Two weighted nonlinear Calderón-Zygmund estimates for nonlinear elliptic equations

Tuoc Phan
University of Tennessee, Knoxville, TN, USA

33rd Southeastern Analysis Meeting, Knoxville, TN

Supported by Simons Foundation: grant # 354889

Outline

- Introduction - Degenerate nonlinear elliptic equations
- Weighted Sobolev spaces, and weak solutions
- Regularity questions, and classical Calderón-Zygmund regularity estimates
- Asymptotically Uhlenbeck type equations, Sawyer's condition on weights
- Our main theorem, and ideas in its proof
- Conclusion, remarks

- Let $\Omega \subset \mathbb{R}^n$ be open, bounded, with boundary $\partial\Omega$ (could be non-smooth).
- We study the regularity of weak solutions of the nonlinear elliptic equations

$$\begin{cases} \operatorname{div}[\mathbb{A}(x, u, \nabla u)] & = & \operatorname{div}[\mathbf{F}(x)] & \text{in} & \Omega, \\ u & = & g(x) & \text{on} & \partial\Omega, \end{cases}$$

- $u : \Omega \to \mathbb{R}^n$ is an unknown solution (in weak sense), $\mathbf{F} : \Omega \to \mathbb{R}^n$ is a given measurable vector field
- $\mathbb{A} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a given vector field:
  - (i) $\mathbb{A}(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$, $\mathbb{A}(\cdot, z, \eta)$ is measurable.
  - (ii) $\mathbb{A}(\cdot, z, \eta)$ is could be singular + degenerate: There exists $\Lambda > 0$ and $\mu \in A_2$, Muckenhoupt class of weights, such that

$$|\mathbb{A}(x, z, \eta)| \leq \Lambda \mu(x) \eta, \quad \Lambda^{-1} \mu(x) |\eta|^2 \leq \langle \mathbb{A}(x, z, \eta), \eta \rangle,$$

for a.e. $x \in \Omega$, and for all $(z, \eta) \in \mathbb{R} \times \mathbb{R}^n$.

Note: The equation is called uniformly elliptic if $\mu = 1$.

Why degenerate equations are of interest?

- Some nice analysis and fun to work on
- Nonlinear equations can be considered as linear degenerate equations. For example: Once we know there is a solution u of the equation

$$\mathrm{div}[\phi(u)\nabla u] = \mathrm{div}(\mathbf{F}),$$

  we can be considered as

$$\mathrm{div}[a(x)\nabla u] = \mathrm{div}(\mathbf{F}), \quad \text{when we take} \quad a(x) = \phi(u(x)).$$

- Some non-local PDEs can be studied through some other generate equations.
- Degenerate equations also appear naturally from some models: Math finance, porous media.

A_p-Muckenhoupt weights

- A non-negative locally integrable function $\mu : \mathbb{R}^n \to \mathbb{R}$ is in A_p, $1 < p < \infty$ if and only if

$$[\mu]_{A_p} := \sup_{\text{ball } B \subset \mathbb{R}^n} \left(\fint_B \mu(x) dx \right) \left(\fint_B \mu^{\frac{1}{1-p}}(x) dx \right)^{p-1} < \infty.$$

- In particular, $\mu \in A_2$ weights if

$$[\mu]_{A_2} := \sup_{\text{ball } B \subset \mathbb{R}^n} \left(\fint_B \mu(x) dx \right) \left(\fint_B \mu^{-1}(x) dx \right) < \infty.$$

- Typical example: $\mu(x) = |x|^\alpha$, then $\mu \in A_p$ if and only if $-n < \alpha < n(p-1)$.
- It turns out that the class $\mathbb{A}_p$ is monotone in p: $A_p \subset A_q$ for $1 < p < q$. Also, observe that $\mu \in A_2$, then $\mu^{-1} \in A_2$.

Weighted Sobolev spaces

Let us fix $p > 1$, $\Omega \subset \mathbb{R}^n$, and $\mu : \mathbb{R}^n \to [0, \infty]$ is a weight function, and $\mu \in A_p$.

