New approach to Balian-Low-type theorems

Mishko Mitkovski

Department of Mathematical Sciences
Clemson University

March 17, 2017

Uncertainty principle

Uncertainty principle in harmonic analysis:

f and $\hat{f}$ cannot be simultaneously be "too big".

This principle has many different manifestations.

Heisenberg uncertainty principle: The position and the momentum of a quantum system cannot be measured simultaneously.

Mathematical formulation: The differentiation operator $Pf(x) = f'(x)$ and multiplication operator $Xf(x) = xf(x)$ don't commute.
More precisely $[P, X] = iI$. These are unique.

Balian Low theorem

Riesz basis = Riesz sequence + frame

A sequence $\{k_n\}$ in a Hilbert space $\mathcal{H}$ is said to be a Riesz sequence if

$$\left\| \sum_n a_n k_n \right\|^2 \simeq \sum |a_n|^2 , \quad \text{for all } \{a_n\} \in l^2,$$

Riesz sequences can be viewed as Hilbert space generalizations of linearly independent sets.

A sequence $\{k_n\}$ in a Hilbert space $\mathcal{H}$ is said to be a frame if

$$\sum_n |\langle f, k_n \rangle|^2 \simeq \|f\|^2 , \quad \text{for all } f \in \mathcal{H},$$

Frames can be viewed as Hilbert space generalizations of spanning sets.

Balian Low theorem

Let $g \in L^2(\mathbb{R})$. Consider the Gabor system

$$g_{m,n}(x) = M_m T_n g(x) = e^{2\pi imx} g(x-n), \qquad m, n \in \mathbb{Z}.$$

Theorem (Balian, Low) If the Gabor system $\{g_{m,n} : m, n \in \mathbb{Z}\}$ is an o.n.b. (Riesz basis) for $L^2(\mathbb{R})$ then either

$$\int (1+x^2)\,|g(x)|^2\,dx = \infty \quad \text{or} \quad \int (1+\xi^2)\,|g(\xi)|^2\,d\xi = \infty.$$

The initial proofs contain gaps which were resolved by Rochberg and Semmes.

Battle's proof

Assume $g \in L^2(\mathbb{R})$ generates an o.n.b..

$$\langle Xg, Pg \rangle = \sum \langle Xg, M_m T_n g \rangle \langle M_m T_n g, Pg \rangle$$

Using that $\langle [M_{-m}T_{-n}, X]g, g \rangle = n \langle M_{-m}T_{-n}g, g \rangle = 0$ we obtain

$$\sum \langle M_{-m}T_{-n}g, Xg \rangle \langle Pg, M_{-m}T_{-n}g \rangle = \langle Pg, Xg \rangle .$$

This implies $\langle [X, P]g, g \rangle = 0$, i.e., $g = 0$ (since $[X, P] = iI$).
Contradiction!

Recent results

Extensions by many mathematicians: Benedetto, Czaja, Gautam, Grochenig, Hardin, Heil, Janssen, Malinikova, Nitzan, Northington, Olsen, Powell, Sterbenz, Walnut,...

Selective results:

Theorem (Benedetto, Powel, Gautam) Let $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$. If the Gabor system $\{g_{m,n} : m, n \in \mathbb{Z}\}$ is a Riesz basis for $L^2(\mathbb{R})$ then either $g \notin H^{p/2}(\mathbb{R})$ or $\hat{g} \notin H^{q/2}(\mathbb{R})$.

Recent results

Theorem (Nitzan, Olsen 2012) If the Gabor system $\{g_{m,n} : m, n \in \mathbb{Z}\}$ is a Riesz basis for $L^2(\mathbb{R})$ then

$$\int_{|x|\geq R} |g(x)|^2 \, dx + \int_{|\xi|\geq L} |g(\xi)|^2 \, d\xi \geq \frac{C}{RL}.$$

Theorem (Grochenig, Malinikova 2011) If $\{f_n\}$ is a Risez basis for $L^2(\mathbb{R})$ and $\epsilon > 0$, then

$$\int (1 + |x - a_n|^2)^{1+\epsilon} |f_n(x)|^2 \, dx + \int (1 + |\xi - b_n|^2)^{1+\epsilon} \left|\hat{f}_n(\xi)\right|^2 d\xi < \infty.$$

Bourgain: We can't take $\epsilon = 0$.

