

The spectrum of the Neumann-Poincaré operator on domains with corners and conical points

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Layer potentials for the Laplacian

- $\Gamma \subset \mathbb{R}^3$ a connected Lipschitz surface with surface measure $d\sigma$, or a Lipschitz curve $\Gamma \subset \mathbb{R}^2$, enclosing a bounded open domain $\text{int}(\Gamma)$.
- Layer potential operator

$$K^\Gamma f(\mathbf{r}) = \frac{1}{2\pi} \text{p.v.} \int_\Gamma K^\Gamma(\mathbf{r}, \mathbf{r}') f(\mathbf{r}') d\sigma(\mathbf{r}'), \quad \mathbf{r} \in \Gamma,$$

based on the normal derivative of the Newtonian kernel

$$K^\Gamma(\mathbf{r}, \mathbf{r}') = \frac{\langle \mathbf{r} - \mathbf{r}', \mathbf{v}_r \rangle}{|\mathbf{r}' - \mathbf{r}|^3} \quad (3\text{D}), \quad K^\Gamma(\mathbf{r}, \mathbf{r}') = 2 \frac{\langle \mathbf{r} - \mathbf{r}', \mathbf{v}_r \rangle}{|\mathbf{r}' - \mathbf{r}|^2} \quad (2\text{D}),$$

where \mathbf{v}_r denotes the outward unit normal of Γ .

- Adjoint $(K^\Gamma)^*$ w.r.t. $L^2(\Gamma)$ -pairing is called the *Neumann–Poincaré operator*.

Double layer potential

- Harmonic double layer potential

$$D^\Gamma f(\mathbf{r}) = -\frac{1}{\pi} \int_\Gamma K^\Gamma(\mathbf{r}', \mathbf{r}) f(\mathbf{r}') d\sigma(\mathbf{r}'), \quad \mathbf{r} \in \mathbb{R}^3 \setminus \Gamma = \text{int}(\Gamma) \cup \text{ext}(\Gamma).$$

- Jump formulas

$$\text{Tr}_{\text{int}} D^\Gamma f = \frac{1}{2}((K^\Gamma)^* f + f), \quad \text{Tr}_{\text{ext}} D^\Gamma f = \frac{1}{2}((K^\Gamma)^* f - f).$$

- Hence the interior/exterior Dirichlet problems (finding a harmonic function in $\text{int}(\Gamma)$ or $\text{ext}(\Gamma)$ with prescribed boundary values f on Γ) can be solved by studying the Fredholm theory of $K^\Gamma + I$ and $K^\Gamma - I$, respectively.

Many spectra

- If Γ is non-smooth, the spectrum of $K^\Gamma: X \rightarrow X$ depends on X , the space for the boundary data.
- If $1 < p \leq 2$ and Γ Lipschitz, then $K^\Gamma \pm I: L^p(\Gamma) \rightarrow L^p(\Gamma)$ is always Fredholm. Verchota '84, after Calderón and Coifman, McIntosh, Meyer '82 proved boundedness.
- If $\Gamma \subset \mathbb{R}^2$ is a curvilinear polygon, then there always exist $p_0 > 2$, depending on the opening angles, such that $K^\Gamma \pm I: L^{p_0}(\Gamma) \rightarrow L^{p_0}(\Gamma)$ is not Fredholm.

Many spectra

- Reason: If $\Gamma \subset \mathbb{R}^2$ is an infinite wedge, the model domain for a corner in 2D. Then $K^\Gamma : L^p(\Gamma) \rightarrow L^p(\Gamma)$ isometrically identifies with a block matrix of Mellin convolution operators with different kernels for different p .
- Fabes, Jodeit, Lewis '87: Similar results for $\Gamma \subset \mathbb{R}^3$ when
 - ▶ Γ is an infinite straight cone (rotationally symmetric conical point).
 - ▶ Γ is an octant of \mathbb{R}^3 (model for a polyhedron). K^Γ identifies with a system of convolution operators against a non-commutative group. Polyhedrons are surprisingly complicated, little is known in the non-elliptic case.

