# The first Szegö theorem of Toeplitz operators on Bergman spaces

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The Hardy space $H^2$ is a closed subspace of $L^2(\mathbb{T})$ consisting of functions whose Fourier coefficients with negative subscripts are all zero.
The Toeplitz operator $\mathbb{T}_f$ with a bounded symbol $f$ on $H^2$ is defined by

$$\mathbb{T}_f(g) = \mathbb{P}(fg)$$

where $\mathbb{P}$ is the orthogonal projection from $L^2(\mathbb{T})$ to $H^2$.

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# Hardy Toeplitz matrices

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$$\mathbb{T}_n[f] = \begin{bmatrix} c_0 & c_{-1} & c_{-2} & \cdots & c_{-n} \\ c_1 & c_0 & c_{-1} & \cdots & c_{-n+1} \\ c_2 & c_1 & c_0 & \cdots & c_{-n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_0 \end{bmatrix}.$$

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Denote $\mathbb{D}_n[f] = \det \, \mathbb{T}_n[f]$.

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**Theorem (Szegö-Widom, 1976)**

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where $\hat{f}$ is the harmonic extension of $f$.

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where $P$ is the orthogonal projection from $L^2(\mathbb{D}, dA)$ to $L^2_a$.

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Let $D_n[f] = \det T_n[\hat{f}]$.
**Q:** What's the asymptotic behavior of $D_n[f]$?

# Asymptotic invertibility

Let $\mathbb{P}_n$ be the orthogonal projection from $H^2$ to $\mathbb{L}_n$, where $\mathbb{L}_n = \overline{\{1, z, \ldots, z^n\}}^{span} \subset H^2$.

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Let $P_n$ be the orthogonal projection from $L^2_a$ to $L_n$, where $L_n = \overline{\{1, z, \ldots, z^n\}}^{span} \subset L^2_a$.

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Let $P_n$ be the orthogonal projection from $L_a^2$ to $L_n$, where $L_n = \overline{\{1, z, \ldots, z^n\}}^{span} \subset L_a^2$.
$T_n[\hat{f}]$ can be identified as the compression $P_n T_{\hat{f}} P_n$.

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## Definition (Asymptotic invertibility)

*Given a bounded invertible operators A on $H^2$ (or $L_a^2$), A is said to be* ***asymptotically invertible*** *if for sufficient large n, the compressions $\mathbb{P}_n A \mathbb{P}_n$ (or $P_n A P_n$) are invertible and*

$$(\mathbb{P}_n A \mathbb{P}_n)^{-1} \mathbb{P}_n \to A^{-1} \quad (or\, (P_n A P_n)^{-1} P_n \to A^{-1})$$

*in the strong operator topology.*

# Identity of Hardy Toeplitz matrices

In 1976, Widom gave an identity between Hardy Toeplitz matrices:

$$\mathbb{T}_n[fg] = \mathbb{T}_n[f]\mathbb{T}_n[g] + \mathbb{P}_n\mathbb{H}_f^*\mathbb{H}_g\mathbb{P}_n + W_nJ\mathbb{H}_f\mathbb{H}_{\bar{g}}^*JW_n. \qquad (1)$$

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Here, $J : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is by $J(f) = \bar{z}f(\bar{z})$,
and $W_n : H^2 \to \mathbb{L}_n$ is by $W_n(f) = z^n\overbrace{\mathbb{P}_n(f)}$, where $\tilde{f}(z) = f(\bar{z})$.

# Asymptotic invertibility of Hardy Toeplitz operators

Using the identity, Widom gave an inversion formula of Hardy Toeplitz matrices:

**Theorem (Widom, 1976)**

*If $f \in H^\infty + C(\mathbb{T})$ and $\mathbb{T}_f$ is invertible on $H^2$, then*

$$\mathbb{T}_n[f]^{-1} = \mathbb{P}_n \mathbb{T}_f^{-1} \mathbb{P}_n + W_n(\mathbb{T}_{\tilde{f}}^{-1} - \mathbb{T}_{\tilde{f}^{-1}})W_n + C_n,$$

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From this, a criterion of the asymptotic invertibility:

