Approximation of Invariant Subspaces by Finite Co-dimensional Ones

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Overview

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Facts:

1. $\mathcal{D}_\alpha$ spaces are RKHP, i.e. $\exists k_w^\alpha \in \mathcal{D}_\alpha$ such that $f(w) = \langle f, k_w^\alpha \rangle$ for all $f \in \mathcal{D}_\alpha$.

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$$\mathcal{M} = I(Z) = \{f \in \mathcal{D}_\alpha : f = 0 \text{ on } Z\},$$

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Question: Can all invariant subspaces be approximated by such invariant subspaces?

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  Example: If Z_n is an increasing sequence of zero sets, then one can apply this and get $I(Z_n) \to I(\bigcup_n Z_n)$.

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$\Longrightarrow$ Hence, if an invariant subspace $\mathcal{M}$ can be approximated by $I(Z_n)$ for some Z_n, then $\mathcal{M}$ can be approximated by finite co-dimensional invariant subspaces.

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$\implies$ (1) and (2) can be put together to show that every invariant subspace is a SOT limit of finite zero set ones.

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Thm(S.M. Shimorin): If $\alpha < 0$, then every index 1 invariant subspace can be approximated by finite co-dimensional ones.

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- For the nontriviality of $\mathcal{M}_E$ it is necessary and sufficient that E is a Carleson thin set.
- It is known that a closed subset E of the unit circle $\mathbb{T}$ is called Carleson thin set if $|E| = 0$ and $\sum_n |I_n| \log(\frac{1}{|I_n|}) < \infty$, where $\mathbb{T} \setminus E = \bigcup_n I_n$, I_n are open disjoint arcs.

Main Result

Theorem 1

Let $\mathcal{M} \in Lat(M_z, \mathcal{D}_2)$ be a nontrivial (closed) subspace. Then there exists a sequence $\mathcal{M}_n \in Lat(M_z, \mathcal{D}_2)$, with $\dim \mathcal{M}_n^\perp < \infty$, such that $P_n \to P$ in the strong operator topology (SOT), where P_n and P are orthogonal projections onto $\mathcal{M}_n$ and $\mathcal{M}$, respectively.

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Idea: By Korenblum's thm, $\mathcal{M} = U\mathcal{M}_E$. If $U = B$ is Blaschke factor, then set $\mathcal{M}_n = B_n \mathcal{M}_{E_n}$, B_n partial Blaschke products, and $E_n \subseteq E$ finite set.

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$\implies \mathcal{M}_n$ has finite co-dimension and since it is decreasing, it will work.
Note: If $\alpha = 1$, then there is no analogue of $\mathcal{M}_{E_n}$ for E_n finite.

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Idea: By Korenblum's thm, $\mathcal{M} = U\mathcal{M}_E$. If $U = B$ is Blaschke factor, then set $\mathcal{M}_n = B_n\mathcal{M}_{E_n}$, B_n partial Blaschke products, and $E_n \subseteq E$ finite set. $\implies \mathcal{M}_n$ has finite co-dimension and since it is decreasing, it will work.

Note: If $\alpha = 1$, then there is no analogue of $\mathcal{M}_{E_n}$ for E_n finite.

That raises the question of whether one can do the $\mathcal{D}_2$ case without use of $E_n \subseteq \mathbb{T}$ and approximate everything (or just $\mathcal{M}_E$) by $I(Z)$, $Z \subseteq \mathbb{D}$ finite.

Main Result

Theorem 2

Let $\emptyset \neq E \subset \mathbb{T}$ *be a Carleson thin set and* $\mathcal{M}_E = \{f \in \mathcal{D}_2 : f(z) = 0,\ \forall z \in E\}$ *be the nontrivial invariant subspace of* $(M_z, \mathcal{D}_2)$. *Then* $\exists Z_k \subseteq \mathbb{D}$ *decreasing such that* $\mathcal{M}_E = span_k I(Z_k)$.

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Note: Here each Z_k contains infinitely many points and by the decreasing property

$$I(Z_1) \subset I(Z_2) \subset ... \subset I(Z_k) \subset ... \subset \mathcal{M}_E.$$

Dirichlet space

Consider $D_E = \{f \in \mathbf{D} : f^* = 0 \;\; q.e. \;\; on \;\; E\}$, $E \subseteq \mathbb{T}$.

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- $D_E = \mathbf{D}$ if E has zero logarithmic capacity, and $D_E = (0)$ if E has positive measure.
- $D_E \in Lat(M_z, \mathbf{D})$.
- Question: If $E \subseteq \mathbb{T}$ is a Carleson thin set, then can we find $Z_k \subseteq \mathbb{D}$ such that $D_E = span_k I(Z_k)$?

