where $p$, $q$ are distinct prime numbers. If no Sylow $p$-subgroup of $G$ is normal and also no Sylow $q$-subgroup of $G$ is normal, then show that $G$ has order 24.

Part II.

Note: Rings are tacitly assumed to be commutative and with $1 \neq 0$.

1. Let $R$ be a ring, $X$ an indeterminate and $h : R[X] \to R[[X]]$ a ring-homomorphism such that $h(a) = a$ for all $a \in R$. Show that $h$ is not surjective.

2. Let $R$ be an integral domain with at least 3 (distinct) maximal ideals. Given maximal ideals $M$ and $N$ of $R$, show that $R_M \cap R_N \neq R$. (Here localization of $R$ at a prime ideal is naturally identified as a ring in between $R$ and the quotient-field of $R$.)

Part III.

Note: Rings are assumed to be commutative and with $1 \neq 0$ and modules are assumed to be unitary.

1. Let $R$ be a ring and let $a \in R$ be a nonzero element of $R$ such that $a^3 = a$. Show that the ideal $Ra$ is a projective $R$-module.

2. Let $R$ be a PID and let $M$ be a finitely generated $R$-module. For a maximal ideal $Q$ of $R$, let $\delta(Q, M)$ denote the dimension of $M \otimes_R R/Q$ as a vector-space over the field $R/Q$. Let $\delta(M)$ denote the sup{$\delta(Q, M)$}, where the supremum is taken over all maximal ideals $Q$ of $R$. Show that as an $R$-module, $M$ has a generating set of cardinality $\delta(M)$ and any generating set of $M$ has cardinality at least $\delta(M)$.

Part IV.

Note: In what follows, $X$ is an indeterminate.

1. Let $f(X)$ be a monic polynomial with rational coefficients. Assume $f(X)$ is irreducible in $Q[X]$ and the Galois-group of $f(X)$ over $Q$ is a group of order 99. What is the degree of $f(X)$?

2. Compute the Galois group of $X^6 - 9$ over $Q$. 
ALGEBRA PRELIMINARY EXAM

JANUARY 2016

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have 1 \neq 0 [and their subrings contain 1] and all modules are unitary.

Part I

1. Let $G$ be a finite group and $H$ be a subgroup of $G$. Prove that $n_p(H) \leq n_p(G)$, where $n_p(X)$ denotes the number of Sylow $p$-subgroups of $X$.

2. Let $G$ be a group of order $p^n$ for some prime $p$ and positive integer $n$. Prove that if $1 \neq H \leq G$, then $Z(G) \cap H \neq 1$. [Here $Z(G)$ denotes the center of $G$.]

Part II

1. Let $R$ be a Boolean ring, i.e., a ring [with 1] for which $a^2 = a$ for all $a \in R$. [You can use without proof the well known fact that if $R$ is Boolean, then it is commutative of characteristic 2.]
   (a) Prove that if $R$ is finite, then its order is a power of 2.
   (b) Prove that every prime ideal of $R$ is maximal.

2. Show that $R \overset{\text{def}}{=} \mathbb{Z}[x_1, x_2, x_3, \ldots]/(x_1 x_2, x_3 x_4, x_5 x_6, \ldots)$ has infinitely many distinct minimal prime ideals. [$P$ is a minimal prime ideal if it is prime and whenever $Q \subseteq P$, with $Q$ also prime, we have $Q = P$.]

Part III

1. Let $F$ be a field and $M$ be a torsion $F[x]$-module. Prove that if there is $m_0 \in M$, with $m_0 \neq 0$, and an irreducible $f \in F[x]$ such that $f \cdot m_0 = 0$, then $\text{Ann}(M) \subseteq (f)$.

2. Let $R$ be an integral domain and $I$ a principal ideal of $R$. Prove that $I \otimes_R I$ has no non-zero torsion element [i.e., if $m \in I \otimes_R I$, with $m \neq 0$, and $r \in R$ with $rm = 0$, then $r = 0$].

