A Generalization of the Fock Space

Southeastern Analysis Meeting

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Definition

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$$\|f\|^2 = \int_{\mathbb{C}} |f(z)|^2 \frac{1}{\pi} e^{-|z|^2} dz < \infty$$

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The reproducing kernels for F^2 are given by:

$$K(z, w) = e^{z\bar{w}}$$

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When $q = 1$:

$$E_1(z) = e^z$$

When $q = 1/2$:

$$E_{1/2}(z) = e^z \mathrm{erfc}(z^{1/2})$$

We define the following generalization of the Fock Space:

Definition

$$ML^2(\mathbb{C};q) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \ : \ \sum_{n=0}^{\infty} |a_n|^2 \Gamma(qn+1) < \infty \right\}$$

is a RKHS with kernel functions given by

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is a RKHS with kernel functions given by

$$K_q(w,z) = E_q(\bar{w}z)$$

Equivalently, we can consider $ML^2(\mathbb{C};q)$ to consist of entire functions for which

$$\|f\|^2 = \int_{\mathbb{C}} |f(z)|^2 \frac{1}{q\pi} |z|^{\frac{2}{q}-2} e^{-|z|^{\frac{2}{q}}} dz < \infty$$

Proposition

If $f \in F^2$*, then* $f \in ML^2(\mathbb{C}; q)$ *for all* $0 < q \leq 1$*. Moreover,*

$$\lim_{q \to 1^-} \|f\|_{ML^2(q)} = \|f\|_{F^2}.$$

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If $f \in F^2$, then $f \in ML^2(\mathbb{C}; q)$ for all $0 < q \leq 1$. Moreover,

$$\lim_{q \to 1^-} \|f\|_{ML^2(q)} = \|f\|_{F^2}.$$

To see this we note that $\left\{ \frac{z^n}{\sqrt{n!}} \right\}_{n=0}^{\infty}$ and $\left\{ \frac{z^n}{\sqrt{\Gamma(qn+1)}} \right\}_{n=0}^{\infty}$ are orthonormal basis for F^2 and $ML^2(q)$ respectively.

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$$\|f\|^2_{ML^2(q)} = \sum |a_n|^2 \left(\frac{\Gamma(qn+1)}{n!} \right) \quad \text{and} \quad \|f\|^2_{F^2} = \sum |a_n|^2$$

By a simple application of Cauchy-Schwarz

$$|f(z)|^2 \leq E_q(|z|^2)\|f\|_{ML^2}$$

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If f is in the Mittag-Leffler space $ML^2(\mathbb{C}; q)$ for $0 < q < 2$ then f is of order less than or equal to $2/q$.

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Let $\{z_n\}$ be the zero sequence, repeated according to multiplicity and arranged so that $0 < |z_1| \leq |z_2| \leq \ldots$, of a function $f \in ML^2_*(\mathbb{C}, q)$ such that $f(0) \neq 0$. If $0 < q < 2$ then there exists a positive constant c such that $|z_n| \geq cn^{\frac{q}{2}}$.

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The Riemann-Liouville fractional integral for $q \in \mathbb{R}_+$ is defined as

$$(J^q f)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau) d\tau$$

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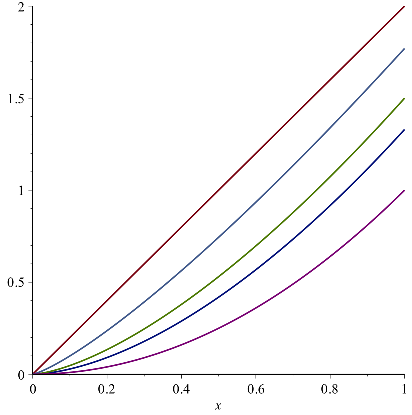
Definition

Let $n \in \mathbb{N}$. For an n-times differentiable function $f : \mathbb{R}_+ \to \mathbb{R}$, the Caputo fractional derivative of order q, where $n-1 < q \leq n$, is given by

$$\begin{aligned} D_*^q f(t) &= J^{n-q} \frac{d^n}{dt^n} f(t) \\ &= \begin{cases} \dfrac{1}{\Gamma(n-q)} \displaystyle\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q+1-n}} d\tau & n-1 < q < n, \ n \in \mathbb{N} \\ f^{(n)}(t) & q = n \in \mathbb{N} \end{cases} \end{aligned}$$



The Caputo derivative has a pointwise interpolation property.

$$\lim_{q \to n} D_*^q f(t) = f^{(n)}(t)$$

On monomials we have that

$$D_*^q t^n = \begin{cases} \frac{\Gamma(n+1)}{\Gamma(n-q+1)} t^{n-q} & m-1 < q < m, \ n > m-1, \ n \in \mathbb{R} \\ 0 & m-1 < q < m, \ n \leq m-1, \ n \in \mathbb{N} \end{cases}$$