- Weighted Lebesgue space $L^p(\Omega, \mu)$ consists of all measurable function $f : \Omega \to \mathbb{R}$ such that

$$\|f\|^p_{L^p(\Omega,\mu)} := \int_\Omega |f(x)|^p \mu(x) dx < \infty.$$

- Weighted Sobolev space $W^{1,p}(\Omega, \mu)$ consists of all measurable function $f \in L^p(\Omega, \mu)$ such that $\partial_{x_k} f \in L^p(\Omega, \mu)$, for $k = 1, 2, \cdots, n$, and

$$\|f\|_{W^{1,p}(\Omega,\mu)} = \|f\|_{L^p(\Omega,\mu)} + \sum_{k=1}^{n} \left\|\partial_{x_k} f\right\|_{L^p(\Omega,\mu)}.$$

- $W^{1,p}_0(\Omega, \mu)$ is the closure of $C^\infty_0(\Omega)$ in $W^{1,p}(\Omega, \mu)$.

Weak solutions

Definition

Let $\mathbf{F}$ such that $|\mathbf{F}/\mu| \in L^2(\Omega, \mu)$ and $g \in W^{1,2}(\Omega, \mu)$ with some fixed $\mu \in A_2$. A function $u \in W^{1,2}\Omega, \mu)$ is a weak solution of

$$\left\{ \begin{array}{lcll} \mathrm{div}[\mathbb{A}(x,u,\nabla u)] & = & \mathrm{div}[\mathbf{F}], & \text{in} \quad \Omega, \\ u & = & g, & \text{on} \quad \partial\Omega, \end{array} \right.$$

if $u - g \in W_0^{1,2}(\Omega, \mu)$ and

$$\int_\Omega \langle \mathbb{A}(x,u,\nabla u), \nabla\varphi \rangle dx = \int_\Omega \langle \mathbf{F}, \nabla\varphi \rangle dx, \quad \varphi \in W_0^{1,2}(\Omega, \mu).$$

Regularity question to study

- For either linear equation

$$\text{div}[\mathbb{A}(x)\nabla u] = \text{div}[\mathbf{F}], \quad \text{in} \quad \Omega$$

or more complicated nonlinear one

$$\text{div}[\mathbb{A}(x, u, \nabla u)] = \text{div}[\mathbf{F}], \quad \text{in} \quad \Omega$$

- Weak solutions: Standard existence theory usually provides solutions in energy space: In our case:

$$u \in W^{1,2}(\Omega, \mu), \quad \text{given} \quad g \in W^{1,2}(\Omega, \mu), \quad |\mathbf{F}/\mu| \in L^2(\Omega, \mu).$$

- Regularity question: Given that the data is more regular, for example $|\mathbf{F}/\mu| \in L^p(\Omega, \omega)$, and $g \in W^{1,p}(\Omega, \omega)$ with some other weight ω and $p \geq 2$, will it true that

$$u \in W^{1,p}(\Omega, \omega)?$$

If yes, quantify the estimate (this is known as Calderón-Zygmund estimate)!

Classical CZ estimates for linear uniformly elliptic equations

- Consider the linear elliptic equation

$$\mathrm{div}[\mathbb{A}(x)\nabla u] = \mathrm{div}[\mathbf{F}], \quad \text{in} \quad B_2.$$

- If $\mathbb{A}$ is uniformly elliptic, $[[\mathbb{A}]]_{\mathrm{BMO}(B_2)} \ll 1$, and $u \in W^{1,2}(B_2)$ is a weak solution, then

$$\fint_{B_1} |\nabla u|^p dx \leq C(n,p)\left[\fint_{B_2} |\mathbf{F}|^p dx + \left(\fint_{B_2} |\nabla u|^2 dx\right)^{p/2}\right].$$

- This is first proved by Calderón-Zygmund with $\mathbb{A} \in C(B_2)$, then extended by many mathematicians to $\mathbb{A} \in \mathrm{BMO}$: F. Chiarenza, M. Frasca, and P. Longo, (1991); L.A. Caffarelli and I. Peral (1998); L. Wang - S-.S. Byun (200s); Krylov (200s), ...

