Carlson's Theorem for Different Measures

SEAM 2017

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Motivation

Carlson's Theorem

Theorem (Carlson's Theorem)

If an ordinary Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges in the right half plane $\mathbb{C}_+$ and is bounded in every half plane $\Re(s) \geq \delta$ for $\delta > 0$, then for each $\sigma > 0$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |f(\sigma + it)|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma}$$

Bohr Correspondence

Carlson's theorem as a special case of the ergodic theorem

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- connects Dirichlet series on $\mathbb{C}_+ = \{s \in \mathbb{C} : \Re(s) > 0\}$ to power series on the infinite polydisk using the fundamental theorem of arithmetic

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- $f(s) = \sum a_n n^{-s} \leftrightarrow F(z) = \sum a_\alpha z^\alpha$
- any vertical line in $\mathbb{C}_+$ maps to an ergodic flow on $\mathbb{T}^\infty$
- imaginary axis ($\sigma = 0$) maps to torus boundary of $\mathbb{D}^\infty$

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- any vertical line in $\mathbb{C}_+$ maps to an ergodic flow on $\mathbb{T}^\infty$
- imaginary axis ($\sigma = 0$) maps to torus boundary of $\mathbb{D}^\infty$
- general ergodic theorem no help in this case

Boundary Behavior

Definition

$\mathcal{H}^\infty = \{\sum_{n=1}^{\infty} a_n n^{-s}$: converges to bounded analytic functions on $\mathbb{C}_+\}$

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Theorem (Saksman-Seip)

The following two statements hold:

(i) *There exists a function f in $\mathcal{H}^\infty$ such that*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |f(it)|^2 dt$$

does not exist.

(ii) *Given $\epsilon > 0$, there exists a singular inner function $g = \sum_{n=1}^\infty b_n n^{-s}$ in $\mathcal{H}^\infty$ such that $\sum_{n=1}^\infty |b_n|^2 \leq \epsilon$.*

$\mathcal{A}(\mathbb{C}_+)$

A better space

- Introduced by Aron, Bayart, *et al*.

$$\mathcal{A}(\mathbb{C}_+) = \left\{ \sum_{n=1}^{\infty} a_n n^{-s} : \text{convergent on } \mathbb{C}_+ \text{ and uniformly continuous there} \right\}$$
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- Closed subspace of $\mathcal{H}^\infty$
- Isometrically isomorphic to disk algebra $A(\mathbb{D}^{\mathbb{N}})$:

$$n : \mathbb{N}^\infty \to \mathbb{N}$$
$$n : (\alpha_1, \dots, \alpha_d, 0, \dots) \mapsto p_1^{\alpha_1} \cdots p_d^{\alpha_d}$$
$$F(z) = \sum_\alpha a_\alpha z^\alpha \mapsto f(s) = \sum_\alpha a_\alpha \left[n(\alpha)\right]^{-s}$$

The Theorem

Theorem (S.)

i *Let* μ *be a Borel probability measure on the infinite torus* $\mathbb{T}^\infty$. *There exists a locally finite Borel measure* ν *on* $\mathbb{R}$, *such that, for all* $\sum a_\alpha [n(\alpha)]^{-s} \in \mathcal{A}(\mathbb{C}_+)$

$$\lim_{T\to\infty} \frac{1}{\nu([0,T])} \int_0^T \left| \sum a_\alpha [n(\alpha)]^{-it} \right|^2 d\nu(t) \quad (1)$$

$$= \int_{\mathbb{T}^\infty} \left| \sum a_\alpha z^\alpha \right|^2 d\mu(z).$$

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$$\lim_{T\to\infty} \frac{1}{\nu([0,T])} \int_0^T \left|\sum a_\alpha [n(\alpha)]^{-it}\right|^2 d\nu(t) = \int_{\mathbb{T}^\infty} \left|\sum a_\alpha z^\alpha\right|^2 d\mu(z). \quad (1)$$

ii *Let ν be a locally finite Borel measure on $\mathbb{R}$ such that the limit (1) exists and is finite for all $\sum a_\alpha [n(\alpha)]^{-s} \in \mathcal{A}(\mathbb{C}_+)$. Then there exists a unique Borel probability measure μ on the infinite torus $\mathbb{T}^\infty$ such that, for all $\sum a_\alpha [n(\alpha)]^{-s} \in \mathcal{A}(\mathbb{C}_+)$, the equality holds.*