Density theorem

Let $\mathcal{F}$ and $\mathcal{G}$ be closed subspaces of $\mathcal{H}$.

Let $\{f_x\}_{x\in(X,d,\mu)}$ and $\{g_x\}_{x\in(X,d,\nu)}$ be continuous frames for $\mathcal{F}$ and $\mathcal{G}$ resp.

For any Borel set $\Omega \subseteq X$ the following equality holds

$$\int_\Omega \left\langle P_{\mathcal{G}}\tilde{f}_y, f_y \right\rangle d\mu - \int_\Omega \langle P_{\mathcal{F}} g_x, \tilde{g}_x \rangle \, d\nu = \int_{\Omega^c} \int_\Omega \langle g_x, f_y \rangle \left\langle \tilde{f}_y, \tilde{g}_x \right\rangle d[\mu, \nu],$$

where $d[\nu, \mu](x, y) = d\nu(x)d\mu(y) - d\mu(y)d\nu(x)$.

We say that they satisfy the localization property (L) if

$$\sup_{a\in X}\left|\int_{B(a,r)^c}\int_{B(a,r)} \langle g_x, f_y\rangle \left\langle \tilde{f}_y, \tilde{g}_x \right\rangle d[\nu,\mu](x,y)\right| = o((\mu+\nu)(B(a,r))),$$

as $r \to \infty$, where $d[\nu,\mu](x,y) = d\nu(x)d\mu(y) - d\mu(y)d\nu(x)$.

Density of measures

Let μ and ν be two (non-degenerate) Borel measures on the same metric space (X, d).

$$D_\nu^+(\mu) := \limsup_{r\to\infty} \sup_{a\in X} \frac{\mu\left(B(a;r)\right)}{\nu\left(B(a;r)\right)}, \qquad D_\nu^-(\mu) := \liminf_{r\to\infty} \inf_{a\in X} \frac{\mu\left(B(a;r)\right)}{\nu\left(B(a;r)\right)}.$$

We recover the classical Beurling densities by taking the measure ν to be the Lebesgue measure, and μ to be the counting measure of the sequence Λ whose density we are computing.

Density result

Let $\{f_x\}_{x\in(X,d,\mu)}$ and $\{g_x\}_{x\in(X,d,\nu)}$ be generalized frames for $\mathcal{F}$ and $\mathcal{G}$ respectively satisfying (L).
Then

(i) If $|\langle P_{\mathcal{F}} g_x, \tilde{g}_x\rangle| \leq 1$ for all $x \in \operatorname{supp}(\nu)$, then

$$D_\mu^-(\nu) \geq \lim_{r\to\infty} \inf_{a\in X} \frac{1}{\mu(B(a;r))} \left| \int_{B(a;r)} \left\langle P_{\mathcal{G}} \tilde{f}_y, f_y \right\rangle d\nu(y) \right|$$

(ii) If $\left\langle P_{\mathcal{G}} \tilde{f}_y, f_y \right\rangle \geq 1$ for all $y \in \operatorname{supp}(\mu)$, then

$$D_\nu^+(\mu) \leq \lim_{r\to\infty} \sup_{a\in X} \frac{1}{\nu(B(a;r))} \left| \int_{B(a;r)} \langle P_{\mathcal{F}} g_x, \tilde{g}_x \rangle\, d\mu(x) \right|.$$

(iii) If $|\langle P_{\mathcal{F}} g_x, \tilde{g}_x\rangle| \leq 1$ for all $x \in \operatorname{supp}(\nu)$, and $\left\langle P_{\mathcal{G}} \tilde{f}_y, f_y \right\rangle \geq 1$ for all $y \in \operatorname{supp}(\mu)$, then

$$D_\mu^-(\nu) \geq 1, \qquad D_\nu^+(\mu) \leq 1.$$

The applicability of the previous result depends heavily on how easy is to verify the localization condition (L).

$$\sup_{a\in X}\left|\int_{B(a,r)^c}\int_{B(a,r)}\langle g_x, f_y\rangle\left\langle \tilde{f}_y, \tilde{g}_x\right\rangle d[\nu,\mu](x,y)\right| = o((\mu+\nu)(B(a,r))),$$

as $r\to\infty$, where $d[\nu,\mu](x,y) = d\nu(x)d\mu(y) - d\mu(y)d\nu(x)$.

