# Density of continuous functions in de Branges-Rovnyak spaces

Bartosz Malman
Joint work with Alexandru Aleman

Lund University

March 17, 2017

33rd Southeastern Analysis Meeting
Knoxville, Tennessee

# de Branges-Rovnyak space $\mathcal{H}(b)$

For $b \in H^\infty = H^\infty(\mathbb{D})$ with $\|b\|_\infty \le 1$ define

$$\mathcal{H}(b) = (1 - T_b T_{\overline{b}})^{1/2} H^2$$

with norm

$$\|f\|_b = \inf_{\substack{g \in H^2, \\ f = (1 - T_b T_{\overline{b}})^{1/2} g}} \|g\|_2.$$

Reproducing kernel of $\mathcal{H}(b)$

$$k_b(z, \lambda) = \frac{1 - \overline{b(\lambda)} b(z)}{1 - \overline{\lambda} z}.$$

# Dichotomy extreme/non-extreme

If $b$ is non-extreme, then $P$, set of polynomials, is contained in $\mathcal{H}(b)$.

If $b$ is extreme, then $P \not\subset \mathcal{H}(b)$.

**Theorem 1 (Sarason, 1986)**

*If $b$ is non-extreme, then $\mathcal{P}$ is dense in $\mathcal{H}(b)$.*

# Special case

If $b$ inner function, then $\mathcal{H}(b) = H^2 \ominus bH^2$ isometrically.

**Theorem 2 (Aleksandrov, 1981)**

*Let $b$ be an inner function. Then the intersection $\mathcal{A} \cap \mathcal{H}(b)$ is dense in $\mathcal{H}(b)$, where $\mathcal{A}$ is the disc algebra.*

# Theorem

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## Theorem 3

*Let $\mathcal{A}$ be the disc algebra. The intersection $\mathcal{A} \cap \mathcal{H}(b)$ is dense in $\mathcal{H}(b)$ for all $b$ in the unit ball of $H^\infty$.*

# Few words about proof: representation of $\mathcal{H}(b)$

**Proposition 4**

*Let $b$ be an extreme point of the unit ball of $H^\infty$ and $E = \{\zeta \in \mathbb{T} : |b(\zeta)| < 1\}$. Then there exists an isometry $J$*

$$\mathcal{H}(b) \ni f \mapsto Jf = (f, g) \in H^2 \oplus L^2(E)$$

*satisfying*

$$J(\mathcal{H}(b))^\perp = \left\{ (bh, \sqrt{1 - |b|^2} h) : h \in H^2 \right\}.$$

## Few words about proof: Duality argument

We study the annihilator of $J(\mathcal{A} \cap \mathcal{H}(b)) \subset \mathcal{A} \oplus L^2(E)$:

$$J(\mathcal{A} \cap \mathcal{H}(b))^{\perp} \subset C \oplus L^2(E),$$

$C = \mathcal{A}'$ is the space of Cauchy transforms of finite measures on $\mathbb{T}$.

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$$J(\mathcal{A} \cap \mathcal{H}(b)) = \cap_{h \in H^2} \ker \phi_h,$$

where

$$\phi_h = \left( hb, h\sqrt{1 - |b|^2} \right) \in (\mathcal{A} \oplus L^2(E))' = C \oplus L^2(E).$$

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$J(\mathcal{A} \cap \mathcal{H}(b))^{\perp}$ is the weak-star closure of $\{\phi_h\}_{h \in H^2}$ (Hahn-Banach theorem).

# Duality argument: the set $S$

$$S = \left\{ (f, h) \in C \oplus L^2(E) : \frac{f}{b} \in N^+, \frac{f}{b} = \frac{h}{\sqrt{1 - |b|^2}} \text{ on } E \right\}.$$

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**Lemma 5**

*The set $S$ is weak-star closed in $C \oplus L^2(E)$. Consequently, the annihilator $J(\mathcal{A} \cap \mathcal{H}(b))^\perp$ is contained in $S$.*

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Given Lemma 5, we can prove Theorem 3.

**Proof of Theorem 3.**

For $Jf = (f,g) \in H^2 \oplus L^2(E) \subset C \oplus L^2(E)$ we have

$$J(\mathcal{A} \cap \mathcal{H}(b)) \perp Jf \Rightarrow Jf = (f,g) \in S,$$

i.e.,
$\exists h \in H^2$ s.t. $Jf = (bh, \sqrt{1-|b|^2}h) \in J(\mathcal{H}(b))^\perp \Rightarrow Jf = 0.$ □

# Sketch of proof of Lemma 5

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$$S \ni (f_n, h_n) \xrightarrow{\text{weak-star}} (f, h).$$

Can assume $h_n \to h$ pointwise a.e on $E$. Then $\lim_n f_n = \phi$ exists a.e on $E$. By a theorem of Khintchin and Ostrowski, $f = \phi$ on $E$.

# Theorem of Khintchin and Ostrowski

**Theorem 6 (Khintchin, Ostrowski [3])**

*Let $f_n$ be a sequence of analytic functions which satisfy*

$$\sup_{r\in(0,1)} \int_{\mathbb{T}} \log^+ |f_n(re^{it})|dt \leq C.$$

*If the boundary values $f_n(\zeta)$ converge on a set of positive measure $E$, then $f_n$ converge uniformly on compact subsets of $\mathbb{D}$ to a holomorphic function $f$, and*

$$\lim_n f_n(\zeta) = f(\zeta)$$

*almost everywhere on $E$.*

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Then, for almost every $\zeta \in E$

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By a theorem of Vinogradov, $C$ has the (contractive) $F$-property. If $f_n, f \in C$ and $I$ is the inner factor of $b$, then

$$f_n \xrightarrow{\text{weak-star}} f \quad \Rightarrow \quad f_n/I \xrightarrow{\text{weak-star}} f/I \in C \subset N^+.$$

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So $f/b \in N^+$, and $S$ is weak-star closed. □

# References


[1] C. Bénéteau, A. A. Condori, C. Liaw, W. T. Ross, and A. A. Sola, *Some open problems in complex and harmonic analysis: Report on problem session held during the conference Completeness problems, Carleson measures, and space of analytic functions*, Contemporary Mathematics, Volume 679

[2] Donald Sarason, *Sub-Hardy Hilbert Spaces in the Unit Disk*, J. Wiley & Sons, 1994

[3] Victor Havin and Burglind Jöricke, *The uncertainty principle in harmonic analysis*, Springer-Verlag, Berlin, 1994