Remarks on the homogeneity in the CZ theory

- The equations

$$\mathrm{div}[\mathbb{A}(x)\nabla u] = \mathrm{div}[\mathbf{F}], \qquad \text{in} \quad B_2$$

is invariant under the dilation: If u is a solution, then $u_\lambda(x) := u(\lambda x)/\lambda$ is also a solution

$$\mathrm{div}[\mathbb{A}_\lambda(x)\nabla u_\lambda] = \mathrm{div}[\mathbf{F}_\lambda], \quad \text{in} \quad B_{2/\lambda}.$$

where $\mathbb{A}_\lambda(x) = \mathbb{A}(\lambda x), \quad \mathbf{F}_\lambda(x) = \mathbf{F}(\lambda x)$.

- The estimate

$$\fint_{B_1} |\nabla u|^p dx \leq C(n,p)\left[\fint_{B_2} |\mathbf{F}|^p dx + \left(\fint_{B_2} |\nabla u|^2 dx\right)^{p/2}\right]$$

is also invariant under this dilation.

$\Longrightarrow$ The CZ reflects the true homogeneity of the PDEs. See also from the Hardy-Littilewood maximal functions.

Difficulties in our questions

For our equation

$$\begin{cases} \operatorname{div}[\mathbb{A}(x,u,\nabla u)] & = & \operatorname{div}[\mathbf{F}(x)] & \text{in} & \Omega, \\ u & = & g(x) & \text{on} & \partial\Omega, \end{cases}$$

- It is non-uniformly elliptic becuase $\mathbb{A}(x,z,\eta) \sim \mu(x)\eta$ with $\mu \in A_2$.
- The homogeneity of the PDE is broken due to the dependent of $\mathbb{A}$ on u. The CZ is not expected in this case by experts.

  Note: However, in case $\mathbb{A}$ is independent on u, the PDE is still invariant with the dilation. The CZ is available in this case indeed.
- We would love to have two weighted estimates: Weak solution $u \in W^{1,2}(\Omega,\mu)$ and the data $|\mathbf{F}/\mu| \in L^p(\Omega,\omega), g \in W^{1,p}(\Omega,\omega)$, with another weight ω. Hope to obtain

$$\nabla u \in L^p(\Omega,\omega)?$$

Asymptotically Uhlenbeck vector fields

Definition

$\mathbb{A} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is asymptotically Uhlenbeck if there is a symmetric measurable matrix $\tilde{\mathbb{A}} : \Omega \to \mathbb{R}^{n \times n}$, and a bounded continuous function $\omega_0 : \overline{\mathbb{K}} \times [0, \infty) \to [0, \infty)$ such that

$$|\mathbb{A}(x, z, \eta) - \tilde{\mathbb{A}}(x)\eta| \leq \omega_0(z, |\eta|)\Big[1 + |\eta|\Big]\mu(x),$$

for for all most every $x \in \Omega, \quad \forall\ (z, \eta) \in \mathbb{R} \times \mathbb{R}^n$, and

$$\lim_{s \to \infty} \omega_0(z, s) = 0, \text{ uniformly in } z, \text{ for } z \in \overline{\mathbb{K}}.$$

A prototypical example is

$$\mathbb{A}(x, z, \eta) = a(x, z, |\eta|)\eta, \text{ with } \lim_{s \to \infty} a(x, z, s) = \tilde{a}(x), \quad \text{uniformly in } z \in \mathbb{R}$$

and

$$\tilde{\mathbb{A}}(x) = \tilde{a}(x)\mathbb{I}_n.$$

Sawyer's condition

Definition

Let μ, ω be any two positive, locally finite Borel measures on $\mathbb{R}^n$ and let $1 < p < \infty$. The pair of measures (μ, ω) is said to satisfy the p-Sawyer's condition if there is a constant $C > 0$ such that

$$\int_B \left(\mathcal{M}_\mu(\chi_B \frac{d\sigma}{d\mu})\right)^p d\omega \leq C\sigma(B), \quad \forall \text{ ball } B \subset \mathbb{R}^n,$$

where χ_B is the characteristic function of the ball B,

$$d\sigma = \left(d\mu/d\omega\right)^{p'} d\omega, \quad \text{and} \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Sawyer two weighted estimates for Hardy-littlewood maximal function