Part IV

1. Let $K/F$ be an algebraic field extension and $\text{Emb}(K/F)$ denote the set of field homomorphisms $\sigma : K \rightarrow \bar{K}$ such that $\sigma(a) = a$ for all $a \in F$. [Here $\bar{K}$ is a fixed algebraic closure of $K$.]
   (a) Prove that if $\alpha$ is a root of a [not necessarily irreducible] non-zero polynomial $f \in F[x]$ with $\deg(f) = n$, then $\text{Emb}(F[\alpha]/F)$ has at most $n$ elements.
   (b) Give an example of an algebraic extension $K/F$ of degree greater than one for which $\text{Emb}(K/F)$ has a single element.

2. Let $F = \mathbb{Q}[\sqrt{2}]$ and $K = \mathbb{Q}[\sqrt{2}, i]$.
   (a) Prove that $K/F$ is Galois with $[K : F] = 8$.
   (b) Prove that $\text{Gal}(K/F)$ has a non-normal subgroup. [This implies that it is the dihedral group of order 8, as it is the only group of order 8 with this property.]
ALGEBRA PRELIMINARY EXAM

AUGUST 2015

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have \(1 \neq 0\) [and their subrings contain 1] and all modules are unitary.

Part I

1. Let \(G\) be a non-Abelian group of order \(p^3\), \([G,G] = \langle xyx^{-1}y^{-1} : x, y \in G \rangle\) be its commutator subgroup and \(Z(G)\) be its center. Show that \(|Z(G)| = p\) and that \(Z(G) = [G,G]\).

2. Let \(G_1\) and \(G_2\) be groups of order \(81\) acting faithfully [i.e., only 1 acts as the identity function] on sets \(X_1\) and \(X_2\), respectively, with 9 elements each. Show that there is an isomorphism \(\psi : G_1 \rightarrow G_2\).

Part II

1. Let \(D\) be a finite division ring. Prove that \(D\) has a prime power number of elements. [Hint: Consider the center \(Z(D) = \{a \in D : ax = xa\ \text{for all } x \in D\}.\]

2. Let \(p \in \mathbb{Z}\) prime and
\[
f = a_{2n+1}x^{2n+1} + \cdots + a_1x + a_0 \in \mathbb{Z}[x].
\]
Prove that if \(p^3 \nmid a_0, p^2 \mid a_0, a_1, \ldots, a_n, p \mid a_{n+1}, a_{n+2}, \ldots, a_{2n}\) and \(p \nmid a_{2n+1}\), then \(f\) is irreducible in \(\mathbb{Q}[x]\).

Part III

1. Let \(R\) be a commutative ring. An \(R\)-module is Artinian if it satisfies the descending chain condition for submodules. [I.e., if \(S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots\) is a chain of submodules, then there is a \(i_0\) such that for all \(i \geq i_0\), we have \(S_i = S_{i_0}\).] Show that if \(L\) and \(N\) are Artinian \(R\)-modules and we have a short exact sequence
\[
0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,
\]
then \(M\) is also Artinian.

2. Let \(R\) be a commutative ring such that every \(R\)-module is free. Prove that \(R\) is a field.

Part IV

1. Let \(\mathbb{F}_p\) be the field with \(p\) elements, and \(t\) be an indeterminate. Let \(f(t), g(t) \in \mathbb{F}_p[t] \setminus \{0\}\), with \(\max\{\deg f, \deg g\} < p\) and \(f(t)/g(t) \notin \mathbb{F}_p\). Show that the extension \(\mathbb{F}_p(t)/\mathbb{F}_p(f(t)/g(t))\) is separable.

2. Suppose that \(f = \prod_{i=1}^{N} (x - \alpha_i) \in \mathbb{Q}[x]\) [with \(\alpha_i \in \mathbb{C}\)] is irreducible in \(\mathbb{Q}[x]\) and let \(f_n \overset{\text{def}}{=} \prod_{i=1}^{N} (x - \alpha_i^n)\). Prove that for each \(n\), there is \(g_n \in \mathbb{Q}[x]\) irreducible and a positive integer \(k_n\) such that \(f_n = g_n^{k_n}\).
ALGEBRA PRELIMINARY EXAMINATION  
Fall 2014

Attempt all four parts. Justify your answers.