Single layer potentials and the energy space

- Single layer potential

$$S^\Gamma f(\mathbf{r}) = \int_\Gamma S^\Gamma(\mathbf{r}, \mathbf{r}') f(\mathbf{r}') d\sigma(\mathbf{r}'), \quad \mathbf{r} \in \mathbb{R}^3,$$

where

$$S^\Gamma(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \text{ (3D)}, \quad S^\Gamma(\mathbf{r}, \mathbf{r}') = \frac{1}{\pi} \log \frac{1}{|\mathbf{r} - \mathbf{r}'|} \text{ (2D)}.$$

- Energy space \mathcal{E} : distributions f such that

$$\|f\|_{\mathcal{E}}^2 = \langle S^\Gamma f, f \rangle_{L^2(\Gamma)} = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla S^\Gamma f|^2 dV < \infty.$$

\mathcal{E} non-isometrically identifies with $H^{-1/2}(\Gamma)$, the Sobolev space of index $-1/2$ on Γ .

Single layer potentials and the energy space

$$\langle f, g \rangle_{\mathcal{E}} = \langle S^{\Gamma} f, g \rangle_{L^2(\Gamma)} = \frac{1}{2} \int_{\mathbb{R}^3} \langle \nabla S^{\Gamma} f, \nabla S^{\Gamma} g \rangle_{\mathbb{R}^3} dV.$$

- Plemelj formula:

$$S^{\Gamma} K^{\Gamma} = (K^{\Gamma})^* S^{\Gamma}$$

- $K^{\Gamma}: \mathcal{E} \rightarrow \mathcal{E}$ is self-adjoint, and hence has a real spectrum (the spectrum on $L^2(\Gamma)$ is only real when Γ is smooth).

Real vs non-real spectrum

$\Gamma \subset \mathbb{R}^3$ a bounded domain with conical point of opening angle 2α .

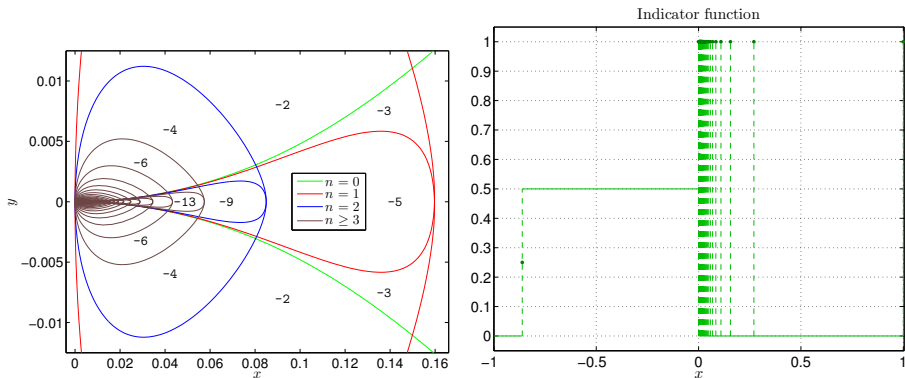


Figure: (a): Spectrum on $L^2(\Gamma)$ ($2\alpha = 5\pi/18$). (b): Spectrum on \mathcal{E} for mode $n=0$ ($2\alpha = 31\pi/18$).

Spectrum of K^Γ and the transmission problem

- Transmission problem

$$\begin{cases} \int_{\mathbb{R}^3} |\nabla U|^2 dV < \infty, \\ \Delta U(\mathbf{r}) = 0, \quad \mathbf{r} \in \mathbb{R}^3 \setminus \Gamma, \\ \text{Tr}_{\text{int}} U(\mathbf{r}) = \text{Tr}_{\text{ext}} U(\mathbf{r}), \quad \mathbf{r} \in \Gamma, \\ \partial_{\mathbf{v}}^{\text{ext}} U(\mathbf{r}) = \epsilon_r \partial_{\mathbf{v}}^{\text{int}} U(\mathbf{r}) - g(\mathbf{r}), \quad \mathbf{r} \in \Gamma. \end{cases}$$

where $\epsilon_r \neq 1$ and $g \in \mathcal{E}$.

- Ansatz $U = S^\Gamma f$: equivalent to find $f \in \mathcal{E}$ such that

$$(K^\Gamma - z)f = \frac{g}{1 - \epsilon_r}, \quad z = -\frac{1 + \epsilon_r}{1 - \epsilon_r}.$$

- Special case: $g(\mathbf{r}) = \mathbf{e} \cdot \mathbf{v}_r$, $\mathbf{e} \in \mathbb{R}^3$. Involved in computing the polarizability tensor of $\text{int}(\Gamma)$.