Theorem (E. R. Sawyer - 1982))

Let μ, ω be any two positive, locally finite Borel measures on $\mathbb{R}^n$ and let $1 < q < \infty$. Then,

$$\left\|\mathcal{M}_\mu\right\|_{L^q(\mathbb{R}^n,\omega)\to L^q(\mathbb{R}^n,\omega)} \leq C,$$

if and only if the pair (μ, ω) satisfies the q-Sawyer's condition, where

$$\mathcal{M}_\mu f(x) = \sup_{\rho>0} \fint_{B_\rho(x)} |f(y)| d(\mu(y)).$$

See also I. Verbitsky (1992), and D. Cruz-Uribe (2000).

Our main results

Theorem (T. P. 2017)

Assume that $\mathbb{A}(x,z,\eta)$ *is degenerate as* $\mu(x)\eta$ *and* $\mathbb{A}$ *is asymptotically Uhlenbeck. Assume also that* $[[\tilde{\mathbb{A}}]]_{\mathrm{BMO}(\Omega,\mu)} \ll 1$, *and* Ω *is sufficiently flat. If* $u \in W^{1,p}(\Omega,\mu)$ *is a weak solution of*

$$\begin{cases} div[\mathbb{A}(x,u,\nabla u)] & = & div[\mathbf{F}(x)] & \text{in} \quad \Omega, \\ u & = & g(x) & \text{on} \quad \partial\Omega, \end{cases}$$

there is a constant $C = C(p,q,\Lambda,M_0,M_1,M_2,\omega_0,n) > 0$ *such that the estimate*

$$\int_{\Omega_R(y_0)} |\nabla u|^p \omega(x)dx \leq C\Bigg[\int_{\Omega_{2R}(y_0)} |\nabla g|^p\omega(x)dx + \int_{\Omega_{2R}(y_0)} |\mathbf{F}/\mu|^p\omega(x)dx + \omega(\Omega_R(y_0))\left\{\left(\frac{1}{\mu(B_{2R}(y_0))}\int_{\Omega_{2R}(y_0)} |\nabla u|^2\mu(x)dx\right)^{p/2} + 1\right\}\Bigg]$$

where $\Omega_R(y_0) = \Omega \cap B_R(y_0)$ *and* $y_0 \in \overline{\Omega}$, (μ,ω) *is the* $\frac{p}{2}$*-Sawyer's*

Ideas and main steps the proof

- Clearly, the main task is to deal with the inhomogeneity of the nonlinear PDE.
- Enlarge the class of equation to study: For $\lambda > 0$, study the class of equations with parameter λ

$$\begin{cases} \mathrm{div}[\mathbb{A}_\lambda(x, u, \nabla u)] = \mathrm{div}[\mathbf{F}(x)] & \text{in} \quad \Omega, \\ u = g(x) & \text{on} \quad \partial\Omega, \end{cases}$$

  where $\mathbb{A}_\lambda(x, z, \eta) = \mathbb{A}(x, \lambda z, \lambda \eta)/\lambda$.
- This class of nonlinear degenerate PDEs are invariant under the dilation: $u_s(x) = u(sx)/s$
- Establish the CZ theory for the above equation when $\lambda \geq \lambda_0$, with some $\lambda_0 = \lambda_0(\Lambda, n, \Omega)$ sufficiently large.
- Use scaling argument to obtain the CZ estimate for $\lambda = 1$.

Conclusion

- We study regularity estimate of Calderón-Zygmund type for nonlinear equations of Uhlenbeck type, with non-homogeneous boundary condition.
- Two weighted nonlinear Calderón-Zygmund type regularity estimates are established.
- Results are new, even for the uniformly elliptic case, because of the dependent of the nonlinearity of the equations on solutions.
- Linear equations, one weighted Calderón-Zygmund type regularity estimates are established earlier, by D. Cao - T. Mengesha - T. P. (2016).

THANK YOU AND QUESTIONS