Equidistribution of zeros of Random Orthogonal polynomials

Koushik Ramachandran (OSU)
(joint work with Igor Pritsker)

SEAM, March 17th, 2017

Introduction

Consider a sequence of random polynomials $P_n(z) = \sum_{i=0}^{n} \xi_i z^i$, where ξ_i are i.i.d random variables. We are interested in the distribution of zeros in the complex plane.

Let Z_{kn}, $1 \leq k \leq n$, denote the zeros of P_n. Form the normalized counting measure

$$\tau_n = \frac{1}{n} \sum_{k=0}^{n} \delta_{Z_{kn}}.$$

The basic question here is to understand when these sequence of random measures converge and if so, to identify the limit.

The following theorem answers this question completely.

Theorem (Ibragimov-Zaporozhets 2013)

Consider a sequence of random polynomials $P_n(z) = \sum_{i=0}^{n} \xi_i z^i$*, where* ξ_i *are i.i.d random variables. Then,* $\tau_n \xrightarrow{w} \mu_{\mathbb{T}}$ *a.s iff* $\mathbb{E}[\log^+ |\xi_0|] < \infty$.

Here $\mu_{\mathbb{T}}$ is the uniform measure on the unit circle.

Following is a rough sketch of proof

- Apply Borel Cantelli to show that $\mathbb{E}[\log^+|\xi_0|] < \infty$ implies $\limsup|\xi_n|^{\frac{1}{n}} = 1$ a.s.
- Therefore $f = \sum_{i=0}^{\infty} \xi_i z^i$ has radius of convergence 1 a.s and is non constant and analytic in the disc.
- $P_n \to f$ uniformly on compacts. Hence zeros cannot accumulate inside the disc.
- The reversed polynomial $Q_n(z) = z^n P_n(\frac{1}{z})$ has same distribution as P_n and hence the zeros do not accumulate outside the disc. So zeros cluster around $\mathbb{T}$ a.s.

Natural question is to consider random polynomials spanned by other basis.

Definition

Let G be bounded Jordan domain in $\mathbb{C}$. Let dm be Lebesgue measure on G. The Bergman orthogonal polynomials B_i are uniquely defined by

$$\int_G B_k(z)\overline{B_l(z)}dm(z) = \delta_{kl}$$

and the requirement that their leading coefficient is positive.

If we start with $d\sigma$ the arclength measure and repeat this procedure, then we get Szegő orthogonal polynomials.

Basic problem

Let G be a bounded Jordan domain. Consider a sequence of random orthogonal polynomials $P_n(z) = \sum_{i=0}^{n} \xi_i B_i(z)$ with ξ_i being i.i.d random variables.

Question: Can we give necessary and sufficient conditions on ξ_i for the measures τ_n to converge and if so, what is the limit?

Theorem (I.P and K. R, 2016)

Let G be a bounded Jordain domain whose boundary $\partial G = E$ is an analytic curve. Let ξ_i be non trivial i.i.d random variables and consider a sequence of random polynomials defined by $P_n(z) = \sum_{i=0}^{n} \xi_i B_i(z)$. Then $\tau_n \xrightarrow{w} \mu_E$ a.s iff $\mathbb{E}[\log^+ |\xi_0|] < \infty$.

Here μ_E denotes equilibrium measure of G i.e, it is the unique probability measure μ which minimizes the energy $I(\mu) = -\int_G \int_G \log|z-w| d\mu(z) d\mu(w)$ among all probability measures supported on $\overline{G}$.

Theorem remains true if B_i is replaced by S_i, the Szegő orthogonal polynomials on G.

The ingredients for the proof are as follows.

1) Borel Cantelli gives $\limsup |\xi_n|^{\frac{1}{n}} = 1$ a.s.

2) On a compact $K \subset G$, it is a classical fact that $\limsup ||B_n||_K^{\frac{1}{n}} < 1$. This fact along with 1) shows that with probability 1, the series $\sum_{i=0}^{\infty} \xi_i B_i(z)$ converges uniformly on compacts to a non constant analytic function f. $P_n \to f$ uniformly on compacts, so zeros cannot accumulate inside G.

3) To show that zeros don't accumulate outside G and that they are distributed according to equilibrium configuration, we use the following theorem of Grothmann.

Theorem G.

If a sequence of polynomials $P_n(z)$, $\deg(P_n) = n$, satisfies

$$\limsup_{n\to\infty} \|P_n\|_E^{1/n} \le 1, \tag{1}$$

for any closed set $K \subset G^\circ$

$$\lim_{n\to\infty} \tau_n(K) = 0, \tag{2}$$

and there is a compact set $S \subset \Omega$ such that

$$\liminf_{n\to\infty} \max_{z\in S} \left(\frac{1}{n} \log |P_n(z)| - g_\Omega(z,\infty) \right) \ge 0, \tag{3}$$

then the zero counting measures τ_n of P_n converge weakly to μ_E as $n \to \infty$.

$\Omega = \mathbb{C}^* \setminus G$ and $g_\Omega(z, \infty)$ denotes the Green function of Ω with pole at infinity.

Heuristically, theorem on distribution of zeros says that for large n the zeros of the random polynomial P_n behave as if they were point charges which are attaining equilibrium configuration (for domains with nice boundary).

Thank you for your attention!!