2D: Curvilinear polygons

- Curvilinear polygon $\Gamma \subset \mathbb{R}^2$ with a finite number of corners $(a_j)_{j=1}^N$ with (interior) angles $(\alpha_j)_{j=1}^N$.
- The (largest) **Fredholm eigenvalue** of Γ is

$$\sup |\sigma(K^\Gamma, \mathcal{E}) \setminus \{1\}|.$$

- Zaremba (1904), Carleman (1916), and Radon (1919): Essential spectral radius of $K^\Gamma: C(\Gamma) \rightarrow C(\Gamma)$ is given by

$$|\sigma_{\text{ess}}(K, C(\Gamma))| = \max_{1 \leq j \leq N} \left| 1 - \frac{\alpha_j}{\pi} \right|.$$

2D: Curvilinear polygons

Shelepov '69, Lewis '91, I. Mitrea '02: Essential spectrum of $K^\Gamma : L^p(\Gamma) \rightarrow L^p(\Gamma)$, $1 < p < \infty$, consists of the union of “bowtie domains”, one for each corner of Γ . The Fredholm index $\text{index}(K^\Gamma - z)$ is the winding number of z w.r.t. the essential spectrum. All eigenvalues must be real.

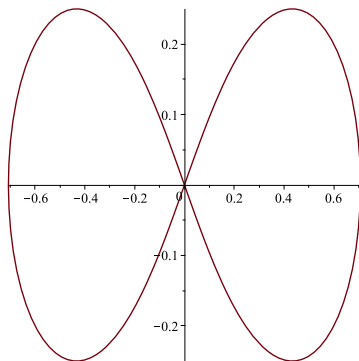


Figure: $p = 2$, one corner of angle $\alpha = \pi/2$. $\text{index}(K^\Gamma - z) = -1$ inside bowties.

2D: The Ahlfors-Beurling transform

Let

$$L_a^2(\text{int}(\Gamma)) = \left\{ f \in \text{Hol}(\text{int}(\Gamma)) : \int_{\text{int}(\Gamma)} |f(z)|^2 dA(z) < \infty \right\}, \quad dA(z) = dx dy.$$

The anti-linear Ahlfors–Beurling transform $T_\Gamma : L_a^2(\text{int}(\Gamma)) \rightarrow L_a^2(\text{int}(\Gamma))$ is given by

$$T_\Gamma f(z) = \frac{1}{\pi} \text{p.v.} \int_{\text{int}(\Gamma)} \frac{\overline{f(\zeta)}}{(\zeta - z)^2} dA(\zeta), \quad f \in L_a^2(\text{int}(\Gamma)), z \in \text{int}(\Gamma).$$

Theorem (Bergman, Schiffer 1951)

If $x \in \mathbb{R}$ and $x \neq 1$, then x is an eigenvalue of K^Γ if and only if it is an eigenvalue of T_Γ .

The Ahlfors-Beurling transform

Theorem (Khavinson, Putinar, Shapiro 2007)

If Γ is Lipschitz, then $K^\Gamma: \mathcal{E}_0 \rightarrow \mathcal{E}_0$ is similar to T_Γ (as an \mathbb{R} -linear operator). In particular

$$\sigma(K^\Gamma, \mathcal{E}) = \sigma_{\mathbb{R}}(T_\Gamma) \cup \{1\}.$$

Theorem (Ahlfors 1952)

The Ahlfors inequality: If Γ is a quasi-circle, then

$$\|T_\Gamma\| \leq q_\Gamma,$$

where q_Γ is the quasi-conformal reflection coefficient of Γ . In particular, if Γ is not a circle or a line, then $\|T_\Gamma\| < 1$.

2D: Essential spectrum

Theorem (P., Putinar, 2017)

Let $K^\Gamma : \mathcal{E} \rightarrow \mathcal{E}$ be the adjoint of the Neumann–Poincaré operator, for a C^2 -smooth curvilinear polygon $\Gamma \subset \mathbb{C}$ with angles $\alpha_1, \dots, \alpha_N$. Then

$$\sigma(K^\Gamma, \mathcal{E}) = \left\{ x \in \mathbb{R} : |x| \leq \max_{1 \leq j \leq N} \left| 1 - \frac{\alpha_j}{\pi} \right| \right\} \cup \{\lambda_k\}.$$

where $\{\lambda_k\}$ is a sequence of **real** eigenvalues with no limit point outside the interval.