Part I.

1. Show that $S_4$ (the group of permutations of $\{1, 2, 3, 4\}$) does not have a subgroup isomorphic to $Q_8$ (the quaternion-group of order 8).

2. Let $G$ be a group of order 2014. Show that $G$ is cyclic if and only if $G$ has a normal subgroup of order 2.

Part II.

Note: Rings are assumed to be commutative and with $1 \neq 0$, subrings are assumed to contain 1 and ring-homomorphisms are assumed to map 1 to 1.

1. Let $R$ be an integral domain with only finitely many units. Show that the intersection of all maximal ideals of $R$ is 0.

2. Let $R$ be a ring such that each non-unit of $R$ is nilpotent. Let $X$ be an indeterminate and let $f \in R[[X]]$. Show that $f^n = f$ for some integer $n \geq 2$ if and only if either $f = 0$ or $f^{n-1} = 1$.

Part III.

Note: Rings are assumed to be commutative and with $1 \neq 0$ and modules are assumed to be unitary.

1. Let $L$ be a module over a ring $R$ and let $M, N$ be $R$-submodules of $L$. Show that if $(M + N)/(M \cap N)$ is a projective $R$-module then $M/(M \cap N)$ is also a projective $R$-module.

2. Let $R$ be a PID with infinitely many prime ideals and let $M$ be a finitely generated $R$-module. Show that $M$ is a torsion $R$-module if and only if $M \otimes_R R/P = 0$ for all but finitely many prime ideals $P$ of $R$.

Part IV.

Note: In what follows, $X$ is an indeterminate.

1. Let $f(X) := X^5 + 3X^3 + X^2 + 3 \in \mathbb{Q}[X]$. Let $K$ be the splitting field of $f(X)$ over $\mathbb{Q}$. Compute $[K : \mathbb{Q}]$.

2. Let $f(X) := X^3 + X + 1 \in \mathbb{Q}[X]$. Let $F$ be a finite Galois extension of $\mathbb{Q}$ such that the Galois group of $F$ over $\mathbb{Q}$ is an Abelian group. Show that $f$ is irreducible in $F[X]$. 
Algebra Preliminary Exam  January 2014

Attempt all problems and justify all your answers. All rings have a 1 ≠ 0, all ring homomorphisms send 1 to 1, and all R-modules are unitary.

Part I. Groups

1. Show that every group of order 1,225 is abelian.
2. Let n ≥ 2. Show that there is a nontrivial homomorphism
   \[ f : S_n \to \mathbb{Z}/n\mathbb{Z} \] (i.e., \( \ker f \neq S_n \)) if and only if n is even.

Part II. Rings

1. Let R be a commutative ring. Show that \( J(R[X]) = \text{nil}(R[X]). \)
   (\( J(A) \) and \( \text{nil}(A) \) are the Jacobson and nil radicals of \( A \).)
2. Let R be a PID.
   (a) Show that R satisfies ACC on ideals.
   (b) Show that every nonzero prime ideal of R is maximal.

Part III. Modules

1. Let \( R \) be a ring and \( M \) a nonzero \( R \)-module. Show that \( M = A \oplus B \) for proper submodules \( A \) and \( B \) of \( M \) if and only if there is a nonzero, nonidentity homomorphism \( f : M \to M \) with \( f^2 = f \).
2. Let \( R \) be a commutative ring, \( I \) a proper ideal of \( R \), and \( M \) an \( R \)-module. Show that \( (R/I) \otimes_R M \) and \( M/IM \) are isomorphic as \( R \)-modules.

Part IV. Fields

1. Let \( K \) a subfield of a field \( F \). Show that there is a subring of \( F \) containing \( K \) that is a PID, but not a field, if and only if the extension \( F/K \) is not algebraic.
2. Determine the Galois group of \( f(X) = X^{10} + X^8 + X^6 + X^2 \) over \( \mathbb{Z}/2\mathbb{Z} \).
Algebra Preliminary Exam
August 2013

Attempt all problems and justify all your answers. All rings have an identity $1 \neq 0$, all ring homomorphisms send 1 to 1, and all $R$-modules are unitary.