The applicability of the previous result depends heavily on how easy is to verify the localization condition (L).

$$\sup_{a\in X}\left|\int_{B(a,r)^c}\int_{B(a,r)}\langle g_x, f_y\rangle\left\langle \tilde{f}_y, \tilde{g}_x\right\rangle d[\nu,\mu](x,y)\right| = o((\mu+\nu)(B(a,r))),$$

as $r\to\infty$, where $d[\nu,\mu](x,y) = d\nu(x)d\mu(y) - d\mu(y)d\nu(x)$.
How is this connected to the Balian-Low theorem?

The applicability of the previous result depends heavily on how easy is to verify the localization condition (L).

$$\sup_{a\in X}\left|\int_{B(a,r)^c}\int_{B(a,r)}\langle g_x, f_y\rangle\left\langle \tilde{f}_y, \tilde{g}_x\right\rangle d[\nu,\mu](x,y)\right| = o((\mu+\nu)(B(a,r))),$$

as $r\to\infty$, where $d[\nu,\mu](x,y) = d\nu(x)d\mu(y) - d\mu(y)d\nu(x)$.
How is this connected to the Balian-Low theorem?

$$\int (1+|x-a|^2)^{1+\epsilon}\,|\phi(x)|^2\,dx + \int (1+|\xi-b|^2)^{1+\epsilon}\left|\hat{\phi}(\xi)\right|^2 d\xi < \infty.$$

iff

$$\int_{\mathbb{R}^2}\left|\left\langle M_b T_a\phi, g_{(x,\xi)}\right\rangle\right|^2(1+x^2+\xi^2)^{1+\epsilon}dxd\xi < \infty,$$

where $g_{(x,\xi)} = M_\xi T_x g, g(x) = Ce^{-x^2}$ classical Gabor system.

Proposition: If $\{f_n\}$ forms a Risez basis which is localized at $\Lambda = (a_n, b_n)$ then $D^{\pm}(\Lambda) = 1$.

Proposition: If $\{f_n\}$ forms a Risez basis which is localized at $\Lambda = (a_n, b_n)$ then $D^{\pm}(\Lambda) = 1$.

Note that the uncertainty principle is not used yet.

Proposition: If $\{f_n\}$ forms a Risez basis which is localized at $\Lambda = (a_n, b_n)$ then $D^{\pm}(\Lambda) = 1$.

Note that the uncertainty principle is not used yet.

We use it in Weyl's form: $M_a T_b = e^{-iab} T_b M_a$.

Proposition: If $\{f_n\}$ forms a Risez basis which is localized at $\Lambda = (a_n, b_n)$ then $D^{\pm}(\Lambda) = 1$.

Note that the uncertainty principle is not used yet.

We use it in Weyl's form: $M_a T_b = e^{-iab} T_b M_a$.

Modified identity for $\delta > 0$ close to 1:

$$\int_{B(a,r)} \left\langle \tilde{f}_y, f_y \right\rangle d\mu(y) - \int_{B(a,r^\delta)} \langle g_x, \tilde{g}_x \rangle \, d\nu(x)$$

$$= \int_{B(a,r^\delta)^c} \int_{B(a,r)} \langle g_x, f_y \rangle \left\langle \tilde{f}_y, \tilde{g}_x \right\rangle d\nu(x) d\mu(y) -$$

$$- \int_{B(a,r)^c} \int_{B(a,r^\delta)} \langle g_x, f_y \rangle \left\langle \tilde{f}_y, \tilde{g}_x \right\rangle d\mu(y) d\nu(x).$$

Cancellations in the right-hand occur due to the uncertainty principle.

Thank you.