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The Mittag-Leffler function of order q is an eigenfunction for Caputo differentiation.

$$D_*^q E_q(\lambda t^q) = \lambda E_q(\lambda t^q)$$

The *real* Mittag-Leffler space is defined as follows:

Definition

For $q > 0$, the (real) Mittag-Leffler space of order q is the (real valued) RKHS corresponding to the kernel function

$$K_q(t, \lambda) = E_q(\lambda^q t^q).$$

We denote this space by $ML^2(\mathbb{R}_+; q)$

Hence,

$$ML^2(\mathbb{R}_+; q) = \left\{ f(t) = \sum_{n=0}^{\infty} a_n t^{qn} \ : \ \sum_{n=0}^{\infty} |a_n|^2 \Gamma(qn+1) < \infty \right\}$$

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For functions in the real valued Mittag-Leffler space we have that

$$D_*^q f(t) = \sum_{n=0}^{\infty} a_{n+1} \frac{\Gamma(q(n+1)+1)}{\Gamma(qn+1)} t^{qn}$$

To extend the Caputo derivative to functions of a complex variable we define the following:

Definition

$$ML^2(\mathbb{C} \setminus \mathbb{R}_-; q) = \{f : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C} \ : \ f(z^{1/q}) \in ML^2(\mathbb{C}; q)\}$$

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We will define

$$\langle f(z),\ g(z) \rangle_{ML^2(\mathbb{C}\setminus\mathbb{R}_-;q)} = \left\langle f(z^{1/q}),\ g(z^{1/q}) \right\rangle_{ML^2(\mathbb{C};q)}.$$

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We can view $ML^2(\mathbb{C} \setminus \mathbb{R}_-; q)$ as the pullback of $ML^2(\mathbb{C}; q)$ using the mapping $\varphi : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$; $\varphi(z) = z^q$.

Definition

Let $f(z) \in ML^2(\mathbb{C} \setminus \mathbb{R}; q)$, then

$$f(z) = \sum_{n=0}^{\infty} a_n z^{qn} \quad \text{where} \quad \sum |a_n|^2 < \infty$$

Define the Caputo derivative of $f(z)$ as follows:

$$D_*^q f(z) = \sum_{n=1}^{\infty} a_n \frac{\Gamma(qn+1)}{\Gamma(q(n-1)+1)} z^{q(n-1)}$$

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Since this definition agrees with the definition of the Caputo fractional derivative for functions in the real valued Mittag-Leffler space on the real line, by an application of the identity theorem this is an appropriate generalization to $\mathbb{C} \setminus \mathbb{R}_-$

Consider the space $\mathcal{L}^2(\mathbb{R}, dx)$ and the densely defined unbounded operators

$$X : \mathrm{Dom}(X) \subset \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R}); \qquad X(f) = xf(x)$$

and

$$D : \mathrm{Dom}(D) \subset \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R}); \qquad D(f) = \frac{\hbar}{i}\frac{d}{dx}f(x)$$

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Define the annihilation and creation operators on $L^2(\mathbb{R}, dx)$ by

$$W = \frac{1}{\sqrt{2}}\left(X + \frac{i}{\hbar}D\right), \quad W^* = \frac{1}{\sqrt{2}}\left(X - \frac{i}{\hbar}D\right)$$

respectively.

Let $\{h_n(x)\}$ be the hermite functions defined by:

Definition

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

and

$$h_n(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} e^{-x^2} H_n(\sqrt{2}x).$$

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$$Wh_n = \sqrt{n}\, h_{n-1} \quad \text{and} \quad W^* h_n = \sqrt{n+1}\, h_{n+1}.$$

In a similar fashion, consider $F^2(\mathbb{C})$ with orthonormal basis $\{e_n\} = \left\{ \frac{z^n}{\sqrt{n!}} \right\}$ given by the scaled monomials.

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Let A and A^* be the unbounded densely defined operators given by

$$Af(z) = \frac{d}{dz} f(z), \quad A^* f(z) = zf(z).$$

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Again we note that

$$Ae_n = \sqrt{n}\, e_{n-1} \quad \text{and} \quad A^*e_n = \sqrt{n+1}\, e_{n+1}.$$

The Bargmann transform, $B : \mathcal{L}^2(\mathbb{R}, dx) \to F^2(\mathbb{C})$ is given by

$$[Bf](z) = \left(\frac{2}{\pi}\right)^{1/4} \int_{\mathbb{R}} f(x) e^{2xz - x^2 - \frac{1}{2}z^2} dx$$

for $z \in \mathbb{C}$.