Part I.

1. (a) Let $p$ and $q$ be (not necessarily distinct) prime numbers. Show that a group $G$ with $|G| = pq$ is either abelian or $Z(G) = \{e\}$.
   (b) Give an example of a nonabelian group $G$ whose order is the product of three (not necessarily distinct) primes and $Z(G) \neq \{e\}$.

2. (a) Let $G$ be a group with $|G| = 100$. Show that $G$ is abelian if and only if its Sylow 2-subgroup is normal.
   (b) Give an example of a nonabelian group of order 100.

Part II.

1. Let $R$ and $S$ be a commutative rings with $1 \neq 0$. Show that every ideal of $R \times S$ has the form $I \times J$ for $I$ an ideal of $R$ and $J$ an ideal of $S$.

2. Let $R$ be a commutative ring with $1 \neq 0$. Show that $f(X) = a_0 + a_1X + \cdots + a_nX^n$ is a unit in $R[X]$ if and only if $a_0$ is a unit in $R$ and $a_1, \ldots, a_n$ are nilpotent.
Part III

1. Let $P$ and $Q$ be finitely generated projective $R$-modules over a commutative ring $R$ with $1 \neq 0$. Show that $\text{Hom}_R(P,Q)$ is a finitely generated projective $R$-module.

2. Let $R$ be a commutative ring with $1 \neq 0$, $S$ a nonempty multiplicatively closed subset of $R$, and $M$ an $R$-module. Show that $(S^{-1}R) \otimes_R M$ and $S^{-1}M$ are isomorphic as $S^{-1}R$-modules.

Part IV.

1. Let $p$ and $q$ be distinct prime numbers, $F$ a subfield of a field $K$, and $f(X), g(X) \in F[X]$ be irreducible with $\deg(f(X)) = p$ and $\deg(g(X)) = q$. Let $a, b \in K$ be roots of $f(x)$ and $g(X)$, respectively. Show that $[F(a,b):F] = pq$.

2. (a) Let $F$ be a splitting field for $f(X) \in \mathbb{Q}[X]$ over $\mathbb{Q}$ with abelian Galois group $G$. Show that every subfield $L$ of $F$ is a splitting field over $\mathbb{Q}$ for some polynomial $g(X) \in \mathbb{Q}[X]$.

(b) Give an example to show that if $G$ is not abelian in part (a), then some $L$ need not be a splitting field.
ALGEBRA PRELIMINARY EXAM

JANUARY 2013

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let $p$ and $q$ be prime numbers such that $q < p$ and $q$ does not divide $p^2 - 1$. Prove that every group of order $p^2q$ is Abelian.

2. Let $G$ be a finite simple group. Show that if $p$ is the largest prime dividing $|G|$, then there is no subgroup $H \leq G$ such that $1 < |G : H| < p$.

Part II

1. Let $R$ be a ring not necessarily having 1 [or commutative], with at least two elements and such that for all non-zero $a \in R$ there is a unique $b \in R$ such that $aba = a$.
   (a) Show that $R$ has no [non-zero] zero divisors.
   (b) Show that for $a$ and $b$ as above, we also have $bab = b$.
   (c) Show that $R$ has 1.

2. Let $R$ be a commutative ring and $a \in R$ such that $a^n \neq 0$ for all positive integers $n$. Let $I$ be an ideal maximal with respect to the property that $a^n \not\in I$ for any positive integer $n$. Show that $I$ is prime.

Part III

1. Let $V = \mathbb{R}^2$ and $\{e_1, e_2\}$ be a basis of $V$. Show that $e_1 \otimes e_2 + e_2 \otimes e_1 \in V \otimes_{\mathbb{R}} V$ cannot be written as a single tensor.

2. Let $R$ be a PID.
   (a) Prove that a finitely generated $R$-module $M$ is free if and only if it is torsion free.
   (b) Prove that if a finitely generated $R$-module $M$ is projective, then it is free.

