Finite Rank Isopairs

Udeni D Wijesooriya

University of Florida

Let $\mathbb{D}$ be the open unit disk, $\mathbb{T}$ be the unit circle and $\mathbb{E}$ be the exterior of the closed unit disk.

Definition

A polynomial $\mathfrak{p} = \mathfrak{p}(z, w) \in \mathbb{C}[z, w]$ is *inner toral* if the zero set of $\mathfrak{p}$,

$$Z(\mathfrak{p}) \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2.$$

Examples:

- $\mathfrak{p}_1(z, w) = z^3 - w^2$
- $\mathfrak{p}_2(z, w) = z^2 - w^2$

Definition

Given an inner toral polynomial $\mathfrak{p}$, the set

$$\mathfrak{V}(\mathfrak{p}) = Z(\mathfrak{p}) \cap \mathbb{D}^2$$

is called a *distinguished variety*.

Definition

A pair $V = (S, T)$ of commuting pure isometries on Hilbert space $\mathcal{K}$ is called a *pure algebraic isopair* if

$$q(S, T) = 0$$

for some polynomial $q \in \mathbb{C}[z, w]$.

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Remark: There exists a **square free inner toral polynomial** $\mathfrak{p}$ such that $\mathfrak{p}(S, T) = 0$ and $\mathfrak{p}$ is minimal in the sense that $\mathfrak{p}|q$ for all q with $q(S, T) = 0$.

- Fix a square free inner toral polynomial $\mathfrak{p}$.
- Consider pure algebraic isopairs with minimal polynomial $\mathfrak{p}$.
- We call them pure $\mathfrak{p}$-isopairs.
- Write $\mathfrak{p} = \mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_s$.
- Let (n, m) and (n_j, m_j) be the bidegree of $\mathfrak{p}$ and $\mathfrak{p}_j$ respectively.

Definition

A pure $\mathfrak{p}$-isopair $V = (S, T)$ acting on the Hilbert space $\mathcal{K}$ is **nearly k-cyclic** if there exist $f_1, ..., f_k \in \mathcal{K}$ such that the closure of

$$\vee \{q_1(S, T)f_1, \ . \ . \ . \ , \ q_k(S, T)f_k : q_j \in \mathbb{C}[z, w]\}$$

is of finite codimension in $\mathcal{K}$, and no set of $k - 1$ members in $\mathcal{K}$ has this property.

Example:

- Let $\mathfrak{p} = z^3 - w^2$.
- Let $\mathcal{H}$ be the Hilbert space with orthonormal basis

$$\left\{z, z^2, z^3, ..., w, zw, z^2 w, ...\right\} \text{ on } \mathfrak{V}(\mathfrak{p}).$$

- Let $S = M_z$ and $T = M_w$.
- $\overline{\{q(S,T){\color{red}z} : q \in \mathbb{C}[z,w]\}} = \mathcal{H} \ominus [w]$.
- z is a nearly 1-cyclic vector for (S,T) and (S,T) is nearly 1-cyclic.

Proposition (Rank of an isopair)

Let $V = (S, T)$ be a pure $\mathfrak{p}$-isopair of finite bimultiplicity (M, N).

(i) *For each j, there exists $\alpha_j \in \mathbb{N}$ such that for $(\lambda, \mu) \in \mathfrak{V}(\mathfrak{p}_j)$ that is a regular point for $\mathfrak{p}$,*

$$\dim\left(\ker(S-\lambda)^* \cap \ker(T-\mu)^*\right) = \alpha_j.$$

(ii) $M = \sum_{j=1}^{s} \alpha_j m_j$ *and* $N = \sum_{j=1}^{s} \alpha_j n_j$.

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We define the *rank* of V as $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s$.

Proposition (stability of the rank)

If

- V *is a pure* $\mathfrak{p}$*-isopair with rank* α *and*
- $\mathcal{H}$ *is a finite codimension* V*-invariant subspace*

then $V|_{\mathcal{H}}$ *also has rank* α*.*

$\mathfrak{p}$- irreducible

Suppose $\mathfrak{p}$ is irreducible (with bidegree (n, m)).

Definition

Fix $\alpha \in \mathbb{N}^+$. A *rank α-admissible kernel K* over $\mathfrak{V}(\mathfrak{p})$ consists of an $\alpha \times m\alpha$ matrix polynomial Q and an $\alpha \times n\alpha$ matrix polynomial P such that, for $(z, w), (\zeta, \eta) \in \mathfrak{V}(\mathfrak{p})$,

$$\frac{Q(z,w)Q(\zeta,\eta)^*}{1 - z\zeta^*} = K((z,w),(\zeta,\eta)) = \frac{P(z,w)P(\zeta,\eta)^*}{1 - w\eta^*}$$

where Q and P have full rank α at some point in $\mathfrak{V}(\mathfrak{p})$.