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for $z \in \mathbb{C}$. The inverse transform is given by

$$\left[B^{-1} f\right](x) = \left(\frac{2}{\pi}\right)^{1/4} \int_{\mathbb{C}} f(z) e^{2x\bar{z} - x^2 - \frac{1}{2}\bar{z}^2} e^{-|z|^2} dz.$$

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Where,

$$\left(\frac{2}{\pi}\right)^{1/4} e^{2xz - x^2 - \frac{1}{2}z^2} = \sum_{n=0}^{\infty} h_n(x) \frac{z^n}{\sqrt{n!}}.$$

$$B : \sum_{n=0}^{\infty} a_n h_n(x) \mapsto \sum_{n=0}^{\infty} a_n \frac{z^n}{\sqrt{n!}}.$$

Hence

Theorem (Bargmann '61)

$$W = B^{-1} A B$$

$$W^* = B^{-1} A^* B$$

We wish to leverage the above result into a result about fractional differentiation and multiplication by z^q on the Mittag-Leffler space on the slitted plane.

Definition

Let $(q, p) \in \mathbb{R}^{2+}$, call

$$H_{p,q}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\Gamma(pn+1)}\sqrt{\Gamma(qn+1)}}$$

the modified Mittag-Leffler function for (p, q).

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Theorem

The operator given by

$$U_q : F^2 \to ML^2(\mathbb{C} \setminus \mathbb{R}_-; q)$$

$$U_q f = \frac{1}{\pi} \int_{\mathbb{C}} f(w) H_{1,q}(\overline{w} z^q) e^{-|w|^2} dw$$

is an isometry between F^2 and $ML^2(\mathbb{C} \setminus \mathbb{R}_-; q)$ such that

$$\frac{z^n}{\sqrt{n!}} \mapsto \frac{z^{qn}}{\sqrt{\Gamma(qn+1)}}$$

Theorem

There exists an isometry from $\mathcal{L}^2(\mathbb{R}, dx)$ *onto* $ML^2(\mathbb{C} \setminus \mathbb{R}^-; q)$ *given by*

$$B_q = U_q B : \mathcal{L}^2(\mathbb{R}, dx) \to ML^2(\mathbb{C} \setminus \mathbb{R}^-; q).$$

Definition

Define the operators Z_q and Y_q on $ML^2(\mathbb{C} \setminus \mathbb{R}; q)$ as follows:

$$Z_q : \mathrm{Dom}(Z_q) \to ML^2(\mathbb{C} \setminus \mathbb{R}; q) \quad (Z_q f)(z) = z^q f(z)$$

and

$$Y_q : \mathrm{Dom}(Y_q) \to ML^2(\mathbb{C} \setminus \mathbb{R}; q) \quad (Y_q f)(z) = D_*^q f(z).$$

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The operators Z_q and Y_q are closed,

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For convenience we will apply the following relabeling:

$$A_q = Y_q \quad \text{and} \quad A_q^* = Z_q$$

We note that on our orthonormal basis for $ML^2(\mathbb{C} \setminus \mathbb{R}_-; q)$ given by

$$\{g_n\} = \left\{ \frac{z^{qn}}{\Gamma(qn+1)} \right\}$$

we have that

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This tells us that the $g_n(z)$ lie in the domains of both operators.

Proposition

Let $S_q : \mathcal{L}^2(\mathbb{R}, dx) \to \mathcal{L}^2(\mathbb{R}, dx)$ *be the operator given by*

$$S_q h_n = \sqrt{\frac{\Gamma(q(n+1)+1)}{(n+1)\Gamma(qn+1)}}\, h_n.$$

S_q *is a compact self-adjoint operator.*

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S_q *is a compact self-adjoint operator.*

We define the following operators:

Definition

$$W_q : \mathrm{Dom}(W) \to \mathcal{L}^2(\mathbb{R}, dx); \qquad W_q = S_q W$$

$$W_q^* : \mathrm{Dom}(W^*) \to \mathcal{L}^2(\mathbb{R}, dx); \qquad W_q^* = W^* S_q$$

Note that $S_q(\mathrm{Dom}(W^*)) \subseteq \mathrm{Dom}(W^*)$

Given the orthonormal basis of the hermite functions we note:

$$W_q h_n = S_q W h_n = \sqrt{\frac{\Gamma(qn+1)}{\Gamma(q(n-1)+1)}}\, h_{n-1}$$

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Hence

Proposition

For $0 < q \leq 1$

$$B_q^{-1} A_q B_q = W_q$$

and

$$B_q^{-1} A_q^* B_q = W_q^*.$$

Proposition

For $0 < q \leq 1$ *the function* $w(z) = \frac{1}{q\pi}|z|^{\frac{2}{q}-2}e^{-|z|^{\frac{2}{q}}}$ *is the unique, continuous, radial weight such that for polynomials* $p(z) = a_n z^{qn} + \ldots + a_0$ *and* $s(z) = b_m z^{qm} + \ldots b_0$ *under the inner product*

$$\langle p(z), s(z) \rangle = \int_{\mathbb{C}} p(z^{1/q})\overline{r(z^{1/q})}w(z)dz$$

the operations of multiplication by z^q *and Caputo differentiation are adjoint, and the function* $f(z) = 1$ *has norm 1.*

Thanks !