# Noncommutative Kernels

Gregory Marx

Department of Mathematics, Virginia Tech, Blacksburg, VA
joint work with
Joseph A. Ball, Virginia Tech and Victor Vinnikov, Ben Gurion University

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# Noncommutative functions

- $\mathcal{V}$ is a vector space over $\mathbb{C}$
- $\mathcal{V}_{\mathrm{nc}} := \amalg_{n=1}^{\infty} \mathcal{V}^{n \times n}$
- $\mathcal{V}_n := \mathcal{V}_{\mathrm{nc}} \cap \mathcal{V}^{n \times n}$
- A nc set is a subset $\Omega \subseteq \mathcal{V}_{\mathrm{nc}}$ that is closed under direct sums: $Z \in \Omega_n,\ W \in \Omega_m \implies Z \oplus W = \left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right] \in \Omega_{n+m}$

# Noncommutative functions

## Example

In the case where $\mathcal{V} = \mathbb{C}^d$, we identify the set $(\mathbb{C}^d)_{\mathrm{nc}}$ with the collection of all d-tuples of $n \times n$ complex matrices and for $Z \in (\mathbb{C}^d)_n$, we write

$$Z = (Z_1, Z_2, \ldots, Z_d)$$

where $Z_i \in \mathbb{C}^{n \times n}$.

# Noncommutative functions

## Noncommutative functions

- A nc function $f : \Omega \subseteq \mathcal{V}_{0,nc} \to \mathcal{V}_{1,nc}$ is graded:

$$f : \Omega_n \to V_{1,n}$$

and respects intertwinings: For $Z \in \Omega_n$, $\widetilde{Z} \in \Omega_{\widetilde{n}}$, $A \in \mathbb{C}^{\widetilde{n} \times n}$,

$$AZ = \widetilde{Z}A \implies Af(Z) = f(\widetilde{Z})A$$

# Noncommutative functions

## Examples of noncommutative functions

Given a polynomial with freely noncommuting indeterminants and complex coefficients, e.g.,

$$p(z_1, z_2, z_3) = z_1 z_2 z_3 + 2z_1 z_2 - 3z_2 z_1$$

we may define a nc function from $(\mathbb{C}^3)_{\mathrm{nc}}$ to $\mathbb{C}_{\mathrm{nc}}$ by

$$p(Z_1, Z_2, Z_3) = Z_1 Z_2 Z_3 + 2Z_1 Z_2 - 3Z_2 Z_1$$

where $(Z_1, Z_2, Z_3) \in (\mathbb{C}^{n \times n})^3$.

## Examples of noncommutative functions

Given a polynomial with freely noncommuting indeterminants and matrix-valued coefficients, e.g.,

$$p(z_1, z_2, z_3) = \left[\begin{smallmatrix} 1 & 2 \\ 0 & 3 \end{smallmatrix}\right] z_1 z_2 z_3 + \left[\begin{smallmatrix} 3 & 5 \\ 1 & 0 \end{smallmatrix}\right] z_1 z_2 - \left[\begin{smallmatrix} 4 & 3 \\ 7 & 4 \end{smallmatrix}\right] z_2 z_1$$

we may define a nc function from $(\mathbb{C}^3)_{\mathrm{nc}}$ to $(\mathbb{C}^{2\times 2})_{\mathrm{nc}}$ by

$$\begin{aligned} p(Z_1, Z_2, Z_3) &= \left[\begin{smallmatrix} 1 & 2 \\ 0 & 3 \end{smallmatrix}\right] \otimes Z_1 Z_2 Z_3 + \left[\begin{smallmatrix} 3 & 5 \\ 1 & 0 \end{smallmatrix}\right] \otimes Z_1 Z_2 - \left[\begin{smallmatrix} 4 & 3 \\ 7 & 4 \end{smallmatrix}\right] \otimes Z_2 Z_1 \\ &= \begin{bmatrix} Z_1 Z_2 Z_3 + 3 Z_1 Z_2 - 4 Z_2 Z_1 & 2 Z_1 Z_2 Z_3 + 5 Z_1 Z_2 - 3 Z_2 Z_1 \\ Z_1 Z_2 - 7 Z_2 Z_1 & 3 Z_1 Z_2 Z_3 - 4 Z_2 Z_1 \end{bmatrix} \end{aligned}$$

where $(Z_1, Z_2, Z_3) \in (\mathbb{C}^{n\times n})^3$.