Part IV

1. Let $\mathbb{F}_p$ be the field with $p$ elements, $\bar{\mathbb{F}}_p$ be a fixed algebraic closure of $\mathbb{F}_p$ and let
   $$L = \{\alpha \in \bar{\mathbb{F}}_p : p \nmid [\mathbb{F}_p[\alpha] : \mathbb{F}_p]\}.$$
   Show that $L$ is a field.

2. Let $p$ be a prime, $F$ be a field of characteristic different from $p$ and $f = x^p - a \in F[x]$ [not necessarily irreducible]. Let $K$ be the splitting field of $x^p - 1$ over $F$ and assume that all roots of $f$ lie in $K$.
   (a) Show that if $f(\alpha) = 0$ with $\alpha \not\in F$, then $F[\alpha] = K$.
   (b) Prove that $f$ has a root in $F$. 
ALGEBRA PRELIMINARY EXAM

AUGUST 2012

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: All rings have $1 
eq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let $G$ and $H$ be finite Abelian groups. Prove that if $G \times H \times H \cong G \times G \times H$, then $G \cong H$.

2. Let $p$ be a prime and $G$ be a group of order $p^n$. For $k \in \{1, 2, 3, \ldots, (n - 1)\}$, let $s_k$ and $n_k$ denote the number of subgroups and normal subgroups of $G$ of order $p^k$ respectively. Show that $s_k - n_k$ is divisible by $p$.

Part II

1. Let $R$ be a commutative ring for which every proper ideal is prime. Show that $R$ is a field.

2. Let $F$ be a field and consider the subring $R$ of $F[t]$ given by polynomials with the coefficient of $t$ equal to zero, i.e., $R = F + t^2 F[t]$.
   (a) Show that $R$ has an irreducible element which is not prime. [Hence, $R$ is not PID.]
   (b) Show that $R$ is Noetherian. [Hint: Consider a connection between $R$ and $F[x, y]$.]

Part III

1. Let $R$ be a commutative ring, $S$ be a subring of $R$, $A$ be an $R$-module and
   \[ \mathcal{H} \overset{\text{def}}{=} \text{Hom}_R(R \otimes_S (S \oplus S), A). \]
   Show that for every surjective homomorphism of $R$-modules $\phi : M \to N$ and $R$-module homomorphism $f : \mathcal{H} \to N$ there is an $R$-module homomorphism $F : \mathcal{H} \to M$ such that $\phi \circ F = f$ if and only if the same if true if we replace $\mathcal{H}$ by $A$.

2. Let $R$ be a commutative ring, $D$, $M$, and $N$ be $R$-modules, $\phi : M \to N$ an $R$-module homomorphism and $1 \otimes \phi : D \otimes_R M \to D \otimes_R N$ be the homomorphism for which
   \[ (1 \otimes \phi)(d \otimes m) = d \otimes \phi(m). \]
   (a) Assume that $\phi$ is injective. Show that if $D$ is free and of finite rank, then $1 \otimes \phi$ is also injective. [The finite rank is not necessary, but we assume it here for simplicity.]
   (b) Show that the above statement is not true for an arbitrary $D$.

Part IV

1. Let $F$ be a field and $K/F$ be an algebraic extension. Show that if $R$ is a subring of $K$ with $F \subseteq R \subseteq K$, then $R$ is a field.

2. Let $F$ be a field, $K/F$ be a Galois extension and $f \in F[z]$ be monic, separable and irreducible. Show that if $f = f_1 \cdots f_k$ is the factorization of $f$ in $K[z]$, with $f_i$ irreducible and monic, then the $f_i$'s are distinct, of the same degree and $G \overset{\text{def}}{=} \text{Gal}(K/F)$ acts transitively on $\{f_1, \ldots, f_k\}$. [i.e., given $\sigma \in G$, the map $f_i \mapsto f_i^\sigma$ is a permutation of the $f_i$'s and given $i, j \in \{1, \ldots, k\}$, there is a $\tau \in G$ such that $f_i^\tau = f_j$.]
ALGEBRA PRELIMINARY EXAMINATION
Spring 2012

Attempt all four parts. Justify your answers.

