Antidifferentiating First Order Noncommutative Functions

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The Noncommutative Space

Let

1. $\mathcal{R}$ be a commutative ring with identity,
2. $\mathcal{M}$ be an $\mathcal{R}$-module, and
3. $\mathcal{M}^{n\times n}$ be the module of all $n \times n$ matrices with entries from $\mathcal{M}$.

Define the noncommutative space over $\mathcal{M}$ to be

$$\mathcal{M}_{nc} := \coprod_{n=1}^{\infty} \mathcal{M}^{n\times n}$$

Noncommutative Sets

For $\Omega \subseteq \mathcal{M}_{nc}$

1. $\Omega_n := \Omega \cap \mathcal{M}^{n \times n}$.
2. Ω is a *noncommutative set* (nc set) if

$$X \in \Omega_n, Y \in \Omega_m \implies X \oplus Y \in \Omega_{n+m}$$

Definition of Noncommutative Function

A function $f : \Omega \to \mathcal{N}_{nc}$ such that

$$f(\Omega_n) \subseteq \mathcal{N}^{n\times n},$$

for $n = 1, 2, \ldots$ is called a noncommutative function if f respects intertwinings:

$$SX = YS \implies Sf(X) = f(Y)S.$$

Example of a Noncommutative Function

Consider the matrix polynomial $f(X) = X^2$. In this case,

$$SX = YS \implies$$
$$Sf(X) = SX^2 = (SX)X = (YS)X = Y(SX) = Y(YS) = Y^2S = f(Y)S$$

The Difference-Differential Operator

Let

1. $f : \Omega \to \mathcal{N}_{nc}$ be an nc function.
2. $\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}$ for some $X \in \Omega_n$, $Y \in \Omega_m$ and $Z \in \mathcal{M}^{n \times m}$.

Define $\Delta_R f(X, Y)(Z)$ implicitly by,

$$f\left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}\right) = \begin{bmatrix} f(X) & \Delta_R f(X, Y)(Z) \\ 0 & f(Y) \end{bmatrix}$$

Proposition 1

$\Delta_R f(X, Y)(Z)$ can be extended to a function linear in Z on $\mathcal{M}^{n \times m}$.

A Difference-Differential Operator Example

If $f(X) = X^2$, then

$$f\left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}\right) = \begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} = \begin{bmatrix} X^2 & XZ + ZY \\ 0 & Y^2 \end{bmatrix}$$

Thus, $\Delta_R f(X,Y)(Z) = XZ + ZY$.

Difference Formula

Theorem 1

Let

1. Ω *be a right admissible nc set and*
2. $f : \Omega \to \mathcal{N}_{nc}$ *be an nc function.*

Then, for all $n, m \in \mathbb{N}$*, all* $X \in \Omega_n$*,* $Y \in \Omega_m$ *and* $S \in \mathcal{R}^{n\times m}$*,*

$$Sf(Y) - f(X)S = \Delta_R f(X, Y)(SY - XS).$$

If $n = m$ *and* $S = I_n$*,*

$$f(Y) - f(X) = \Delta_R f(X, Y)(Y - X)$$

Properties of the Difference-Differential Operator

The difference-differential operator respects intertwinings as follows:

$$SX = WS \implies S\Delta f(X,Y)(Z) = \Delta f(W,Y)(SZ),$$
$$TY = WT \implies \Delta f(X,Y)(ZT) = \Delta f(X,W)(Z)T.$$

First Order NC Functions

Let

1. $\mathcal{M}_0, \mathcal{M}_1, \mathcal{N}_0, \mathcal{N}_1$ be $\mathcal{R}$-modules and
2. $\Omega^{(0)} \subseteq \mathcal{M}_{0,}, \Omega^{(1)} \subseteq \mathcal{M}_{1,}$ are nc sets.