Outline of the Proof

Part *i*

$$\lim_{T\to\infty} \frac{1}{\nu([0,T])} \int_0^T \left|\sum a_\alpha \left[n(\alpha)\right]^{-it}\right|^2 d\nu(t) = \int_{\mathbb{T}^\infty} \left|\sum a_\alpha z^\alpha\right|^2 d\mu(z)$$

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- a single point mass

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So, construct the measure ν for when μ is

- a single point mass
- a linear combination of point masses
- a Borel measure

$$|F(\lambda)|^2 = \lim_{T\to\infty} \frac{1}{\nu_\lambda([0,T])} \int_0^T |f(t)|^2 d\nu_\lambda$$

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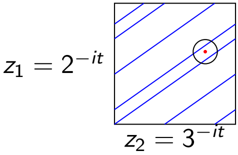

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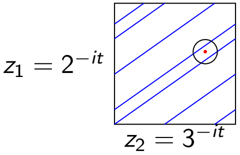

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- passes "near" λ infinitely many times

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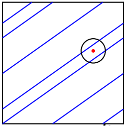

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$$|f(t)|^2 = \sum |a_\alpha|^2 + \sum_\alpha \sum_{\beta\neq\alpha} a_\alpha \overline{a_\beta} n(\alpha-\beta)^{-it}$$

$$\lambda^\alpha \overline{\lambda^\beta} = e^{i\theta_1(\alpha_1-\beta_1)} e^{i\theta_2(\alpha_2-\beta_2)} \ldots e^{i\theta_d(\alpha_d-\beta_d)}$$

$$n(\alpha-\beta)^{-it} = e^{-i(\alpha_1-\beta_1)t\log(p_1)} \ldots e^{-i(\alpha_d-\beta_d)t\log(p_d)}$$

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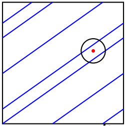

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$$-t\log p_j \approx \theta_j \; mod \; 2\pi$$

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Lemma

Fix k, $M \in \mathbb{N}$, $T > 0$ and let $\epsilon > 0$. Then there exists $T' > T$ such that there exist M distinct points $t_m^k \in (T, T')$ such that for each t_m^k there exist $q_j \in \mathbb{Z}$ for $1 \leq j \leq k$ satisfying

$$|-t_m^k \log(p_j) - \theta_j - 2\pi q_j| < \epsilon.$$

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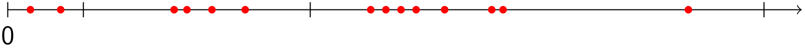
$$|-t_m^k \log(p_j) - \theta_j - 2\pi q_j| < \epsilon.$$

- For larger T make it work for more primes and shrink ϵ
- weight so (better estimate) + (more primes) = (matters more)



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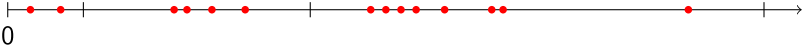

- Build ν_λ inductively from these

$$\begin{aligned}|\nu_k([0, T_k])| = \|\nu_k\| &= \|\nu_{k-1}\| + 2^k\|\nu_{k-1}\| \\ &= (2^k + 1)\|\nu_{k-1}\|.\end{aligned}$$

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Lemma

Given $\epsilon > 0$, there exists T_ϵ such that for any finite $T > T_\epsilon$ and for all $j = 1, \ldots, N$

$$\left| \frac{1}{\nu_{\lambda_j}([T_\epsilon, T])} \int_{T_\epsilon}^T |f(t)|^2 d\nu_{\lambda_j} - |F(\lambda_j)|^2 \right| < \epsilon.$$

$$\nu^{(n)} = \sum_{j=1}^N \frac{c_j \nu_{\lambda_j}}{\nu_{\lambda_j}([T_{n-1}, T_n])} \qquad \nu = \nu^{(1)}\Big|_{[0,T_1]} + \sum_{n=1}^\infty \nu^{(n)}\Big|_{[T_{n-1},T_n]}.$$