Part I.

1. Show that a group of order 455 is necessarily cyclic.

2. Let $G$ be a group of order 56. Show that $G$ is solvable.

Part II.

1. Let $f : \mathbb{Q} \to \mathbb{Z}$ be a function such that $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{Q}$. Show that the image of $f$ has at most three elements and there exist an infinite number of such functions whose image has three elements.

2. Let $R$ be a PID and let $J$ denote the intersection of all maximal ideals of $R$. If $a^2 - a$ is in $J$ for all $a \in R$, then show that $R$ has only finitely many maximal ideals.

Part III.

Note: Rings are assumed to be commutative and with $1 \neq 0$ and modules are assumed to be unitary.

1. Let $R$ be an integral domain and let $M, N$ be projective $R$-modules. Show that $M \otimes_R N$ is a projective $R$-module.

2. Suppose $R$ is a principal ideal domain that is not a field. Suppose $M$ is a finitely generated $R$-module such that for every maximal ideal $P$ of $R$, $M/PM$ is a cyclic $R/P$-module. Show that $M$ itself is cyclic.

Part IV.

1. Let $f(X)$ be a monic polynomial of degree 9 having rational coefficients. Assume that $f(X)$ is irreducible in $\mathbb{Q}[X]$. Let $K$ denote the splitting field of $f$ over $\mathbb{Q}$ and let $u \in K$ be a root of $f$. If $[K : \mathbb{Q}] = 27$, then show that $\mathbb{Q}[u]$ has a subfield $L$ with $[L : \mathbb{Q}] = 3$.

2. Let $F, K$ be fields such that $K$ is a finite Galois extension of $F$ with Galois group $G$. Suppose $a \in K$ is such that $\sigma(a) - a \in F$ for all $\sigma \in G$. If the characteristic of $F$ does not divide the order of $G$, then show that $a \in F$. Assuming $F$ to be the field of two elements, construct a quadratic field extension $K := F[a]$ of $F$ such that $\sigma(a) - a \in F$ for all $\sigma \in G$. 

Algebra Preliminary Exam

January 2011

Attempt all problems and justify all your answers. All rings have an identity \( 1 \neq 0 \), all ring homomorphisms send \( 1 \) to \( 1 \), and all \( R \)-modules are unitary.

I. 1. Let \( G \) be a finite simple group. Show that if \( G \) has a subgroup \( H \) with \( [G:H] = n \geq 2 \), then \( |H| |(n - 1)! \).

2. List, up to isomorphism, all groups of order 153. Justify your answer.

II. 1. Let \( R \) be a commutative ring and \( I \) an ideal of \( R \). Let \( I^* = (I, X) \) be an ideal of the polynomial ring \( R[X] \). Determine, in terms of \( I \), when \( I^* \) is a prime ideal of \( R[X] \) and when \( I^* \) is a maximal ideal of \( R[X] \). Justify your answers.

2. (a) Show that if a commutative ring \( R \) satisfies DCC on ideals (i.e., \( R \) is Artinian), then \( R \) has only a finite number of maximal ideals.

(b) Give an example to show that (a) may be false if DCC is replaced by ACC (i.e., if \( R \) is Noetherian).
III. 1. Let $f: M \to M$ be an $R$-module homomorphism with $f \cdot f = f$. Show that the following statements are equivalent.

(a) $f$ is injective.
(b) $f$ is surjective.
(c) $f = 1_M$.

2. (a) Let $G$ and $H$ be finitely generated abelian groups such that $\mathbb{Z}_n \otimes G \cong \mathbb{Z}_n \otimes H$ for every integer $n \geq 2$. Show that $G \cong H$.

(b) Give an example to show that (a) may be false if $G$ and $H$ are not both finitely generated.

IV. 1. Let $F$ be a subfield of a field $L$. Show that $L/F$ is an algebraic extension if and only if every subring $R$ of $L$ containing $F$ is a field.