- K is an $\alpha \times \alpha$ matrix function.

Example:

- Let $\mathfrak{p}(z,w) = z^3 - w^2$.
- Let K be the kernel defined on $\mathfrak{V}(\mathfrak{p}) \times \mathfrak{V}(\mathfrak{p})$ by:

$$\frac{1 + w\overline{\eta}}{1 - z\overline{\zeta}} \;=\; K((z,w),(\zeta,\eta)) \;=\; \frac{1 + z\overline{\zeta} + z^2\overline{\zeta}^2}{1 - w\overline{\eta}}.$$

- Let $Q(z,w) = (1 \quad w)$ and $P(z,w) = (1 \quad z \quad z^2)$.
- K is a rank 1-admissible kernel.

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Proof:

- There exists a rank α-admissible kernel K such that V is unitarily equivalent to (M_z, M_w) on $\mathcal{H}^2(K)$.

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Proof:

- There exists a rank α-admissible kernel K such that V is unitarily equivalent to (M_z, M_w) on $\mathcal{H}^2(K)$.
- (M_z, M_w) is nearly α-cyclic.

Suppose $\mathfrak{p} = \mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_s$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_s)$.

Theorem

If V is a pure $\mathfrak{p}$-isopair of rank α, then V is at most nearly $\max\{\alpha_1, \ldots, \alpha_s\}$-cyclic.

Suppose $\mathfrak{p} = \mathfrak{p}_1 \, \mathfrak{p}_2 \cdots \mathfrak{p}_s$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_s)$.

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If V is a pure $\mathfrak{p}$-isopair of rank α, then V is at most nearly $\max\{\alpha_1, \ldots, \alpha_s\}$-cyclic.

Proof:

- (Agler, Knese and McCarthy) There exists a finite codimensional invariant subspace $\mathcal{H}$ and pure $\mathfrak{p}_j$-isopairs V_j such that

$$V|_{\mathcal{H}} = V_1 \oplus V_2 \oplus ... \oplus V_s.$$

- $V_1 \oplus V_2 \oplus ... \oplus V_s$ has rank α.

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- $V_1 \oplus V_2 \oplus ... \oplus V_s$ has rank α.
- Each V_j is nearly α_j-cyclic.
- $V_1 \oplus V_2 \oplus ... \oplus V_s$ is nearly $\max\{\alpha_1, \ldots, \alpha_s\}$-cyclic.

Proposition

If $V = (S, T)$ is a k-cyclic pure $\mathfrak{p}$-isopair then for each $(\lambda, \mu) \in \mathfrak{V}(\mathfrak{p})$,

$$\dim\left(\ker(S-\lambda)^* \cap \ker(T-\mu)^*\right) \leq k.$$

In particular, if V has rank α, then $k \geq \max\{\alpha_1, \alpha_2, ..., \alpha_s\}$.

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In particular, if V has rank α, then $k \geq \max\{\alpha_1, \alpha_2, ..., \alpha_s\}$.

Corollary

If V is a pure $\mathfrak{p}$-isopair with rank α, then V is at least nearly $\max\{\alpha_1, \ldots, \alpha_s\}$-cyclic.

Proposition

If V is a pure $\mathfrak{p}$-isopair with rank $\alpha = (\alpha_1, \ldots, \alpha_s)$, then V is nearly $\max\{\alpha_1, \ldots, \alpha_s\}$-cyclic.

Theorem

Any nearly k-cyclic pure $\mathfrak{p}$-isopair is unitarily equivalent to a k-cyclic pure $\mathfrak{p}$-isopair restricted to a finite codimensional invariant subspace.

Remark: Agler,Knese and McCarthy proved this for $k = 1$ case.

Summary

If V is a finite bimultiplicity pure $\mathfrak{p}$-isopair with rank $\alpha = (\alpha_1, ..., \alpha_s)$ and $k = \max\{\alpha_1, ..., \alpha_s\}$, then

(i) *V is nearly k-cyclic;*

(ii) *There exists a finite codimensional invariant subspace $\mathcal{H}$ for V such that $V|_{\mathcal{H}}$ is k-cyclic;*

(iii) *V is a k-cyclic pure $\mathfrak{p}$-isopair restricted to a finite codimensional invariant subspace.*

Question (Agler, Knese and McCarthy)

Suppose V and V' are both nearly k-cyclic pure algebraic isopairs with the same minimal polynomial. Are they nearly unitarily equivalent?

- TRUE for $k = 1$ (Agler, Knese and McCarthy).

Refined Question

Suppose V and V' are both pure algebraic isopairs with the same minimal polynomial and the same rank. Are they nearly unitarily equivalent?

THANK YOU