The first Szegö theorem of Toeplitz operators on Bergman spaces

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Joint work with X. Zhao and D. Zheng

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Let \( T \) be the unit circle and \( L^2(T) \) be the Hilbert space of square-integrable functions.
Let $\mathbb{T}$ be the unit circle and $L^2(\mathbb{T})$ be the Hilbert space of square-integrable functions. The Hardy space $H^2$ is a closed subspace of $L^2(\mathbb{T})$ consisting of functions whose Fourier coefficients with negative subscripts are all zero.
Let $\mathbb{T}$ be the unit circle and $L^2(\mathbb{T})$ be the Hilbert space of square-integrable functions. The Hardy space $H^2$ is a closed subspace of $L^2(\mathbb{T})$ consisting of functions whose Fourier coefficients with negative subscripts are all zero. The Toeplitz operator $T_f$ with a bounded symbol $f$ on $H^2$ is defined by

$$T_f(g) = P(fg)$$

where $P$ is the orthogonal projection from $L^2(\mathbb{T})$ to $H^2$. 

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The first Szegö theorem of Toeplitz operators on Bergman spaces
Matrix representations of Hardy Toeplitz operators

Let \( f(z) = \sum_{k=-\infty}^{+\infty} c_k z^k \in L^\infty(\mathbb{T}) \).
Let \( f(z) = \sum_{k=-\infty}^{+\infty} c_k z^k \in L^\infty(\mathbb{T}) \).

\( T_f \) has a matrix representation under an orthonormal basis \( \{ z^n \} \) of \( H^2 \) as

\[
T_f = \begin{bmatrix}
c_0 & c_{-1} & c_{-2} & c_{-3} & \cdots \\
c_1 & c_0 & c_{-1} & c_{-2} & \cdots \\
c_2 & c_1 & c_0 & c_{-1} & \cdots \\
c_3 & c_2 & c_1 & c_0 & \ddots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}.
\]
The upper left corner

\[ T_n[f] = \begin{bmatrix}
  c_0 & c_{-1} & c_{-2} & \cdots & c_{-n} \\
  c_1 & c_0 & c_{-1} & \cdots & c_{-n+1} \\
  c_2 & c_1 & c_0 & \cdots & c_{-n+2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  c_n & c_{n-1} & c_{n-2} & \cdots & c_0
\end{bmatrix}. \]
Hardy Toeplitz matrices

The upper left corner

\[
\begin{bmatrix}
    c_0 & c_{-1} & c_{-2} & \cdots & c_{-n} \\
    c_1 & c_0 & c_{-1} & \cdots & c_{-n+1} \\
    c_2 & c_1 & c_0 & \cdots & c_{-n+2} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    c_n & c_{n-1} & c_{n-2} & \cdots & c_0 \\
\end{bmatrix}
\]

We call \( T_n[f] \) the Hardy Toeplitz matrix.
The upper left corner

\[ T_n[f] = \begin{bmatrix} c_0 & c_{-1} & c_{-2} & \cdots & c_{-n} \\ c_1 & c_0 & c_{-1} & \cdots & c_{-n+1} \\ c_2 & c_1 & c_0 & \cdots & c_{-n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_0 \end{bmatrix}. \]

We call \( T_n[f] \) the Hardy Toeplitz matrix.
Denote \( \mathbb{D}_n[f] = \det T_n[f] \).
First Szegö theorem of Hardy Toeplitz operators

The asymptotic behavior of $D_n[f]$ was first described by Szegö in 1919, as the first Szegö theorem.
First Szegö theorem of Hardy Toeplitz operators

The asymptotic behavior of $\mathbb{D}_n[f]$ was first described by Szegö in 1919, as the first Szegö theorem. One version given by Widom (translated to scalar-valued case):

**Theorem (Szegö-Widom, 1976)**

If $f \in H^\infty + C(\mathbb{T})$ and $T_f$ is invertible on $H^2$, then

$$\lim_{n \to +\infty} \frac{\mathbb{D}_n[f]}{\mathbb{D}_{n-1}[f]} = G[f].$$
The asymptotic behavior of $\mathbb{D}_n[f]$ was first described by Szegö in 1919, as the first Szegö theorem. One version given by Widom (translated to scalar-valued case):