2. Compute the Galois group of $f(X) = X^4 + X + 1 \in \mathbb{Z}_2[X]$. 
Attempt all four parts. Justify your answers.

Part I.

1. How many Sylow 2-subgroups does $S_5$ (the group of permutations of \{1, 2, 3, 4, 5\}) have?

2. Let $G$ be a group of order 231. Show that $G$ is Abelian if and only if $G$ has an Abelian subgroup of order 21.

Part II.

Note: Rings are assumed to be commutative and with $1 \neq 0$, subrings are assumed to contain 1 and ring-homomorphisms are assumed to map 1 to 1.

1. Let $R$ be a UFD such that each maximal ideal of $R$ is a principal ideal. Prove that $R$ is a PID.

2. Let $\mathbb{R}[[X]]$ denote the power-series ring in a single indeterminate $X$ over the field of real numbers $\mathbb{R}$. If $T$ is a multiplicative subset of $\mathbb{R}[[X]]$ containing 1 but not containing 0, then show that either $T^{-1}\mathbb{R}[[X]] = \mathbb{R}[[X]]$ or $T^{-1}\mathbb{R}[[X]]$ is a field.

Part III.

Note: Rings are assumed to be commutative and with $1 \neq 0$ and modules are assumed to be unitary.

1. Let $R$ be an integral domain and $I$ an ideal of $R$. Show that there exists a surjective $R$-module homomorphism $f : I \rightarrow R$ if and only if $I$ is a nonzero principal ideal.

2. Let $K$ be a field, $X$ an indeterminate over $K$ and $M$ a finitely generated $K[X]$-module. Show that $M$ is a projective $K[X]$-module if and only if $M$ is $K[X]$-module isomorphic to $K[X] \otimes_K V$ for some finite dimensional $K$-vector space $V$.

Part IV.

1. Let $K$ be a field and $F$ a subfield of $K$. The group of units of $K$ is denoted by $K^\times$. Suppose $f \in F[X]$ is a monic irreducible polynomial and $a, b \in K^\times$ are such that $f(a) = 0 = f(b)$. Show that the subgroup of $K^\times$ generated by $a$, is isomorphic to the subgroup of $K^\times$ generated by $b$.

2. Let $f \in \mathbb{Q}[X]$ be a polynomial of degree 4 such that the Galois group of $f$ (over $\mathbb{Q}$) is a group of order 6. Show that $f$ has a root in $\mathbb{Q}$. 
Algebra Preliminary Exam  August 2010

Attempt all problems and justify all answers. All rings have an identity 1 ≠ 0, ring homomorphisms send 1 to 1, and all R-modules are unitary.

I. 1. Let f : G → H be a surjective homomorphism of finite groups and y ∈ H with |y| = n. Show that there is an x ∈ G with |x| = n.

2. Let p and q be primes, p ≥ q, n ≥ 1, and G a group with |G| = p^n q. Show that G has a normal subgroup H of order p^n. (Hint: do the p > q and p = q cases separately.)

II. 1. Let R be a commutative ring with distinct prime ideals P and Q with P ∩ Q = {0}. Show that R is isomorphic to a subring of the direct product of two fields.

2. Let p and q be positive primes. Show that the polynomial f(X) = X^3 + pX^2 + q ∈ Z[X] is irreducible in Q[X].

III. 1. Let A and B be finite abelian groups with |A| = m and |B| = n. Show that HomZ(A, B) = 0 if and only if gcd(m, n) = 1.

2. Let A be a submodule of a projective R-module B. Show that A is projective if B/A is projective.
IV. 1. Let $K \subseteq F$ and $K \subseteq L$ be subfields of a field $M$ with
$[F:K] = p$ and $[L:K] = q$ for distinct primes $p$ and $q$.
Show that $F \cap L = K$, and that $F = K(\alpha)$ and $L = K(\beta)$
for any $\alpha \in F - K$ and $\beta \in L - K$.

2. Let $K$ be a field and $f(X) \in K[X]$ be irreducible and
separable with $\deg(f(X)) = n$. Show that if the Galois
group $G$ of $f(X)$ over $K$ is abelian, then $|G| = n$. 