2D: Proof strategy

- Step 1 (**complex analysis**): Compute the spectrum of T_{W_α} for the wedge

$$W_\alpha = \{z \in \mathbb{C} : |\arg z| = \alpha/2\}.$$

Based on Fourier analysis of $L^2_a(\text{int}(W_\alpha))$ via group of unitaries

$$U_t f(z) = e^t f(e^t z).$$

Due to homogeneity of kernel:

$$U_t T_{W_\alpha} = T_{W_\alpha} U_t.$$

- Step 2 (**trick**): Localize the operator K^{W_α} to each corner (the wedge has two corners!).

2D: Proof strategy

- Step 3 (**complex analysis**): Perturbation argument to deal with curvilinear corners, not just corners arising from two straight lines. Uses conformal mapping and results on their regularity.
- Step 4 (**real analysis, operator theory**): Localize the operator K^Γ to the corners of any curvilinear polygon.

2D: refinements

Theorem (H. Kang, M. Lim, S. Yu, arXiv 2015)

For the wedge W_α , explicit spectral resolution (i.e. diagonalization) of $K^{W_\alpha} : \mathcal{E} \rightarrow \mathcal{E}$. Real variable techniques, by understanding explicitly the scalar product

$$\langle f, g \rangle_{\mathcal{E}} = \langle S^{W_\alpha} f, g \rangle_{L^2(W_\alpha)}.$$

Theorem (E. Bonnetier, H. Zhang, arXiv 2017)

A different approach to the theorem in the case that the corner is made of straight lines, leading to an understanding how the singular part of the solution ρ to $(K^\Gamma - z)\rho = g$ behaves when $z \notin \sigma_{\text{ess}}(K^\Gamma, \mathcal{E})$.

3D: Rotationally symmetric conical points

Let Γ be a closed surface of revolution with a conical point of opening angle 2α , obtained by revolving a C^4 -curve γ .

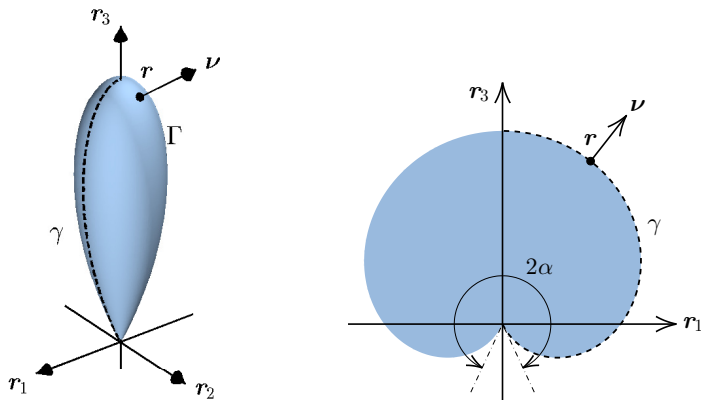


Figure: (a): An axially symmetric surface Γ with a conical point of opening angle $2\alpha = 5\pi/18$. (b): A cross-section of $\text{int}(\Gamma)$ with opening angle $2\alpha = 31\pi/18$.

3D: Modal operators

- Parametrization:

$$\mathbf{r}(t, \theta) = (\gamma_1(t) \cos \theta, \gamma_1(t) \sin \theta, \gamma_2(t)), \quad \theta \in [0, 2\pi], 0 \leq t \leq 1.$$

- Kernel on K^Γ rotationally invariant,

$$K^\Gamma(t, \theta, t', \theta') = K^\Gamma(t, \theta - \theta', t', 0).$$

- Fourier mode kernels:

$$K_n^\gamma(t, t') = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} K^\Gamma(t, \theta, t', 0) d\theta, \quad 0 < t, t' \leq 1.$$

defining operators

$$K_n^\gamma f_n(t) = \int_0^1 K_n^\gamma(t, t') f_n(t') \gamma_1(t') |\gamma'(t')| dt', \quad 0 \leq t \leq 1.$$

3D: Modal operators

- For $K^\Gamma: L^2(\Gamma) \rightarrow L^2(\Gamma)$,

$$K^\Gamma \simeq_{\text{ue}} \bigoplus_{n=-\infty}^{\infty} K_n^\gamma,$$

where $K_n^\gamma: L^2(\gamma_1(t)|\gamma'(t)| dt) \rightarrow L^2(\gamma_1(t)|\gamma'(t)| dt)$.