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$$\lim_{n \to +\infty} \frac{\mathbb{D}_n[f]}{\mathbb{D}_{n-1}[f]} = G[f].$$

Here, $G[f]$ is the geometric mean of $f$ as

$$G[f] = \lim_{r \to 1} \exp \left[ \frac{1}{2\pi} \int_{0}^{2\pi} \ln \hat{f}(r e^{i\theta}) d\theta \right],$$

where $\hat{f}$ is the harmonic extension of $f$. 
Let $\mathbb{D}$ be the unit disk and $dA$ be the normalized Lebesgue measure on $\mathbb{D}$. Let $L^2(\mathbb{D}, dA)$ be the Hilbert space of square-integrable functions on $\mathbb{D}$. 
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Let $\mathbb{D}$ be the unit disk and $dA$ be the normalized Lebesgue measure on $\mathbb{D}$. Let $L^2(\mathbb{D}, dA)$ be the Hilbert space of square-integrable functions on $\mathbb{D}$. The Bergman space $L^2_a$ is a closed subspace of $L^2(\mathbb{D}, dA)$ consisting of analytic functions on $\mathbb{D}$. The Toeplitz operator $T_f$ with a bounded symbol $f$ on $L^2_a$ is defined by

$$T_f(g) = P(fg)$$

where $P$ is the orthogonal projection from $L^2(\mathbb{D}, dA)$ to $L^2_a$. 
Matrix representations of Bergman Toeplitz operators with harmonic symbols

Let \( f(z) = \sum_{k=-\infty}^{+\infty} c_k z^k \in L^\infty(\mathbb{T}) \).
Matrix representations of Bergman Toeplitz operators with harmonic symbols

Let $f(z) = \sum_{k=-\infty}^{+\infty} c_k z^k \in L^\infty(\mathbb{T})$. Then $\hat{f} = \sum_{k=0}^{+\infty} c_k z^k + \sum_{k=1}^{+\infty} c_k \bar{z}^k$. 
Matrix representations of Bergman Toeplitz operators with harmonic symbols

Let \( f(z) = \sum_{k=-\infty}^{+\infty} c_k z^k \in L^\infty(\mathbb{T}) \). Then \( \hat{f} = \sum_{k=0}^{+\infty} c_k z^k + \sum_{k=1}^{+\infty} c_k \bar{z}^k \).

\( T_{\hat{f}} \) has a matrix representation under an orthonormal basis \( \{ e_n = \sqrt{n+1} z^n \}_{n=0}^{+\infty} \) of \( L^2_a \),

\[
T_{\hat{f}} = \begin{bmatrix}
c_0 & \sqrt{\frac{1}{2}} c_{-1} & \sqrt{\frac{1}{3}} c_{-2} & \sqrt{\frac{1}{4}} c_{-3} & \cdots \\
\sqrt{\frac{1}{2}} c_1 & c_0 & \sqrt{\frac{2}{3}} c_{-1} & \sqrt{\frac{2}{4}} c_{-2} & \cdots \\
\sqrt{\frac{1}{3}} c_2 & \sqrt{\frac{2}{3}} c_{-1} & c_0 & \sqrt{\frac{3}{4}} c_{-1} & \cdots \\
\sqrt{\frac{1}{4}} c_3 & \sqrt{\frac{2}{4}} c_{-2} & \sqrt{\frac{3}{4}} c_{-1} & c_0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]
Bergman Toeplitz matrices

The upper left corner

\[ T_n[\hat{f}] = \begin{bmatrix}
    c_0 & \sqrt{\frac{1}{2}} c_{-1} & \sqrt{\frac{1}{3}} c_{-2} & \cdots & \sqrt{\frac{1}{n+1}} c_{-n} \\
    \sqrt{\frac{1}{2}} c_1 & c_0 & \sqrt{\frac{2}{3}} c_{-1} & \cdots & \sqrt{\frac{2}{n+1}} c_{-n+1} \\
    \sqrt{\frac{1}{3}} c_2 & \sqrt{\frac{2}{3}} c_1 & c_0 & \cdots & \sqrt{\frac{3}{n+1}} c_{-n+2} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    \sqrt{\frac{1}{n+1}} c_n & \sqrt{\frac{2}{n+1}} c_{n-1} & \sqrt{\frac{3}{n+1}} c_{n-2} & \cdots & c_0
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  \sqrt{\frac{1}{n+1}} c_n & \sqrt{\frac{2}{n+1}} c_{n-1} & \sqrt{\frac{3}{n+1}} c_{n-2} & \cdots & c_0
\end{bmatrix}.
\]