# Noncommutative kernels

- $\mathcal{S}_0$, $\mathcal{S}_1$ are operator systems (so equipped with a $*$)
- A nc kernel $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{S}_1, \mathcal{S}_0)_{\rm nc}$ is graded:

$$Z \in \Omega_n,\ W \in \Omega_m \Rightarrow K(Z,W) \in \mathcal{L}(\mathcal{S}_1^{n\times m}, \mathcal{S}_0^{n\times m})$$

and respects intertwinings:

$$Z \in \Omega_n,\ \widetilde{Z} \in \Omega_{\widetilde{n}},\ A \in \mathbb{C}^{\widetilde{n}\times n} \text{ such that } AZ = \widetilde{Z}A,$$
$$W \in \Omega_m,\ \widetilde{W} \in \Omega_{\widetilde{m}},\ B \in \mathbb{C}^{\widetilde{m}\times m} \text{ such that } BW = \widetilde{W}B,$$
$$P \in \mathcal{S}_1^{n\times m} \Rightarrow A\,K(Z,W)(P)\,B^* = K(\widetilde{Z},\widetilde{W})(APB^*)$$

# Noncommutative kernels

## Definition

A nc kernel $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{S}_1, \mathcal{S}_0)_{\mathrm{nc}}$ is completely positive if for all $n \in \mathbb{N}$:

$$Z \in \Omega_n,\ P \succeq 0 \text{ in } \mathcal{S}_1^{n\times n} \Rightarrow K(Z,Z)(P) \succeq 0 \text{ in } \mathcal{S}_0^{n\times n}.$$

## Proposition

Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras.
A nc kernel $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{B})_{nc}$ is completely positive iff

$$\sum_{i,j=1}^{N} b_i^* K(Z_i, Z_j)(P_i^* P_j) b_j \geq 0$$

for any $Z_j \in \Omega_{n_j}$, $P_j \in \mathcal{A}^{M\times n_j}$, $n_j \in \mathbb{N}$, $b_j \in \mathcal{B}^{n_j}$

# Noncommutative kernels

## Theorem (Ball-M.-Vinnikov)

Suppose that $\Omega$ is a nc subset of $\mathcal{V}_{\mathrm{nc}}$, $\mathcal{E}$ is a Hilbert space, $\mathcal{A}$ is a $C^*$-algebra and $K \colon \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{E}))_{\mathrm{nc}}$ is a given function. Then the following are equivalent.

1. $K$ is a cp nc kernel from $\Omega \times \Omega$ to $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{E}))_{\mathrm{nc}}$.
2. There exists a reproducing kernel Hilbert space $\mathcal{H}(K)$ which consists of nc functions $f : \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{E})_{\mathrm{nc}}$ and contains kernel elements with the reproducing property. Furthermore, $\mathcal{H}(K)$ is equipped with a unital $*$-representation $\sigma$ mapping $\mathcal{A}$ to $\mathcal{L}(\mathcal{H}(K))$.
3. $K$ has a Kolmogorov decomposition: there is a Hilbert space $\mathcal{X}$ equipped with a unital $*$-representation $\sigma \colon \mathcal{A} \to \mathcal{L}(\mathcal{X})$ together with a nc function $H \colon \Omega \to \mathcal{L}(\mathcal{X}, \mathcal{E})_{\mathrm{nc}}$ so that

$$K(Z, W)(P) = H(Z)(\mathrm{id}_{\mathbb{C}^{n \times m}} \otimes \sigma)(P)H(W)^*$$

for all $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathcal{A}^{n \times m}$.

# Noncommutative kernels

Theorem (Stinespring's dilation theorem)

Let $\mathcal{A}$ be a $C^*$-algebra and let $\mathcal{H}$ be a Hilbert space. For a linear map $\phi : \mathcal{A} \to \mathcal{L}(\mathcal{E})$, the following are equivalent:

1. $\phi$ is completely positive:

$$A = [a_{i,j}] \in \mathcal{A}^{n \times n} \geq 0 \Rightarrow [\phi(a_{i,j})] \geq 0$$

   for any $n \in \mathbb{N}$.

2. There is a Hilbert space $\mathcal{X}$, a map $H \in \mathcal{L}(\mathcal{X}, \mathcal{E})$, and a unital $*$-representation $\sigma : \mathcal{A} \to \mathcal{L}(\mathcal{X})$ such that

$$\phi(a) = H\sigma(a)H^*.$$

# Arveson's extension theorem

### Theorem (Arveson)

Let $\mathcal{S}$ be an operator system contained in a $C^*$-algebra $\mathcal{A}$. For every completely positive map $\phi : \mathcal{S} \to \mathcal{L}(\mathcal{E})$, there is a completely positive map $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{E})$ such that $\Phi|_{\mathcal{S}} = \phi$.