**Theorem (Widom, 1976)**

*If $f \in H^\infty + C(\mathbb{T})$, then $\mathbb{T}_f$ is asymptotically invertible if and only if $\mathbb{T}_f$ is invertible on $H^2$.*

# Asymptotic identity of Bergman Toeplitz matrices

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# Criterion of asymptotic invertibility and Inversion formula

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A criterion of asymptotic invertibility and An asymptotic inversion formula:

**Theorem (1)**

*Let $f \in H^{\infty}(\mathbb{D}) + C(\overline{\mathbb{D}})$. Then $T_f$ is asymptotically invertible if and only if $T_f$ is invertible on $L_a^2$.*
*Moreover, if $T_f$ is invertible, then for sufficiently large $n$ one has*

$$T_n[f]^{-1} = P_n T_f^{-1} P_n + U W_n (\mathbb{T}_{\tilde{f^*}}^{-1} - \mathbb{T}_{\tilde{f^*}^{-1}}) W_n U^* + C_n,$$

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# Idea of the proof of asymptotic invertibility

Some lemmas:

**Lemma**

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**Lemma**

*Let $f \in H^\infty(\mathbb{D}) + C(\overline{\mathbb{D}})$. If $T_f$ is invertible on $L_a^2$, then both $\mathbb{T}_{f^*}$ and $\mathbb{T}_{\tilde{f}^*}$ are invertible on $H^2$.*

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Silbermann's theorem: A criterion of asymptotic invertibility for general bounded linear operators on $H^2$.

# First Szegö theorem of Bergman Toeplitz operaotrs

The first Szegö theorem for Bergman case:

**Theorem (2)**

*If $f \in H^\infty(\mathbb{D}) + C(\overline{\mathbb{D}})$ and $T_f$ is invertible, then*

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Then the entry in the lower right corner is

$$< T_n[f]^{-1} e_n, e_n >= \frac{D_{n-1}[f]}{D_n[f]}.$$

# Sketch of proof of first Szegö theorem of Bergman case

Apply asymptotic inversion formula to the above formula to get

$$\lim_{n\to+\infty}\frac{D_{n-1}[f]}{D_n[f]}$$
$$=\lim_{n\to+\infty}\langle[P_nT_f^{-1}P_n+UW_n(\mathbb{T}_{\tilde{f}^*}^{-1}-\mathbb{T}_{\tilde{f}^{*-1}})W_nU^*+C_n]e_n,e_n\rangle.$$

# Sketch of proof of first Szegö theorem of Bergman case

Apply asymptotic inversion formula to the above formula to get

$$\lim_{n\to+\infty} \frac{D_{n-1}[f]}{D_n[f]} = \lim_{n\to+\infty} \langle [P_n T_f^{-1} P_n + U W_n (\mathbb{T}_{\tilde{f^*}}^{-1} - \mathbb{T}_{\tilde{f^*}^{-1}}) W_n U^* + C_n] e_n, e_n \rangle.$$

Similar argument gives the following identity of Hardy case

$$\lim_{n\to+\infty} \frac{\mathbb{D}_{n-1}[f^*]}{\mathbb{D}_n[f^*]} = \lim_{n\to+\infty} \langle [\mathbb{P}_n \mathbb{T}_{f^*}^{-1} \mathbb{P}_n + W_n (\mathbb{T}_{\tilde{f^*}}^{-1} - \mathbb{T}_{\tilde{f^*}^{-1}}) W_n + C_n] e_n, e_n \rangle.$$

# Sketch of proof of first Szegö theorem of Bergman case

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Estimate the above identities and use the first Szegö theorem for Hardy case.

# First Szegö theorem of Bergman Toeplitz operaotrs

The first Szegö theorem for Bergman case:

**Theorem (2)**

*If $f \in H^\infty(\mathbb{D}) + C(\overline{\mathbb{D}})$ and $T_f$ is invertible, then*

$$\lim_{n \to +\infty} \frac{D_n[f]}{D_{n-1}[f]} = G[f^*].$$

Here, $G[f^*]$ is the geometric mean of $f^*$ as

$$G[f] = \lim_{r \to 1} \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \ln \widehat{f^*}(re^{i\theta}) d\theta\right].$$

# Thank You!