We call \(T_n[\hat{f}]\) the Bergman Toeplitz matrix.
First Szegö theorem of Hardy Toeplitz operators
Asymptotic invertibility of Hardy and Bergman Toeplitz operators
Asymptotic invertibility of Bergman Toeplitz operators with $H^\infty (\mathbb{D})$
First Szegö theorem of Bergman Toeplitz operators

Bergman Toeplitz matrices

The upper left corner

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  \sqrt{\frac{1}{2}} \, c_1 & c_0 & \sqrt{\frac{2}{3}} \, c_{-1} & \cdots & \sqrt{\frac{2}{n+1}} \, c_{-n+1} \\
  \sqrt{\frac{1}{3}} \, c_2 & \sqrt{\frac{2}{3}} \, c_1 & c_0 & \cdots & \sqrt{\frac{3}{n+1}} \, c_{-n+2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  \sqrt{\frac{1}{n+1}} \, c_n & \sqrt{\frac{2}{n+1}} \, c_{n-1} & \sqrt{\frac{3}{n+1}} \, c_{n-2} & \cdots & c_0
\end{bmatrix}. $$

We call $T_n[\hat{f}]$ the Bergman Toeplitz matrix. Let $D_n[f] = \det T_n[\hat{f}]$. 

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The first Szegö theorem of Toeplitz operators on Bergman spaces
First Szegö theorem of Hardy Toeplitz operators
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\end{bmatrix}.$$ 

We call $T_n[\hat{f}]$ the Bergman Toeplitz matrix.

Let $D_n[f] = \det T_n[\hat{f}]$.

**Q:** What’s the asymptotic behavior of $D_n[f]$?
Asymptotic invertibility

Let $P_n$ be the orthogonal projection from $H^2$ to $\mathbb{L}_n$, where
$\mathbb{L}_n = \overline{\{1, z, \ldots, z^n\}^{\text{span}}} \subset H^2$. 

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The first Szegö theorem of Toeplitz operators on Bergman spaces
Asymptotic invertibility

Let $\mathbb{P}_n$ be the orthogonal projection from $H^2$ to $\mathbb{L}_n$, where

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$T_n[f]$ can be identified as the compression $\mathbb{P}_n T_f \mathbb{P}_n$. 

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Asymptotic invertibility

Let $P_n$ be the orthogonal projection from $H^2$ to $L_n$, where
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$T_n[f]$ can be identified as the compression $P_n T f P_n$.

Let $P_n$ be the orthogonal projection from $L^2_a$ to $L_n$, where
\[ L_n = \overline{\{1, z, \ldots, z^n\}}^{\text{span}} \subset L^2_a. \]
Asymptotic invertibility

Let $P_n$ be the orthogonal projection from $H^2$ to $\mathbb{L}_n$, where $\mathbb{L}_n = \text{span} \{1, z, \ldots, z^n\} \subset H^2$. $\mathbb{T}_n[f]$ can be identified as the compression $P_n T_f P_n$.

Let $P_n$ be the orthogonal projection from $L^2_a$ to $L_n$, where $L_n = \text{span} \{1, z, \ldots, z^n\} \subset L^2_a$. $T_n[\hat{f}]$ can be identified as the compression $P_n T_\hat{f} P_n$. 
Asymptotic invertibility

Let $\mathbb{P}_n$ be the orthogonal projection from $H^2$ to $\mathbb{L}_n$, where $\mathbb{L}_n = \text{span}\{1, z, \ldots, z^n\} \subset H^2$. $\mathcal{T}_n[f]$ can be identified as the compression $\mathbb{P}_n \mathcal{T}_f \mathbb{P}_n$.