### Theorem (nc kernel version)

Let $\mathcal{S}$ be an operator system contained in a $C^*$-algebra $\mathcal{A}$, and let $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{S}, \mathcal{L}(\mathcal{E}))_{\mathrm{nc}}$ be a cp nc kernel. Then there is a cp nc kernel $\widetilde{K} : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{E}))_{\mathrm{nc}}$ so that $\widetilde{K}(Z, W)(P) = K(Z, W)(P)$ for all $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathcal{S}^{n \times m}$.

# Bimodule maps

**Definition**

Consider a linear map $\phi : \mathcal{S}^{n\times n} \to \mathcal{L}(\mathcal{E})^{n\times n}$ and a $C^*$-algebra $\mathcal{C} \subseteq \mathbb{C}^{n\times n}$. We say that $\phi$ is a $\mathcal{C}$-bimodule map if for all $P \in \mathcal{S}^{n\times n}$

$$\phi(\alpha P \beta) = \alpha \phi(P) \beta$$

for all $\alpha, \beta \in \mathcal{C}$.

**Lemma**

*Let $\phi : \mathcal{S}^{n\times n} \to \mathcal{L}(\mathcal{E})^{n\times n}$ be a completely positive $\mathcal{C}$-bimodule map, and let $\Phi : \mathcal{A}^{n\times n} \to \mathcal{L}(\mathcal{E})^{n\times n}$ be an extension of $\phi$. Then $\Phi$ is a $\mathcal{C}$-bimodule map.*

# Bimodule maps and nc kernels

**Lemma**

Let $K$ be a completely positive nc kernel and fix a $Z$ in $\Omega_n$. Let $\mathcal{D}$ denote the subset of $\mathbb{C}^{n\times n}$ such that for any $\alpha \in \mathcal{D}$

$$\alpha K(Z,Z)(I) = K(Z,Z)(I)\alpha$$

Then $\mathcal{D}$ is a $C^*$-algebra and $K(Z,Z)$ is a $\mathcal{D}$-bimodule map.

**Remark**

When $\Omega = \{Z^{(1)}, Z^{(2)}, \dots, Z^{(n)}\}$ and $Z = \oplus_{i=1}^{n} Z^{(i)}$, the noncommutative kernel structure of $K$ is "encoded" into $\mathcal{D}$.

# Arveson's extension theorem (nc kernel version)

Strategy for Arveson's extension theorem (nc kernel version)

1. Consider the special case with finite set $\Omega = \{Z^{(1)}, Z^{(2)}, \cdots, Z^{(n)}\}$ and a finite-dimensional $\mathcal{E}$.
2. Let $Z = \oplus_{i=1}^{n} Z^{(i)}$ and construct the completely positive $\mathcal{D}$-bimodule map $\phi = K(Z, Z)$
3. Use Arveson's extension theorem to extend $\phi$ to a completely positive $\mathcal{D}$-bimodule map $\Phi$.
4. Derive the extended kernel $\widetilde{K}$ from $\Phi$
5. Use compactness argument to extend to general $\Omega$ and $\mathcal{E}$, i.e., Kurosh's Theorem: The limit of an inverse spectrum of nonempty compacta is a nonempty compacta

# Globally bounded nc kernels

Let $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{S}, \mathcal{L}(\mathcal{E}))_{\mathrm{nc}}$ be a nc kernel.

- $K$ is globally bounded if $\|K(Z, Z)\|_{\mathrm{cb}} \leq M$ for any $Z \in \Omega$ with $M$ fixed
- The adjoint of $K$ is given by

$$K^*(Z, W)(P) = K(W, Z)(P^*)^*.$$

If $K^*(Z, W) = K(Z, W)$, the kernel is Hermitian.

# Wittstock's decomposition theorems

Theorem (Wittstock 1981)

- Completely bounded Hermitian maps are the difference of completely positive maps.
- Completely bounded maps are the linear span of completely positive maps.

Theorem (nc kernel version)

- Globally bounded Hermitian nc kernels are the difference of globally bounded completely positive nc kernels.
- Globally bounded nc kernels are the linear span of globally bounded completely positive nc kernels.

# Wittstock's decomposition theorems (nc kernel version)

Strategy for Wittstock's decomposition theorems

1. Use Paulsen's off-diagonal set up to convert the given problem to that of extending a completely positive nc kernel
2. Use the nc kernel version of Arveson's extension theorem to extend the completely positive nc kernel

# Citations and comments

- Completely bounded kernels (Bhattacharyya, Dritschel, Todd, 2013)
- Noncommutative reproducing kernel Hilbert spaces (Ball, M., Vinnikov, 2016)
- *Foundations of Free Noncommutative Function Theory* (Kaliuzhnyi-Verbovetskyi, Vinnikov, 2014)
- *Completely Bounded Maps and Operator Algebras* (Paulsen, 2002)

# The end

Thank you!