- For $K^\Gamma: \mathcal{E} \rightarrow \mathcal{E}$,

$$K^\Gamma \simeq_{\text{ue}} \bigoplus_{n=-\infty}^{\infty} K_n^\gamma,$$

where $K_n^\gamma: \mathcal{E}_n \rightarrow \mathcal{E}_n$. Here

$$\|f_n\|_{\mathcal{E}_n}^2 = \langle S_n^\gamma f_n, g_n \rangle_{L^2(\gamma_1(t)|\gamma'(t)| dt)},$$

where S_n^γ is the modal version of S^Γ .

3D: Spectrum on $L^2(\Gamma)$

For $n \in \mathbb{Z}$, denote by Π_n the closed curve

$$\Pi_n = \left\{ \frac{P_{i\xi}^n(\cos \alpha) \frac{dP_{i\xi}^n}{dx}(-\cos \alpha) - P_{i\xi}^n(-\cos \alpha) \frac{dP_{i\xi}^n}{dx}(\cos \alpha)}{P_{i\xi}^n(-\cos \alpha) \frac{dP_{i\xi}^n}{dx}(\cos \alpha) + P_{i\xi}^n(\cos \alpha) \frac{dP_{i\xi}^n}{dx}(-\cos \alpha)} : -\infty \leq \xi \leq \infty \right\},$$

with orientation given by the ξ -variable. Here $P_{\lambda}^n(x)$ denotes an associated Legendre function of the first kind.

3D: Spectrum on $L^2(\Gamma)$

Theorem (Helsing, P., arXiv 2017)

The operator $K^\Gamma: L^2(\Gamma, d\sigma) \rightarrow L^2(\Gamma, d\sigma)$ has essential spectrum

$$\sigma_{\text{ess}}(K^\Gamma, L^2) = \bigcup_{n=-\infty}^{\infty} \Pi_n.$$

If $z \notin \sigma_{\text{ess}}(K^\Gamma, L^2)$, then $K^\Gamma - z$ has Fredholm index

$$\text{index}(K^\Gamma - z) = \sum_{n=-\infty}^{\infty} W(z, \Pi_n) = W(z, \Pi_0) + 2 \sum_{n=1}^{\infty} W(z, \Pi_n)$$

where $W(z, \Pi_n) \leq 0$ denotes the winding number of z with respect to Π_n and the right-hand side is always a finite sum.

All eigenvalues z must be **real**.

3D: Spectrum on $L^2(\Gamma)$

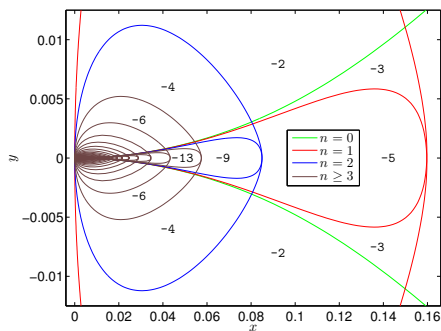
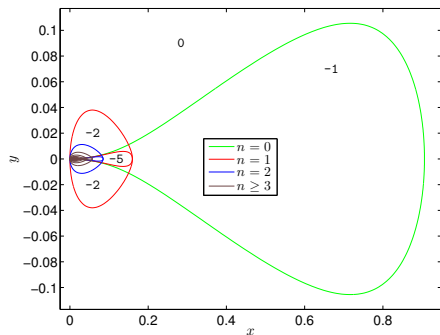


Figure: The essential spectrum of K^Γ for a surface Γ with a conical point of opening angle $2\alpha = 5\pi/18$. $\text{index}(K^\Gamma - z)$ is marked in the regions of non-essential spectrum. Zoom to the right.