Let $P_n$ be the orthogonal projection from $L^2_a$ to $L_n$, where $L_n = \text{span}\{1, z, \ldots, z^n\} \subset L^2_a$. $T_n[\hat{f}]$ can be identified as the compression $P_n T \hat{f} P_n$.

**Definition (Asymptotic invertibility)**

Given a bounded invertible operators $A$ on $H^2$ (or $L^2_a$), $A$ is said to be **asymptotically invertible** if for sufficient large $n$, the compressions $\mathbb{P}_n AP_n$ (or $P_n AP_n$) are invertible and

$$(\mathbb{P}_n AP_n)^{-1} \mathbb{P}_n \rightarrow A^{-1} \quad (\text{or} \quad (P_n AP_n)^{-1} P_n \rightarrow A^{-1})$$

in the strong operator topology.
In 1976, Widom gave an identity between Hardy Toeplitz matrices:

\[ T_n[fg] = T_n[f]T_n[g] + P_n H_f^* H_g P_n + W_n J H_f H_g^* J W_n. \quad (1) \]
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Here, $J : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is by $J(f) = \bar{z} f(\bar{z})$, where $\bar{z}$ denotes the complex conjugate of $z$. 
In 1976, Widom gave an identity between Hardy Toeplitz matrices:

\[ T_n[fg] = T_n[f]T_n[g] + P_n H_f^* H_g P_n + W_n J H_f H_g^* J W_n. \]  \hspace{1cm} (1)

Here, \( J : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T}) \) is by \( J(f) = \overline{zf(\overline{z})} \), and \( W_n : H^2 \rightarrow L_n \) is by \( W_n(f) = z^n \overline{\mathbb{P}_n(f)} \), where \( \tilde{f}(z) = f(\overline{z}) \).
Asymptotic invertibility of Hardy Toeplitz operators

Using the identity, Widom gave an inversion formula of Hardy Toeplitz matrices:

**Theorem (Widom, 1976)**

If \( f \in H^\infty + C(T) \) and \( T_f \) is invertible on \( H^2 \), then

\[
T_n[f]^{-1} = P_n T_f^{-1} P_n + W_n (T_{\bar{f}}^{-1} - T_{\bar{f}-1}) W_n + C_n,
\]

for sufficient large \( n \), where \( \|C_n\| \to 0 \) as \( n \to +\infty \).
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Using the identity, Widom gave an inversion formula of Hardy Toeplitz matrices:

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for sufficient large \( n \), where \( \|C_n\| \to 0 \) as \( n \to +\infty \).

From this, a criterion of the asymptotic invertibility:

\[ \text{Theorem (Widom, 1976)} \]
\[ \text{If } f \in H^\infty + C(\mathbb{T}) \text{ and } T_f \text{ is invertible on } H^2, \text{ then} \]
\[ T_f \text{ is asymptotically invertible if and only if } T_f \text{ is invertible on } H^2. \]
In 1990, Böttcher studied Bergman Toeplitz operators with symbols in $C_{N \times N}(\mathbb{D})$. 
Asymptotic identity of Bergman Toeplitz matrices

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An asymptotic identity of Bergman Toeplitz matrices:

**Theorem (Böttcher, 1990)**

If $f, g \in C_{N \times N}(\mathbb{D})$, then

$$T_n[fg] = T_n[f]T_n[g] + P_nH_f^*H_gP_n + UW_nJH_f^*H_g^*JW_nU^* + C_n,$$

where $\|C_n\| \to 0$ as $n \to +\infty$. 
In 1990, Böttcher studied Bergman Toeplitz operators with symbols in $C_{N \times N} \left( \overline{D} \right)$. An asymptotic identity of Bergman Toeplitz matrices:

**Theorem (Böttcher, 1990)**

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where $\| C_n \| \to 0$ as $n \to +\infty$.

Here, $U$ is an unitary operator from $H^2$ to $L^2_a$ by $U(z^m) = e_m$ where $e_m = \sqrt{m+1} z^m$. 