A function

$$f : \Omega^{(0)} \times \Omega^{(1)} \to \hom_{\mathcal{R}}(\mathcal{N}_1^{n_0 \times n_1}, \mathcal{N}_0^{n_0 \times n_k})$$

is an nc function of order 1 if

NC Functions Respect Intertwinings

$$SX^0 = Y^0 S \implies \mathbf{S} f(\mathbf{X^0}, X^1)(Z^1) = f(\mathbf{Y^0}, X^1)(\mathbf{S} Z^1),$$

$$TX^1 = Y^1 T \implies f(X^0, \mathbf{X^1})(Z^1 \mathbf{T}) = f(X^0, \mathbf{Y^1})(Z^1)\mathbf{T}.$$

Notation

By this definition $\Delta_R f(X,Y)(Z)$ is a first order function while f is considered a zero order function.

Let

$$\mathcal{T}^1(\Omega^{(0)},\Omega^{(1)};\mathcal{N}_{0,nc},\mathcal{N}_{1,nc})$$

be the set of all nc functions of order 1 on $\Omega^{(0)} \times \Omega^{(1)}$.

Example of First Order NC Function

Let $f(X^0, X^1)(Z) = (X^0)^2 ZX^1 + X^0 ZX^1$. Then,

$$SX^0 = Y^0 S \implies$$
$$Sf(X^0, X^1)(Z) = S((X^0)^2 ZX^1 + X^0 ZX^1) = (SX^0)X^0 ZX^1 + SX^0 ZX^1$$
$$= Y^0(SX^0)ZX^1 + Y^0(SZ)X^1 = (Y^0)^2(SZ)X^1 + Y^0(SZ)X^1$$
$$= f(Y^0, X^1)(SZ)$$

and

$$TX^1 = Y^1 T \implies$$
$$f(X^0, X^1)(ZT) = (X^0)^2 Z(TX^1) + X^0 Z(TX^1) = (X^0)^2 ZY^1 T + X^0 ZY^1 T$$
$$= ((X^0)^2 ZY^1 + X^0 ZY^1)T = f(X^0, Y^1)(Z)T$$

First Order Difference-Differential Operators

Proposition 2

Let $f \in \mathcal{T}^1(\Omega^{(0)}, \Omega^{(1)}; \mathcal{N}_{0,nc}, \mathcal{N}_{1,nc})$,

$$f\left(\begin{bmatrix} X_1^0 & Z \\ 0 & X_2^0 \end{bmatrix}, X^1\right)\left(\begin{bmatrix} Z_1^1 \\ Z_2^1 \end{bmatrix}\right) = \begin{bmatrix} f(X_1^0, X^1)(Z_1^1) + {}_0\Delta_R f(X_1^0, X_2^0, X^1)(Z, Z_2^1) \\ f(X_2^0, X^1)(Z_2^1) \end{bmatrix}$$

$$f\left(X^0, \begin{bmatrix} X_1^1 & Z \\ 0 & X_2^1 \end{bmatrix}\right)\left(\begin{bmatrix} Z_1^1 & Z_2^1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} f(X^0, X_1^1)(Z_1^1) & {}_1\Delta_R f(X^0, X_1^1, X_2^1)(Z_1^1, Z) + f(X^0, X_2^1)(Z_2^1) \end{bmatrix}$$

The Problem

Question: Given an order 1 nc function, F, does there exist an order 0 nc function f, such that $\Delta_R f = F$?

The Theorem

Theorem 2

Let $F \in \mathcal{T}^1(\Omega, \Omega; \mathcal{M}, \mathcal{N})$*.*

Then, there exists an nc function $f \in \mathcal{T}^0(\Omega, \mathcal{N})$ *if and only if* ${}_1\Delta F = {}_0\Delta F$*.*

Further, f *is unique up to a constant* $c \in \mathcal{N}$*. That is, given a solution* f*,*

$$\tilde{f} : \Omega \to \mathcal{N}$$
$$\tilde{f}(X) = f(X) + cI_n,$$

is also a solution.