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$$
\begin{aligned}
&\frac{1}{\nu([0,T])} \int_0^T |f(t)|^2 d\nu \\
&= \frac{1}{\nu([0,T])} \left[\int_0^{T_1} |f(t)|^2 d\nu^{(1)} + \int_{T_1}^{T_2} |f(t)|^2 d\nu^{(2)} + \cdots + \int_{T_{k-1}}^{T} |f(t)|^2 d\nu^{(k)} \right] \\
&= \frac{1}{\nu([0,T])} \sum_{j=1}^{N} c_j \left[\frac{1}{\nu_{\lambda_j}([0,T_1])} \int_0^{T_1} |f(t)|^2 d\nu_{\lambda_j} + \cdots \right. \\
&\qquad \left. + \frac{1}{\nu_{\lambda_j}([T_{k-1},T_k])} \int_{T_{k-1}}^{T} |f(t)|^2 d\nu_{\lambda_j} \right] \\
&< \frac{1}{\nu([0,T])} \sum_{j=1}^{N} c_j \left[\left(|F(\lambda_j)|^2 + 2^{-1}\right) + \left(|F(\lambda_j)|^2 + 2^{-2}\right) + \cdots + \left(|F(\lambda_j)|^2 + 2^{-k}\right) \right]
\end{aligned}
$$

Borel Measures

$\lim_{T\to\infty} \frac{1}{\nu([0,T])} \int_0^T \left|\sum a_\alpha \left[n(\alpha)\right]^{-it}\right|^2 d\nu(t) = \int_{\mathbb{T}^\infty} \left|\sum a_\alpha z^\alpha\right|^2 d\mu(z)$

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Point masses are weak-$*$ dense:

$$\left|\int |F(z)|^2 d\mu - \int |F(z)|^2 d\mu_n\right| < \frac{1}{2}2^{-n}.$$

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Lemma

Given $\epsilon > 0$, there exists for each n, T_ϵ such that for each finite $T > T_\epsilon$

$$\left|\frac{1}{\nu_n([T_\epsilon, T])} \int_{T_\epsilon}^T |f(t)|^2 d\nu_n - \int |F(z)|^2 d\mu_n\right| < \epsilon.$$

$$\nu = \frac{\nu_1\big|_{[0,T^{(1)}]}}{\nu_1([0,T^{(1)}])} + \sum_{n=2}^{\infty} \frac{\nu_n\big|_{[T^{(n-1)},T^{(n)}]}}{\nu_n([T^{(n-1)},T^{(n)}])}$$

Proof of the Converse

$\lim_{T\to\infty} \frac{1}{\nu([0,T])} \int_0^T \left|\sum a_\alpha \left[n(\alpha)\right]^{-it}\right|^2 d\nu(t) = \int_{\mathbb{T}^\infty} \left|\sum a_\alpha z^\alpha\right|^2 d\mu(z)$

Theorem (Riesz Representation Theorem)

Let X be a compact Hausdorff space. If ϕ is a positive bounded linear functional on $C(X, \mathbb{R})$, then there exists a unique regular Borel measure μ on X such that

$$\phi(f) = \int_X f d\mu$$

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$$\phi : |F|^2 \mapsto \lim_{T\to\infty} \frac{1}{\nu([0,T])} \int_0^T |f(it)|^2 \, d\nu(t).$$

Thank You!

Summary

$$\mathcal{H}^\infty = \left\{ \sum_{n=1}^{\infty} a_n n^{-s} : \text{converges to bounded analytic functions on } \mathbb{C}_+ \right\}$$

$$\mathcal{A}(\mathbb{C}_+) = \left\{ \sum_{n=1}^{\infty} a_n n^{-s} : \text{convergent on } \mathbb{C}_+ \text{ and uniformly continuous there} \right\}$$

$$= \overline{span\{n^{-s} : n \in \mathbb{N}\}}^{\|\cdot\|_\infty}$$

For $\sum a_\alpha \left[n(\alpha)\right]^{-s} \in \mathcal{A}(\mathbb{C}_+)$

$$\lim_{T\to\infty} \frac{1}{\nu([0,T])} \int_0^T \left| \sum a_\alpha \left[n(\alpha)\right]^{-it} \right|^2 d\nu(t) = \int_{\mathbb{T}^\infty} \left| \sum a_\alpha z^\alpha \right|^2 d\mu(z)$$