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Here, $U$ is an unitary operator from $H^2$ to $L^2_a$ by $U(z^m) = e_m$ where $e_m = \sqrt{m+1} z^m$, and $f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$. 

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The first Szegö theorem of Toeplitz operators on Bergman spaces
Also a criterion of the asymptotic invertibility:

**Theorem (Böttcher, 1990)**

If \( f \in C_{N \times N}(\overline{\mathbb{D}}) \), then \( T_f \) is asymptotically invertible if and only if \( T_f \) is invertible on \( (L^2_\alpha)^N \) and \( \tilde{T}_{f^*} \) is invertible on \( H^2_N \).
Criterion of asymptotic invertibility and Inversion formula

Inspired by these work, we consider Bergman Toeplitz operators with symbols in $H^\infty(\mathbb{D}) + C(\mathbb{D})$. 

Theorem (1)

Let $f \in H^\infty(\mathbb{D}) + C(\mathbb{D})$. Then $T_f$ is asymptotically invertible if and only if $T_f$ is invertible on $L^2_a$. Moreover, if $T_f$ is invertible, then for sufficiently large $n$ one has

$$T_n[f]^{-1} = P_n T^{-1} f P_n + U W_n (T^{-1} \tilde{f}^* - T{\tilde{f}^*}^{-1}) W_n^* + C_n,$$

where $||C_n|| \to 0$ as $n \to +\infty$. 

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A criterion of asymptotic invertibility and an asymptotic inversion formula:

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$$T_n[f]^{-1} = P_n T_f^{-1} P_n + U W_n (T_{f^*}^{-1} - T_{f^*}^{-1}) W_n U^* + C_n,$$

where $\|C_n\| \to 0$ as $n \to +\infty$. 

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First Szegő theorem of Hardy Toeplitz operators
Asymptotic invertibility of Hardy and Bergman Toeplitz operators
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Criterion of asymptotic invertibility and Inversion formula

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A criterion of asymptotic invertibility and An asymptotic inversion formula:

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where $\|C_n\| \to 0$ as $n \to +\infty$. 

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The first Szegő theorem of Toeplitz operators on Bergman spaces
Idea of the proof of asymptotic invertibility

Some lemmas:

Lemma

If \( f, g \in H^\infty(\mathbb{D}) + C(\mathbb{D}) \), then

\[
T_n[fg] = T_n[f]T_n[g] + P_nH_f^*H_gP_n + UW_nJH_f^*H_g^*JW_nU^* + C_n, \tag{2}
\]

where \( \|C_n\| \to 0 \) as \( n \to +\infty \).
Idea of the proof of asymptotic invertibility

Some lemmas:

**Lemma**

*If* $f, g \in H^\infty(\mathbb{D}) + C(\overline{\mathbb{D}})$, *then*

$$T_n[fg] = T_n[f]T_n[g] + P_nH_f^*H_gP_n + UW_nJH_f^*H_g^*JW_nU^* + C_n, \quad (2)$$

*where* $\|C_n\| \to 0$ *as* $n \to +\infty$.

**Lemma**

*Let* $f \in H^\infty(\mathbb{D}) + C(\overline{\mathbb{D}})$. *If* $T_f$ *is invertible on* $L^2_a$, *then both* $T_f^*$ *and* $T_{\tilde{f}}^*$ *are invertible on* $H^2$. 
Idea of the proof of asymptotic invertibility

Some lemmas:

**Lemma**

If \( f, g \in H^\infty(D) + C(D) \), then

\[
T_n[fg] = T_n[f]T_n[g] + P_nH^*_fH_gP_n + UW_nJH^*_fH^*_gJW_nU^* + C_n, \tag{2}
\]

where \( \|C_n\| \to 0 \) as \( n \to +\infty \).

**Lemma**

Let \( f \in H^\infty(D) + C(D) \). If \( T_f \) is invertible on \( L^2_a \), then both \( T_{f^*} \) and \( T_{\bar{f}^*} \) are invertible on \( H^2 \).