${}_0\Delta F = {}_1\Delta F$

Suppose that there exists an nc function $f \in \mathcal{T}^0(\Omega, \mathcal{N})$. Then,

$$f\left(\begin{bmatrix} X^0 & \begin{bmatrix} Z_1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} X^1 & Z_2 \\ & X^2 \end{bmatrix} \end{bmatrix}\right) = f\left(\begin{bmatrix} \begin{bmatrix} X^0 & Z_1 \\ 0 & X^1 \end{bmatrix} & \begin{bmatrix} 0 \\ Z_2 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & X^2 \end{bmatrix}\right)$$

$$\begin{bmatrix} f(X^0) & F\left(X^0, \begin{bmatrix} X^1 & Z_2 \\ & X^2 \end{bmatrix}\right)(\begin{bmatrix} Z_1 & 0 \end{bmatrix}) \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & f\left(\begin{bmatrix} X^1 & Z_2 \\ & X^2 \end{bmatrix}\right) \end{bmatrix}$$

$$= \begin{bmatrix} f\left(\begin{bmatrix} X^0 & Z_1 \\ 0 & X^1 \end{bmatrix}\right) & F\left(\begin{bmatrix} X^0 & Z_1 \\ 0 & X^1 \end{bmatrix}, X^2\right)\left(\begin{bmatrix} 0 \\ Z_2 \end{bmatrix}\right) \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & f(X^2) \end{bmatrix}$$

${}_0\Delta F = {}_1\Delta F$

$$\begin{bmatrix} f(X^0) & F(X^0,X^1)(Z_1) & {}_1\Delta_R F(X^0,X^1,X^2)(Z^1,Z) \\ 0 & f(X^1) & F(X^1,X^2)(Z^2) \\ 0 & 0 & f(X^2) \end{bmatrix}$$
$$= \begin{bmatrix} f(X^0) & F(X^0,X^1)(Z_1) & {}_0\Delta_R F(X^0,X^1,X^2)(Z^1,Z) \\ 0 & f(X^1) & F(X^1,X^2)(Z^2) \\ 0 & 0 & f(X^2) \end{bmatrix}$$

The upper, right hand corner gives the desired equality.

Defining a Possible Anti-Derivative

For the converse, fix $Y \in \Omega_s$ and assume $f(Y)$ is known.

For any $X \in \Omega_{ns}$, define the anti-derivative of F,

$$f : \bigsqcup_{n=1}^{\infty} \Omega_{ns} \to \mathcal{N},$$

by

$$f(X) = \left(\bigoplus_{i=1}^{n} f(Y) \right) + F\left(\bigoplus_{i=1}^{n} Y, X \right) \left(X - \bigoplus_{i=1}^{n} Y \right).$$

Condition on $f(Y)$

Proposition 3

As defined,

1. f *is an nc function and*
2. $\Delta_R f = F$,

if and only if

$$Sf(Y) - f(Y)S = F(Y,Y)(SY - YS), \tag{1}$$

for all $S \in \mathcal{R}^{s\times s}$.

What Remains

It remains to be shown that for any matrix Y, there exists some matrix $f(Y)$ that satisfies (1).

A Natural Basis

Let

1. E_{ij} be a matrix with a 1 in position i, j and zeros elsewhere.
2. $\mathcal{B} = \{E_{ij} \mid i, j = 1, \ldots, s\}$.

Then,

1. $\mathcal{B}$ is a basis of $\mathcal{R}^{s \times s}$ and
2. Property (1) holds for all $S \in \mathcal{R}^{s \times s}$ iff it holds for $S \in \mathcal{B}$.

Derivations

A derivation the Lie algebra structure on $\mathcal{R}$ is a linear map,

$$D : \mathcal{R}^{s \times s} \to \mathcal{N}^{s \times s},$$

such that

$$[S, D(T)] + [D(S), T] = D([S, T]).$$

Derivation Identity

Proposition 4

For $F \in \mathcal{T}^1(\Omega, \Omega; \mathcal{M}, \mathcal{N})$*:*

$$\begin{aligned} &SF(Y,Y)(TY-YT) - F(Y,Y)(TY-YT)S \\ &\quad + F(Y,Y)(SY-YS)T - TF(Y,Y)(SY-YS) \\ &= F(Y,Y)((ST-TS)Y - Y(ST-TS)) \end{aligned}$$

Define,

$$D_Y(S) = F(Y,Y)(SY-YS).$$

Then $D_Y(S)$ is a derivation.