3D: Spectrum on \mathcal{E}

Theorem (Helsing, P., arXiv 2017)

For $n \in \mathbb{Z}$, denote by $\Sigma_n \subset \mathbb{R}$ the closed interval

$$\Sigma_n = \left\{ \frac{P_{i\xi-1/2}^n(\beta) \frac{dP_{i\xi-1/2}^n}{dx}(-\beta) - P_{i\xi-1/2}^n(-\beta) \frac{dP_{i\xi-1/2}^n}{dx}(\beta)}{P_{i\xi-1/2}^n(-\beta) \frac{dP_{i\xi-1/2}^n}{dx}(\beta) + P_{i\xi-1/2}^n(\beta) \frac{dP_{i\xi-1/2}^n}{dx}(-\beta)} : -\infty \leq \xi \leq \infty \right\},$$

where $\beta = \cos \alpha$. Then the self-adjoint operator $K^\Gamma: \mathcal{E} \rightarrow \mathcal{E}$, where \mathcal{E} is the energy space of Γ , has essential spectrum

$$\sigma_{\text{ess}}(K^\Gamma, \mathcal{E}) = \bigcup_{n=-\infty}^{\infty} \Sigma_n.$$

Hence, the spectrum of K^Γ consists of this interval and a sequence of real eigenvalues $\{\lambda_k\}$ with no limit point outside of it,

$$\sigma(K^\Gamma, \mathcal{E}) = \{\lambda_k\} \cup \sigma_{\text{ess}}(K^\Gamma, \mathcal{E}).$$

3D: Spectrum on \mathcal{E}

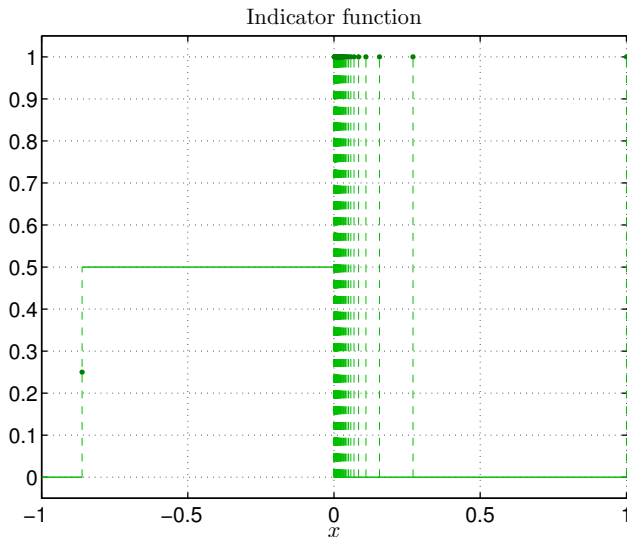


Figure: Spectrum on \mathcal{E} for mode $n = 0$ ($2\alpha = 31\pi/18$).

Step 1: Model case

- Model case: Γ_α a straight infinite circular cone. Homogeneity:

$$K_n^{\gamma_\alpha}(\lambda t, \lambda t') = \frac{1}{\lambda^2} K_n^{\gamma_\alpha}(t, t'), \quad \lambda > 0.$$

- Hence

$$K_n^{\gamma_\alpha} f(t) = \sin(\alpha) \int_0^\infty K_n^{\gamma_\alpha}\left(\frac{t}{t'}, 1\right) f(t') \frac{dt'}{t'},$$

a convolution against the group \mathbb{R}^+ with Haar measure $\frac{dt}{t}$. Let

$$F_n(\zeta) = \sin(\alpha) \mathcal{M} K_n^{\gamma_\alpha}(\zeta) = \sin(\alpha) \int_0^\infty t^\zeta K_n^{\gamma_\alpha}(t, 1) \frac{dt}{t}.$$

It turns out that

$$\Pi_n = \{F_n(1 + i\xi) : \xi \in \mathbb{R}\}, \quad \Sigma_n = \{F_n(3/2 + i\xi) : \xi \in \mathbb{R}\}$$

Step 2: Localization of model case

- Localize K^{Γ_α} to each conical point. The cone Γ_α has **two conical points**, equal but running in opposite directions.

Step 3: Perturbation

- Show that difference between K^{Γ_α} and K^Γ is compact, where Γ has a curvilinear corner.
- For $L^2(\Gamma)$: write difference of kernels as

$$\sum (\text{smooth kernels with decay at corner}) \times (\text{Riesz kernels}).$$

- The Riesz kernels do **not** define bounded operators on \mathcal{E} . How does one see that $K^\Gamma: \mathcal{E} \rightarrow \mathcal{E}$ is bounded? Use algebraic identities + real interpolation.

Step 4: Localization in general

- On $L^2(\Gamma)$, use the symbolic calculus for pseudodifferential Mellin type operators. Developed by Elschner, Lewis, Parenti. Index formula comes from their work.
- On \mathcal{E} we use Weyl sequences. Very weak method that yields no information on the character of the spectrum.