Silbermann’s theorem: A criterion of asymptotic invertibility for general bounded linear operators on \( H^2 \).
The first Szegö theorem for Bergman case:

**Theorem (2)**

If \( f \in H^\infty(\mathbb{D}) + C(\overline{\mathbb{D}}) \) and \( T_f \) is invertible, then

\[
\lim_{n \to +\infty} \frac{D_n[f]}{D_{n-1}[f]} = G[f^*].
\]
The first Szegő theorem for Bergman case:

**Theorem (2)**

If \( f \in H^\infty(D) + C(D) \) and \( T_f \) is invertible, then

\[
\lim_{n \to +\infty} \frac{D_n[f]}{D_{n-1}[f]} = G[f^*].
\]

Here, \( G[f^*] \) is the geometric mean of \( f^* \) as

\[
G[f] = \lim_{r \to 1} \exp \left[ \frac{1}{2\pi} \int_0^{2\pi} \ln \hat{f}^*(re^{i\theta}) d\theta \right].
\]
Sketch of proof of first Szegö theorem of Bergman case

Let $M = [m_{ij}]_{i,j=1}^n$ be an invertible $n$ by $n$ matrix.
Sketch of proof of first Szegö theorem of Bergman case

Let $M = [m_{ij}]_{i,j=1}^n$ be an invertible $n$ by $n$ matrix. Then

$$M^{-1} = \frac{\text{adj } M}{\det M}.$$
Sketch of proof of first Szegö theorem of Bergman case

Let $M = [m_{ij}]_{i,j=1}^n$ be an invertible $n$ by $n$ matrix. Then

$$M^{-1} = \frac{\text{adj } M}{\det M}.$$ 

Here, the adjugate matrix $\text{adj } M = [M_{ij}]'$, where $M_{ij}$ is the cofactor of $m_{ij}$. 
Sketch of proof of first Szegö theorem of Bergman case

Let $M = [m_{ij}]_{i,j=1}^n$ be an invertible $n$ by $n$ matrix. Then

$$M^{-1} = \frac{adj\, M}{\det M}.$$ 

Here, the adjugate matrix $adj\, M = [M_{ij}]'$, where $M_{ij}$ is the cofactor of $m_{ij}$.

Apply to invertible Bergman Toeplitz matrices to get

$$T_n[f]^{-1} = \frac{adj\, T_n[f]}{D_n[f]}.$$
Sketch of proof of first Szegö theorem of Bergman case

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$$T_n[f]^{-1} = \frac{\text{adj} T_n[f]}{D_n[f]}.$$ 

Then the entry in the lower right corner is

$$< T_n[f]^{-1} e_n, e_n > = \frac{D_{n-1}[f]}{D_n[f]}.$$
Sketch of proof of first Szegö theorem of Bergman case

Apply asymptotic inversion formula to the above formula to get

\[
\lim_{n \to +\infty} \frac{D_{n-1}[f]}{D_n[f]} = \lim_{n \to +\infty} \left\langle [P_n T_f^{-1} P_n + U W_n (T_{\tilde{f}^*}^{-1} - T_{\tilde{f}^*^{-1}}) W_n U^* + C_n] e_n, e_n \right\rangle.
\]
Sketch of proof of first Szegő theorem of Bergman case

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Similar argument gives the following identity of Hardy case

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\lim_{n \to +\infty} \frac{D_{n-1}[f^*]}{D_n[f^*]} = \lim_{n \to +\infty} \langle [P_n T_{f^*}^{-1} P_n + W_n (T_{f_*}^{-1} - T_{f_*^{-1}}) W_n + C_n] e_n, e_n \rangle.
\]

Estimate the above identities and use the first Szegő theorem for Hardy case.
The first Szegő theorem for Bergman case:

**Theorem (2)**

If \( f \in H^\infty(D) + C(\overline{D}) \) and \( T_f \) is invertible, then

\[
\lim_{n \to +\infty} \frac{D_n[f]}{D_{n-1}[f]} = G[f^*].
\]

Here, \( G[f^*] \) is the geometric mean of \( f^* \) as

\[
G[f] = \lim_{r \to 1} \exp \left[ \frac{1}{2\pi} \int_0^{2\pi} \ln \hat{f}^*(re^{i\theta}) d\theta \right].
\]
Thank You!