Dirichlet space

Theorem 3

Let $E \subset \mathbb{T}$ be a Carleson thin set with positive logarithmic capacity and let f_E be the corresponding outer function constructed by Korenblum that is zero on E, and smooth elsewhere. Then there exists a decreasing sequence of zero sets $Z_k \subseteq \mathbb{D}$ such that

$$D_E \supseteq span_k I(Z_k) \supseteq [f_E].$$

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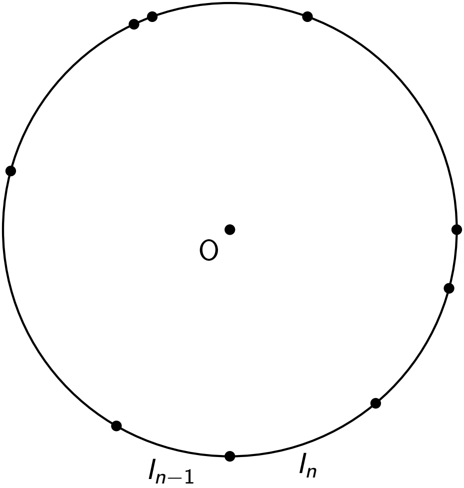
Lemma 4

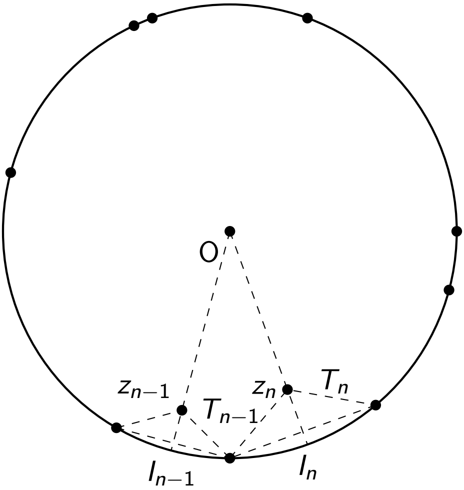
Let $E \subset \mathbb{T}$ be a compact set with Lebesgue measure $|E| = 0$. Then there exists $\mathcal{A} \subset \mathbb{D}$ a Blaschke sequence such that every point of E is a non-tangential limit point from $\mathcal{A}$.

Dirichlet space

Idea



Dirichlet space



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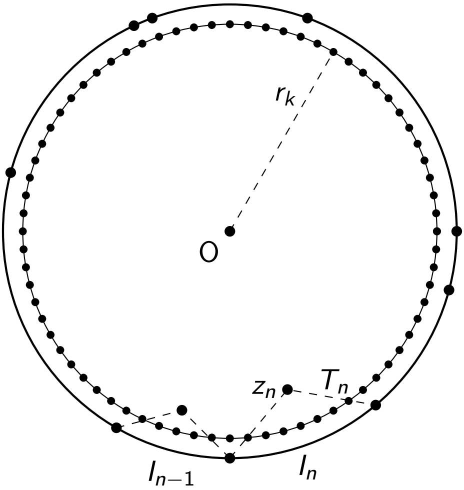

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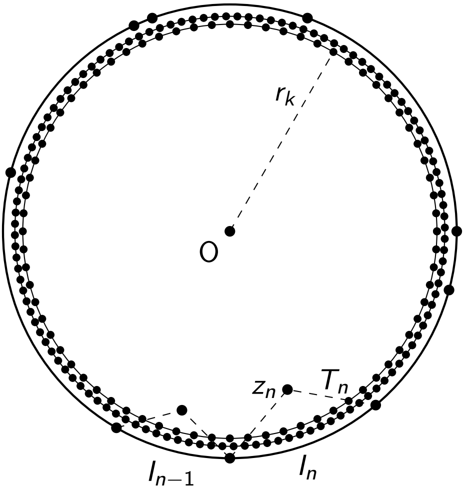

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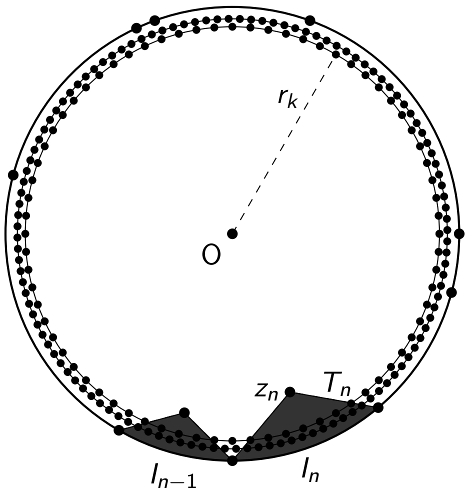

References

L. Brown and Allen L. Shields (1984)
Cyclic vectors in the Dirichlet space
Trans. Amer. Math. Soc. 285(1), 269–303.

B.I. Korenblum (1972)
Invariant subspaces of the shift operator in a weighted Hilbert space
Mat. Sb. (N.S.) 89(131), 110 – 137.

S.M. Shimorin (2000)
Approximate spectral synthesis in the Bergman space
Duke Math. J. 101(1), 1–39.

Thank You