Derivation Theorem

Theorem 3

Let

1. $D : \mathcal{R}^{s \times s} \to \mathcal{N}^{s \times s}$ *be a derivation and*
2. $D^{ij} = D(E_{ij})$.

Then, D is inner, i.e. $D(S) = SN - NS$, if and only if

$$D^{ii}_{kk} = 0 \quad \text{and} \quad N = \sum_{i=1}^{n} (E_{ii} D^{ii} + E_{i1} D^{1i} E_{ii}) + cI_s, \tag{2}$$

for all $i, k = 1, \ldots, s$ and some $c \in \mathcal{N}$.

The Initial Calculation

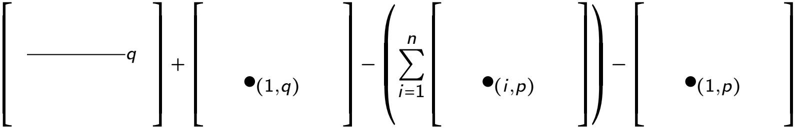
Suppose that condition (2) holds.

$$
\begin{aligned}
&E_{pq}N - NE_{pq} \\
&= E_{pq}\left(\sum_{i=1}^{n} E_{ii}D^{ii} + E_{i1}D^{1i}E_{ii} + cI_s\right) - \left(\sum_{i=1}^{n} E_{ii}D^{ii} + E_{i1}D^{1i}E_{ii} + cI_s\right)E_{pq} \\
&= E_{pq}E_{qq}D^{qq} + E_{pq}E_{q1}D^{1q}E_{qq} + cE_{pq} \\
&\qquad - \left(\sum_{i=1}^{n} E_{ii}D^{ii}E_{pq}\right) - E_{p1}D^{1p}E_{pp}E_{pq} - cE_{pq} \\
&= E_{pq}D^{qq} + E_{p1}D^{1q}E_{qq} - \left(\sum_{i=1}^{n} E_{ii}D^{ii}E_{pq}\right) - E_{p1}D^{1p}E_{pq}.
\end{aligned}
$$

Visualizing the Equation

Writing these out as matrices, we get,

$$E_{pq}D^{qq} \quad + \quad E_{p1}D^{1q}E_{qq} \quad - \left(\sum_{i=1}^{n} \quad E_{ii}D^{ii}E_{pq} \quad \right) - \quad E_{p1}D^{1p}E_{pq}$$



The Equations

This reduces to the following equations:

$$\begin{aligned} D^{qq}_{qm} &= D^{pq}_{pm} && \text{where } m \neq q \\ -D^{ll}_{lp} &= D^{pq}_{lq} && \text{where } l \neq p \\ D^{qq}_{qq} - D^{pp}_{pp} + D^{1q}_{1q} - D^{1p}_{1p} &= D^{pq}_{pq} && \\ 0 &= D^{pq}_{lm} && \text{where } l \neq p \text{ and } m \neq q \end{aligned}$$

Finishing the Proof

The proof now reduces to choosing r, s, u and v in such a way that we get the desired equalities. It uses the fact that $D^{ii}_{kk} = 0$.

This shows that if the diagonals are 0 and N can have the given form, then

$$D(S) = SN - NS.$$

The Last Computation

Proposition 5

Let,

1. $F \in \mathcal{T}^1(\Omega, \Omega; \mathcal{M}, \mathcal{N})$ *and*
2. ${}_0\Delta F = {}_1\Delta F$.

Then, for $P \in \mathcal{R}^{s\times s}$ *such that* $P^2 = P$,

$$PF(Y,Y)(PY - YP)P = 0.$$

Proof of Main Theorem

By Proposition 5, the diagonal entries of the derivation $D_Y(S)$ are zero.

Thus by Theorem 3, $F(Y,Y)(SY - YS) = D_Y(S) = Sf(Y) - f(Y)S$ as long as,

$$f(Y) = \sum_{i=1}^{s} E_{ii} D_Y(E_{ii}) + E_{i1} D_Y(E_{1i}) E_{ii} + cI_s.$$

Thus, we have a value for $f(Y)$ that satisfies condition (1) as desired.

THANK YOU